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MINIMAL PERMUTATION REPRESENTATIONS OF FINITE SIMPLE ORTHOGONAL GROUPS

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In the paper, nontrivial permutation representations of minimal degree are studied for finite simple orthogonal groups. For them, we find degrees, ranks, subdegrees, point stabilizers and their pairwise intersections.

For a finite group, a faithful permutation representation of minimal degree is called a *minimal permutation representation.* A classical group is a general linear, symplectic, orthogonal, or unitary group over a field; a simple classical group is one which is isomorphic to the (unique) non-Abelian composition factor of some classical group.

This paper continues [1], where minimal permutation representations of $L_m(q)$, $S_m(q)$, and $U_m(q)$ were studied. Our present goal is to obtain a description of minimal permutation representations of finite simple orthogonal groups. We follow the notation and terminology of [1].

Let V be a vector space of dimension m over a finite field k of order q. A *quadratic form* on V is a map $F\colon V \to \mathrm{k}$ such that $F(\lambda x) = \lambda^2 F(x)$ and $F(x+y) = F(x) + f(x,y) + F(y)$, where f is some *symmetric bilinear* form. Throughout the paper, the value $f(x,y)$ of the form f on a pair x, $y \in V$ will be denoted by (x,y). A form f is symmetric if $(x,y) = (y,x)$ for all $x, y \in V$. A vector v is said to be *isotropic* with respect to F if it is isotropic with respect to f, and $F(v) = 0$. A subspace is *isotropic with respect to* F if it is isotropic with respect to f and all of its elements are isotropic with respect to F.

A quadratic form F is said to be *nondegenerate* if the kernel $\ker F = \{x | x \in \ker f,\ F(x) = 0\}$ of F contains only the zero vector. A subspace is said to be *anisotropic with respect to* F if the restriction of F to the subspace is nondegenerate. It is clear that f is determined by F. On the other hand, if the characteristic of k is odd, then $4 \cdot F(x) = F(2x) = 2F(x) + (x,x)$, i.e., $F(x) = (x,x)/2$, and F is also uniquely determined by f. The latter is not true for a field of even characteristic.

In what follows, we will assume that F is a nondegenerate form.

LEMMA 1. If the dimension m of the space V is odd, $m = 2t+1$, then V has a basis $u_1, v_1, \dots, u_t, v_t, a$ such that $F(a) \neq 0$, $F(u_i) = F(v_i) = (u_i, v_j) = (u_i, a) = (v_i, a) = 0$, and $(u_i, v_i) = 1$ for i, $j = 1, 2, \dots, t$, $i \neq j$.

If the dimension m of V is even, $m = 2t$, then V has a basis of one of the following types:

(a) a basis u_1, $v_1, \dots, u_t, v_t$ such that $F(u_i) = F(v_i) = (u_i, v_j) = 0$ and $(u_i, v_i) = 1$ for i, $j = 1, 2, \dots, t$, $i \neq j$

or

(b) a basis u_1, $v_1, \dots, u_{t-1}, v_{t-1}$, a, b such that $F(u_i) = F(v_i) = (u_i, v_j) = (u_i, a) = (v_i, a) = (u_i, b) = (v_i, b) = 0$, $(u_i, v_i) = F(a) = (a, b) = 1$ for i, $j = 1, 2, \dots, t-1$, $i \neq j$, and $F(b) = \lambda$, where the polynomial $x^2 + x + \lambda$ has no roots in k.

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Proof. See [2, Sec. 2.5].

The basis of V of one of the types (a) or (b), specified above, is called *standard*. When the dimension of V is even, we call it a *space of type* $+$ if its basis is of type (a); otherwise, V is a *space of type* $-$. In the general case we speak of *spaces of type* ε, where ε refers either to one of the symbols $+$, $-$ or to one of the numbers $1, -1$.

A one-dimensional subspace X is called a *plus point* if its orthogonal complement $X^\perp$ has even dimension and is of type $+$ with respect to the restrictions of f and F to $X^\perp$. Similarly, X is a *minus point* if $X^\perp$ is of type $-$. When a transformation φ of the space V is written in matrix form $[\varphi]$, it is assumed that, unless specified otherwise, a standard basis is chosen in V; the order of columns of $[\varphi]$ corresponds to the order of vectors in the basis specified in Lemma 1.

LEMMA 2. Let $i(V)$ be the number of vectors in V which are isotropic with respect to F.

If $\dim(V) = 2t$ is even and V is of type ε, then $i(V) = q^{2t+1} + \varepsilon q^{t-1}(q-1)$.

If $\dim(V) = 2t+1$ is odd, then $i(V) = q^{2t}$.

Proof. We proceed by induction on $\dim(V)$. If $\dim(V) = 1$ or $\dim(V) = 2$ and V is a space of type $-$, then only the zero vector is isotropic with respect to F. If $\dim(V) = 2$ and V is of type $+$, then a vector $x = \alpha u + \beta v$, where u and v form a standard basis, is isotropic if and only if $\alpha\beta = 0$. It follows that there exist $2q-1$ isotropic vectors in V.

Let $\dim(V) \geq 3$. Denote by W the orthogonal complement to the subspace spanned by elements $u = u_1$ and $v = v_1$ of a standard basis. Then W has the same type as V, and for every isotropic vector $x = \alpha u + \beta v + w$, where $w \in W$, $\alpha\beta = -F(w)$. For every isotropic vector w, there exist $2q-1$ such vectors x; for every anisotropic vector, there exist $q-1$ vectors. Thus $i(V) = i(W)(2q-1) + (|W| - i(W))(q-1)$, and the lemma is proved by induction.

LEMMA 3. Let $i_d(V)$ be the number of subspaces of dimension d in V which are isotropic with respect to F.

If $\dim(V) = 2t$ is even and V is of type ε, then

$$i_d(V) = \prod_{j=0}^{d-1} (q^{t-j-1} + \varepsilon)(q^{t-j} - \varepsilon)/(q^{j+1} - 1).$$

If $\dim(V) = 2t+1$ is odd, then

$$i_d(V) = \prod_{j=0}^{d-1} (q^{2t-2j} - 1)/(q^{j+1} - 1).$$

Proof. The conclusion of Lemma 2 provides the necessary basis for a proof by induction. Now, if W is an isotropic subspace of dimension $d \geq 2$ in V and x is a nonzero vector in W, then $W \subseteq x^\perp$; since the kernel of the restriction of F to $x^\perp$ coincides with kx, the form obtained by restricting F to $x^\perp$ and then using induction on $x^\perp/kx$ is nondegenerate. Thus, W/kx is an isotropic subspace of dimension $d-1$ in the orthogonal space $X = x^\perp/kx$ of dimension $m-2$ which has the same type as V. It follows that every isotropic vector x of V is contained in $i_{d-1}(X)$ isotropic subspaces of dimension d in V. On the other hand, every such subspace contains $q^d - 1$ nonzero vectors all of which are isotropic. Therefore, $i_d(V) = i_{d-1}(X)(i(V)-1)/(q^d-1)$, from which the conclusion of the lemma follows by induction.

The orthogonal group of a quadratic form F is the group $GO(F) = \{\varphi \in GL(V) | \,\forall x \in V\ F(x\varphi) = F(x)\}$. It is clear that $GO(F) \subseteq I(f)$, and that $GO(F) = I(f)$ if the characteristic of the field is odd. If the

characteristic is equal to 2, then $(x, x) = 0$ for every x in V, and $GO(F)$ is a subgroup of the symplectic group $Sp(V)$.

Witt's LEMMA. Let $\varphi\colon U \to W$ be a nondegenerate linear map of a subspace U of V onto a subspace W of V. If φ satisfies the condition $F(x\varphi) = F(x)$ for all $x \in U$, then φ can be extended to an element of the group $GO(F)$.

Proof. See [39, Sec. 20].

It follows easily from Lemma 1 that, for a space V of odd dimension over $\mathbf{k}$ and for all quadratic forms F, groups $GO(F)$ are isomorphic. We will denote each of them by $GO(V)$, and the corresponding matrix group by $GO_m(\mathbf{k})$ or $GO_m(q)$.

For a space V of even dimension over $\mathbf{k}$, there exist two classes of isomorphic groups $GO(F)$, depending on which of the cases of Lemma 1 is realized. Every representative of the class corresponding to case (a) will be denoted by $GO^+(V)$, and the corresponding matrix groups, by $GO^+_m(\mathbf{k})$ and $GO^+_m(q)$. The groups corresponding to case (b) are denoted by $GO^-(V)$, $GO^-_m(\mathbf{k})$, and $GO^-_m(q)$. In the general case we write $GO^\varepsilon(V)$, where ε is either $+$ or $-$ if the dimension of V is even, and ε is an empty symbol if the dimension of V is odd.

Considering those elements of the orthogonal group whose determinant equals 1, we obtain the *special orthogonal group* $SO^\varepsilon(V)$ [resp. $SO^\varepsilon_m(q)$]. The *projective special orthogonal group* $PSO^\varepsilon(V)$ $[PSO^\varepsilon_m(q)]$ is the factor group of $SO^\varepsilon(V)$ $[SO^\varepsilon_m(q)]$ w.r.t. the central subgroup. In the general case $PSO^\varepsilon(V)$ is not simple. Except for finitely many cases, however, it contains a certain normal subgroup which is non-Abelian and simple. This simple subgroup will be denoted by $O^\varepsilon(V)$ [in matrix form, by $O^\varepsilon_m(q)$].

Following [4], we now give the exact definition of $O^\varepsilon(V)$. Let $v \in V$ and $F(v) \neq O$. A transformation r_v of $GL(V)$ defined by the rule $x \cdot r_v = x - \frac{(x,v)}{F(v)} \cdot v$ has order 2, belongs to $GO^\varepsilon(V)$, and is called a *reflection*. For odd q, every element of $GO^\varepsilon(V)$ can be written as a product of reflections. Let $g \in GO^\varepsilon(V)$; then $g = r_{v_1} \cdot \ldots \cdot r_{v_l}$, where $v_1, \ldots, v_l \in V$. This expression is not unique for g, but the map $\Theta\colon GO^\varepsilon(V) \to \mathbf{k}^*/(\mathbf{k}^*)^2$ given by the rule $\Theta(g) = F(v_1) \cdot \ldots \cdot F(v_l) \cdot (\mathbf{k}^*)^2$ does not depend on the choice of $r_{v_1}, \ldots, r_{v_l}$. Thus, Θ is a well-defined function, which we call the *spinor norm*.

Denote by $\Omega^\varepsilon(V)$ the set of all $g \in SO^\varepsilon(V)$ such that $\Theta(g) = 1 \cdot (\mathbf{k}^*)^2$. Since, for odd q, the order of $\mathbf{k}^*/(\mathbf{k}^*)^2$ is equal to 2, the set $\Omega^\varepsilon(V)$ is a subgroup of index 2 in $SO^\varepsilon(V)$. Its image $P\Omega^\varepsilon(V)$ in $PSO^\varepsilon(V)$ will be denoted by $O^\varepsilon(V)$.

Suppose now that q is even and the dimension m of V is odd, $m = 2t + 1$. Then $GO(V) = SO(V)$ is isomorphic to the simple symplectic group $S_{2t}(q)$, and we define $O(V)$ by the equalities $O(V) = \Omega(V) = SO(V) = GO(V)$.

Finally, let q be even and let $m = 2t$, i.e., the dimension of V is also even. We define the map from $SO^\varepsilon(V)$ into $\{\pm 1\}$ which takes every $g \in SO^\varepsilon(V)$ to $(-1)^d$, where d is the dimension of the subspace of fixed points of g. (Hereinafter, for convenience we denote this map by Θ.) The map is a homomorphism, and its kernel is a subgroup of index 2 in $SO^\varepsilon(V)$, denoted by $\Omega^\varepsilon(V)$. Denote by $O^\varepsilon(V)$ the group $P\Omega^\varepsilon(V) = \Omega^\varepsilon(V)$.

When $m = 2t + 1$, the orthogonal groups have the following orders: $|GO_m(q)| = d \cdot N$, $|SO_m(q)| = |PSO_m(q)| = N$, $|\Omega_m(q)| = |P\Omega_m(q)| = |O_m(q)| = N/d$, where $N = q^{t^2}(q^{2t} - 1)(q^{2t-2} - 1) \cdot \ldots \cdot (q^2 - 1)$, $d = (2, q - 1)$.

When $m = 2t$, the orthogonal groups have the following orders: $|GO^\varepsilon_m(q)| = 2N$, $|SO^\varepsilon_m(q)| = 2N/e$, $|PSO^\varepsilon_m(q)| = 2N/e^2$, $|\Omega^\varepsilon_m(q)| = N/e$, $|P\Omega^\varepsilon_m(q)| = |O^\varepsilon_m(q)| = N/d$, where $N = q^{t(t-1)}(q^t - \varepsilon)(q^{2t-2} - 1)(q^{2t-4} - 1) \cdot \ldots \cdot (q^2 - 1)$, $d = (4, q^t - \varepsilon)$, $e = (q - 1, 2)$.

It has already been noted that minimal permutation representations of other simple classical groups are

described in [1]. Since $O_3(q)$ is isomorphic to $L_2(q)$, $O_4^+(q)$ to $L_2(q)L_2(q)$, $O_4^-(q)$ to $L_2(q^2)$, $O_5(q)$ to $S_4(q)$, $O_6^+(q)$ to $L_4(q)$, and $O_6^-(q)$ to $U_4(q)$, throughout the paper we can assume that $m \geq 7$. Furthermore, for q even, the groups $O_{2t+1}(q)$ and $S_{2t}(q)$ will be isomorphic; their minimal permutation representations are also described in [1].

Consider some permutation representations of $GO^\varepsilon(V)$ and $\Omega^\varepsilon(V)$, from which we can obtain minimal permutation representations of simple orthogonal groups. Denote the vectors u_1 and v_1 of a standard basis by u and v, respectively, and set $U = \mathrm{k}u$. Then $U^\perp = U \oplus W$, where W is spanned by all elements of the standard basis, except u and v.

Let $P_1 = P_1(GO(V))$ be the stabilizer of U in $GO(V)$. It is clear that P_1 preserves the series $O < U < U^\perp < V$. Denote by M the normal subgroup of P_1 which consists of elements acting trivially on the factors of the series, and set $L = \{\varphi \in GL(V) | u\varphi = \lambda u,\ v\varphi = \lambda^{-1}v,\ \varphi|_W \in GO(F|_W)\}$.

It is easy to verify that L is a subgroup and $P_1 = M \cdot L$. Let $\psi \in M$. Elementary calculations show that there exists $w(\psi) \in W$ such that $v\psi = v + w(\psi) - F(w(\psi))u$ and $w\psi = w - (w, w(\psi))u$ for every $w \in W$.

Conversely, every linear transformation ψ lies in M whenever it satisfies these equalities and the equality $u\psi = u$.

Thus, the correspondence $\psi \to w(\psi)$ is a one-to-one map of M onto W. Moreover, it is obviously an isomorphism of M onto the additive group of the space W. Now, if $\varphi \in L$ and $u\varphi = \lambda u$, then $w(\varphi^{-1}\psi\varphi) = \lambda w(\psi)\varphi$, and so P_1 is isomorphic to the semidirect product of the additive subgroup of W and $\mathrm{k}^* \times GO(W)$, on which the action of a pair $(\lambda, \varphi) \in \mathrm{k}^* \times GO(W)$ is defined by the equality $w^{(\lambda,\varphi)} = \lambda w\varphi$.

By assumption, the dimension of W is greater than 2, and so W contains a vector w isotropic with respect to F. The stabilizer of $\mathrm{k}w$ in L is isomorphic to $L_0 = \mathrm{k}^* \times P_1(GO(W))$ and the stabilizer of $\mathrm{k}w$ in $M \simeq W$ coincides with the orthogonal complement W_0 of w in W. The stabilizer D_2 of the one-dimensional subspace $\mathrm{k}w$ in P_1 is isomorphic to $W_0 \cdot L_0$.

Obviously, the stabilizer D_3 of $\mathrm{k}v$ in P_1 coincides with L. Simple calculations show that

$$|GO(V) \colon P_1| = 1 + |P_1 \colon D_2| + |P_1 \colon D_3|,$$

and hence we obtain the following:

Proposition 1. The rank of the permutation representation of $GO_m^\varepsilon(q)$ on the set of one-dimensional subspaces isotropic with respect to a nondegenerate quadratic form F is equal to 3. If n is a degree and n_2, n_3 are nontrivial subdegrees of this representation, then for $m = 2t+1$ we have

$n = (q^{2t} - 1)/(q - 1)$,
$n_2 = (q^{2t-1} - q)/(q - 1)$,
$n_3 = q^{2t-1}$;

for $m = 2t$,

$n = (q^t - \varepsilon)(q^{t-1} + \varepsilon)/(q - 1)$,
$n_2 = (q^t - \varepsilon q)(q^{t-2} + \varepsilon)/(q - 1)$,
$n_3 = q^{2t-2}$.

If H is a point stabilizer and if D_2 and D_3 are two-point stabilizers in this representation, then

$H = p^{s(m-2)} \cdot (GO_{m-2}^\varepsilon(q) \times (q - 1))$,
$D_2 = p^{s(m-3)} \cdot ((q - 1) \times p^{s(m-4)} \cdot (GO_{m-4}^\varepsilon(q) \times (q - 1)))$,
$D_3 = GO_{m-2}^\varepsilon(q) \times (q - 1)$,

where p is a prime and $q = p^s$.

Now we find the same invariants for $\Omega_m^\varepsilon(q)$, preserving the notation of Proposition 1.

If q is even, then $GO^\varepsilon(V) = SO^\varepsilon(V)$ and the stabilizer $P_1(SO^\varepsilon(V))$ of the one-dimensional isotropic subspace U is equal to P_1. The group $\Omega^\varepsilon(V)$ coincides with $GO^\varepsilon(V)$ for $m = 2t+1$. The same is true for stabilizers. Let $m = 2t$. The reflection r_d generated by $d = u_2 + v_2$ lies in P_1, because $u \cdot r_d = u$, and r_d does not lie in $\Omega^\varepsilon(V)$ because it stabilizes a subspace of dimension $2t-1$. Hence, the index in $\Omega^\varepsilon(V)$ of the stabilizer $\bar{P}_1 = P_1(\Omega^\varepsilon(V))$ is equal to the index of P_1 in $GO^\varepsilon(V)$. It is easy to see that $M \leq \bar{P}_1$ and $L \cap \bar{P}_1 = \bar{L} = \{\varphi \in GL(V) |\ u\varphi = \lambda u,\ v\varphi = \lambda^{-1}v,\ \varphi|_W \in \Omega(F|_W)\}$. Comparing the orders, we see that $\bar{P}_1 = M \cdot \bar{L}$. Since $m \geq 7$, a similar argument shows that $D_2(\Omega^\varepsilon(V)) = W_0 \cdot \bar{L}_0$, where $L_0 \simeq \mathbf{k}^* \times \bar{P}_1(W)$ and $D_3(\Omega^\varepsilon(V)) = \bar{L}$.

The case where q is odd is slightly more complicated. The reflection r_d (as any reflection) has determinant -1 and stabilizes U. Therefore, $|GO^\varepsilon(V) \colon P_1| = |SO^\varepsilon(V) \colon P_1(SO^\varepsilon(V))|$. An argument similar to the case where q is even shows that $P_1(SO^\varepsilon(V)) = M \cdot \widetilde{L}$, where $\widetilde{L} = \{\varphi \in GL(V) |\ u\varphi = \lambda u,\ v\varphi = \lambda^{-1}v,\ \varphi|_W \in SO(F|_W)\}$, $D_2(SO^\varepsilon(V)) = W_0 \cdot \widetilde{L}_0$, where $\widetilde{L}_0 = \mathbf{k}^* \times P_1(SO^\varepsilon(W))$, and $D_3(SO^\varepsilon(V)) = \widetilde{L}$.

Now consider the element $g_1 \in GO^\varepsilon(V)$ equal to the product of two reflections, r_c and $r_{\bar{c}}$, generated by the vectors $c = u + v$ and $\bar{c} = u + \alpha v$, where α lies in $\mathbf{k}^*$ but does not lie in $(\mathbf{k}^*)^2$. Since $\det g_1 = 1$ and $u \cdot g_1 = \alpha u$, the element g_1 belongs to $P_1(SO^\varepsilon(V))$ but does not belong to $\Omega^\varepsilon(V)$, because $\Theta(g_1) = \Theta(r_c) \cdot \Theta(r_{\bar{c}}) = \alpha \cdot (\mathbf{k}^*)^2 \neq 1 \cdot (\mathbf{k}^*)^2$. Hence, the index of the stabilizer $\bar{P}_1 = P_1(\Omega^\varepsilon(V))$ in $\Omega^\varepsilon(V)$ is equal to the index of P_1 in $GO^\varepsilon(V)$. As before, $M \leq \bar{P}_1$. Define $\bar{L}$ as the semidirect product of $\bar{L}^*$ and the group of order 2 generated by h, where $\bar{L}^* = \{\varphi \in GL(V) |\ u\varphi = \lambda u,\ v\varphi = \lambda^{-1}v,\ \lambda \in (\mathbf{k}^*)^2,\ \varphi|_W \in \Omega(F|_W)\}$ and $h = g_1 \cdot g_2$. In the last equality, g_1 is defined as above and g_2 is the product of two reflections generated by $d = u_2 + v_2$ and $\bar{d} = u_2 + \alpha \cdot v_2$. By comparing the orders and taking into account the inclusion $\bar{L} \leq \widetilde{L} \cap \Omega^\varepsilon(V)$, we obtain the desired equality $\bar{P}_1 = M \cdot \bar{L} = M \cdot ((\Omega^\varepsilon(W) \times (\mathbf{k}^*)^2) \cdot 2)$. A similar argument shows that $D_2(\Omega^\varepsilon(V)) = W_0 \cdot \bar{L}_0$, where $\bar{L}_0 = (P_1(W) \times (\mathbf{k}^*)^2) \cdot 2$ and $D_3(\Omega^\varepsilon(V)) = \bar{L}$.

Obviously, $|\Omega^\varepsilon(V) \colon \bar{P}_1| = 1 + |\bar{P}_1 \colon D_2(\Omega^\varepsilon(V))| + |\bar{P}_1 \colon D_3(\Omega^\varepsilon(V))|$. Hence, what we have proved so far is the following:

Proposition 1′. The rank of the permutation representation of $\Omega^\varepsilon_m(q)$ on the set of one-dimensional subspaces isotropic with respect to a nondegenerate quadratic form F is equal to 3. If n is a degree and n_2, n_3 are nontrivial subdegrees of this representation, their values are the same as the respective values in Proposition 1.

If H is a point stabilizer and D_2, D_3 are two-point stabilizers in this representation, then for even $q = 2^s$ we have

$H = 2^{s(m-2)} \cdot (\Omega^\varepsilon_{m-2}(q) \times (q-1))$,

$D_2 = 2^{s(m-3)} \cdot ((2^{s(m-4)} \cdot (\Omega^\varepsilon_{m-4}(q) \times (q-1))) \times (q-1))$,

$D_3 = \Omega^\varepsilon_{m-2}(q) \times (q-1)$;

for $q = p^s$, where p is an odd prime,

$H = p^{s(m-2)} \cdot ((\Omega^\varepsilon_{m-2}(q) \times (q-1)/2) \cdot 2)$,

$D_2 = p^{s(m-3)} \cdot (((p^{s(m-4)} \cdot ((\Omega^\varepsilon_{m-4}(q) \times (q-1)/2) \times (q-1)/2) \cdot 2)$,

$D_3 = (\Omega^\varepsilon_{m-2}(q) \times (q-1)/2) \cdot 2$.

Let V be a space of type $+$, let $\dim(V) = m = 2t$, and let U be its isotropic subspace of dimension t spanned by vectors $u_1, \ldots, u_t$ of a standard basis. Denote by W the subspace spanned by $v_1, \ldots, v_t$.

First, we will find the stabilizer $P_2 = P_2(GO^+(V))$ of U in $GO^+(V)$. By Witt's lemma, $GO^+(V)$ acts transitively on the set of isotropic subspaces of dimension t. Hence $|GO^+(V) \colon P_2| = i_t(V)$ (see Lemma 3).

Next, we describe two subgroups M and L in P_2 such that P_2 is their semidirect product. Let M be a subgroup consisting of those elements of $GO^+(V)$ which act identically on U. Obviously, $M \trianglelefteq P_2$. Let

$\varphi \in M$. In the basis $u_1, u_2, \dots, u_t, v_1, \dots, v_t$, the element φ corresponds to the matrix $[\varphi]$ which has the form $\left[\begin{array}{c|c} E & 0 \\ \hline C & E \end{array}\right]$, where $C = \begin{bmatrix} 0 & a_{12} & a_{13} & \dots \\ -a_{12} & 0 & a_{23} & \dots \\ -a_{13} & -a_{23} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$. It is clear that M is an elementary Abelian group of order $q^{(t-1)t/2}$. Obviously, $M \le SO^+(V)$ and $M \le \Omega^+(V)$. Let $L = \{\psi \oplus \eta \mid \psi \in GL(U), \eta \in GL(W)\} \cap GO^+(V)$. We will prove that for $\psi \in GL(U)$, there exists a unique element η in $GL(W)$ such that $\psi \oplus \eta \in GO^+(V)$. Every matrix in $GL(U)$ can be written as a product of transvections, i.e., matrices of the form $\mathrm{tr}_{ij}(\alpha) = \begin{bmatrix} 1 & & \alpha \\ & \ddots & \\ & & 1 \end{bmatrix}$ (α lies at the intersection of the ith row and the jth column; all other elements are equal to 0), and diagonal matrices. Hence, it suffices to prove the uniqueness of η for transvections and diagonal matrices. If $[\psi] = \mathrm{diag}(\alpha_1, \dots, \alpha_t)$, then $[\eta] = \mathrm{diag}(\alpha_1^{-1}, \dots, \alpha_t^{-1})$. Note that $\det(\psi \oplus \eta) = 1$. If $[\psi] = \mathrm{tr}_{ij}(\alpha)$, then $[\eta] = \mathrm{tr}_{ij}(-\alpha)$ and again $\det(\varphi \oplus \eta) = 1$. Thus, $L \simeq GL_t(q)$.

It is clear that $L \cap M = 1$ and $M \cdot L \le P_2(GO^+(V))$. But $|GO^+(V) : M \cdot L| = i_t(V) = |GO^+(V) : P_2|$, and so $P_2 = M \cdot L$. Since M and L are in $SO^+(V)$, we have $P_2 \le SO^+(V)$. Therefore, if q is odd, the set of isotropic subspaces of dimension t splits, under the action of $SO^+(V)$, into two orbits of equal size. Moreover, two such subspaces U_1 and U_2 are in the same orbit if and only if the parity of the dimension of $U_1 \cap U_2$ coincides with the parity of t, since in this case, one subspace can be transformed in the other by an even number of reflections. In the present case, the transformation g_1 defined in the proof of Proposition 1′ does not lie in $\Omega^+(V)$ but stabilizes U and has a determinant equal to 1. Hence $|SO^+(V) : P_2| = |\Omega^+(V) : P_2(\Omega^+(V))|$. Moreover, the stabilizer $\bar{P}_2 = P_2(\Omega^+(V))$ is equal to a semidirect product of M and $\bar{L}$, where $\bar{L} \simeq SL_t(q) \cdot (q-1)/2$.

If q is even, we have $GO^+(V) = SO^+(V)$. Then $P_2 = P_2(GO^+(V)) \le \Omega^+(V)$ since an element $\psi \oplus \eta$ of L stabilizes a subspace of dimension $2d$, if $[\psi] = \mathrm{diag}(\alpha_1, \dots, \alpha_t)$ and d is the number of units in $[\psi]$, and a subspace of dimension $2t-2$ if $[\psi]$ is a transvection. Hence, $\bar{P}_2 = P_2(\Omega^+(V)) = P_2$, and under the action of $\Omega^+(V)$, the set of isotropic subspaces of dimension t splits into two orbits. The condition for two subspaces to belong to one orbit is similar to the corresponding condition for odd q. The results obtained are summed up as follows.

Proposition 2. In the group $\Omega_m^+(q)$, $m = 2t$, the stabilizer of a t-dimensional subspace isotropic with respect to a nondegenerate quadratic form F has the form $p^{s(t-1)t/2} \cdot (SL_t(q) \cdot (q-1)/2)$ if $q = p^s$, where p is an odd prime, and the form $2^{s(t-1)t/2} \cdot GL_t(q)$ if $q = 2^s$. The index of this stabilizer is equal to $i_t(V)/2$ in both cases.

In the above notation, let $m = 8$. Then $i_4(V) = (q^3+1)(q^2+1)(q+1)(q^0+1) = 2 \cdot \frac{(q^4-1)(q^3+1)}{q-1} = 2 \cdot |GO^+(V) : P_1|$. Therefore, $|\Omega^+(V) : \bar{P}_1| = |\Omega^+(V) : \bar{P}_2|$. We shall find the other invariants for the representation of $\Omega^+(V)$ w.r.t. $\bar{P}_2$. Note that $\bar{P}_2$ acts transitively on each of the sets Δ_U and Γ_U, where $\Delta_U = \{X \mid X$ is an isotropic subspace, $\dim(X) = 4$, and $X \cap Y = 0\}$ and $\Gamma_U = \{Y \mid Y$ is an isotropic subspace, $\dim(Y) = 4$, and $Y \cap U = 2\}$. It is obvious that the stabilizer D_3 of a subspace $X \in \Delta_U$ in $\bar{P}_2$ is equal to $\bar{L}$, where $\bar{L} = L$ if q is even. Hence $|\Delta_U| = q^6$. The stabilizer D_2 of a subspace Y from Γ_U is equal to the semidirect product of M_0 and $\bar{L}_0$, where M_0 is an elementary Abelian group of order q^5, and $\bar{L}_0 = p^{4s} \cdot (SL_2(q) \cdot (q-1)/2 \times GL_2(q))$ for $q = p^s$, with p an odd prime; for even $q = 2^s$, $\bar{L}_0 = L_0 = 2^{4s} \cdot (GL_2(q) \times GL_2(q))$. In both cases $|\Gamma_U| = |\bar{P}_2 : \bar{D}_2| = q(q^3-1)(q^2+1)/(q-1)$. Elementary

calculations show that $|\Omega^+(V)\colon \bar{P}_2| = 1 + |\Delta_U| + |\Gamma_U|$. Therefore, the rank of this representation is equal to 3, and we obtain the following:

Proposition 2′. If a permutation representation of $\Omega_8^+(q)$ on the set of four-dimensional subspaces isotropic w.r.t. a nondegenerate quadratic form F is such that all of the subspaces constitute one orbit, then its rank is equal to 3.

If n is a degree and n_2, n_3 are nontrivial subdegrees of the representation, then

$n = (q^4-1)(q^3+1)/(q-1)$,

$n_2 = q(q^3-1)(q^2+1)/(q-1)$,

$n_3 = q^6$.

If H is a point stabilizer and D_2, D_3 are two-point stabilizers in this representation, then, for $q = 2^s$, we have

$H = 2^{6s}\cdot GL_4(q)$,

$D_2 = 2^{5s}\cdot(2^{4s}\cdot(GL_2(q)\times GL_2(q)))$,

$D_3 = GL_4(q)$;

for $q = p^s$, where p is an odd prime,

$H = p^{6s}\cdot(SL_4(q)\cdot(q-1)/2)$,

$D_2 = p^{5s}\cdot(p^{4s}\cdot(GL_2(q)\times(SL_2(q)\cdot(q-1)/2)))$,

$D_3 = SL_4(q)\cdot(q-1)/2$.

Slightly modifying the argument used in the proof of the last three propositions, we can prove the following auxiliary result.

LEMMA 4. For every d such that V contains an isotropic subspace of dimension d, the group $\Omega^\varepsilon(V)$ acts transitively on the set of all such subspaces, except for the case where $\dim(V) = m = 2t$, V is of type $+$, and $d = t$. In this exceptional case, the set splits into two orbits of the same length, and two subspaces of dimension t will be in the same orbit iff the parity of the dimension of their intersection coincides with the parity of t.

Proof. By Witt's lemma, $GO^\varepsilon(V)$ acts transitively on the set of all isotropic subspaces of a fixed dimension. Furthermore, we have already proved the lemma for $d = 1$ and the exceptional case (see Propositions 1′ and 2.) Consider the other cases.

Let U be an isotropic subspace of dimension d in a space V of dimension m. For $H \leq GL(V)$, we denote by $H(U)$ the stabilizer of U in H. If $d \leq t-2$ whenever $m = 2t$ and V is a space of type $-$ and if $d \leq t-1$ in other cases, then in the orthogonal complement $U^\perp$ to U there exist two vectors u and v such that $u, v \notin U$, $F(u) = F(v) = 0$, and $(u,v) = 1$. Complete the set $\{u, v\}$ to a standard basis for V. Note that if $q = 2^s$ and $m = 2t+1$, then the group $GO(V)$ coincides with $\Omega(V)$, and there is nothing to prove. Let $q = 2^s$ and $m = 2t$. Then the reflection generated by $u+v$ fixes a subspace of dimension $2t-1$ and stabilizes U. Hence, the stabilizer of $GO^\varepsilon(V)$ is not contained in $\Omega^\varepsilon(V)$. Thus, $|\Omega^\varepsilon(V)\colon \Omega^\varepsilon(U)| = |GO^\varepsilon(V)\colon GO^\varepsilon(U)| = i_d(V)$.

If q is odd, then the same reflection has determinant -1 and lies in $GO^\varepsilon(U)$. The product of this reflection and the reflection generated by $u + \alpha v$, where $0 \neq \alpha \notin (\mathrm{k}^*)^2$, lies in the intersection of $GO^\varepsilon(U)$ and $SO^\varepsilon(V)$ but does not lie in $\Omega^\varepsilon(V)$, since the spinor norm of this product equals $\alpha\cdot(\mathrm{k}^*)^2$. Thus,

$$|\Omega^\varepsilon(V)\colon \Omega^\varepsilon(U)| = |SO^\varepsilon(V)\colon SO^\varepsilon(U)| = |GO^\varepsilon(V)\colon GO^\varepsilon(U)| = i_d(V).$$

Now let $m = 2t+1$ and $d = t$. In view of the above, we can assume that q is odd. The transformation that takes the vector a of a standard basis to $-a$ and fixes all other elements of the basis stabilizes U and does not belong to $SO^\varepsilon(V)$. Let $u \in U$ and let v be a vector in V such that $(u,v) = 1$. The product of two

reflections generated by $u+v$ and $u+\alpha v$, $\alpha \in \mathrm{k}^*\backslash(\mathrm{k}^*)^2$, stabilizes U, lies in $SO^\varepsilon(V)$ and does not lie in $\Omega^\varepsilon(V)$. Thus, the conclusion of the lemma is true in this case.

It remains to consider the case where $m=2t$, $d=t-1$, and V is of type $-$. For even q, we have $GO^-(V)=SO^-(V)$. Furthermore, there exist an element α in k such that $F(\alpha b)=F(a)=1$, where a and b are elements of the standard basis. This follows from the equality $\mathrm{k}^*=(\mathrm{k}^*)^2$ which holds if q is even. The transformation that transposes a and αb and fixes all other elements of the standard basis does not lie in $\Omega^\varepsilon(V)$ and stabilizes U, because $U\cap\langle a,b\rangle=0$. Thus,

$$|GO^\varepsilon(V)\colon GO^\varepsilon(U)|=|\Omega^\varepsilon(V)\colon\Omega^\varepsilon(U)|.$$

If q is odd, then the reflection generated by a lies in $GO^\varepsilon(U)$ but does not lie in $SO^\varepsilon(V)$. A transformation that lies in $SO^\varepsilon(U)$ but does not lie in $\Omega^\varepsilon(V)$ can be defined as in the previous case. The lemma is proved.

Let the order of k be $q=p^s$, where p is an odd prime, and let C be an anisotropic subspace of V. Then $V=C+W$, where $W=C^\perp$. Therefore, if $P=P(GO^\varepsilon(V))$ is the stabilizer of C in $GO^\varepsilon(V)$, and $\varphi\in P$, then $\varphi=\psi\oplus\eta$, where $\psi\in GO^{\varepsilon_1}(C)$ and $\eta\in GO^{\varepsilon_2}(W)$. It follows that $P\simeq GO_d^{\varepsilon_1}(q)\times GO_{m-d}^{\varepsilon_2}(q)$. Here, $d=\dim(C)$ and ε_1, ε_2 depend on the choice of C. An essential case is one where $C=\mathrm{k}c$ is a one-dimensional subspace spanned by an anisotropic vector c. Using the fact that $GO_1(q)=\{1,-1\}$, along with some elementary calculations, we obtain the following:

Proposition 3. In the group $GO_m^\varepsilon(q)$ with q odd, the stabilizer of a one-dimensional subspace C anisotropic with respect to a nondegenerate quadratic form F has the form $GO_{m-1}^{\varepsilon'}(q)\times 2$, where ε' is empty, if $m=2t$, and equals $+1$ or -1 if C is a plus point or a minus point, respectively, and $m=2t+1$. If n is the index of the stabilizer in $GO_m^\varepsilon(q)$, then $n=q^{t-1}(q^t-\varepsilon)/2$ for $m=2t$ and $n=q^t(q^t+\varepsilon')/2$ for $m=2t+1$.

Since q is odd, $|GO^\varepsilon(V):SO^\varepsilon(V)|=2$. The reflection r_c generated by c stabilizes C and does not belong to $SO^\varepsilon(V)$. Hence, $|P(GO^\varepsilon(V))\colon P(SO^\varepsilon(V))|=2$, and so $P(SO^\varepsilon(V))\simeq SO_{m-1}^{\varepsilon'}(q)\cdot 2$. The product $g=r_d\cdot r_{\bar d}$, where $r_{\bar d}$ and r_d are the reflections generated by $d=u+v$ and $\bar d=u+\alpha v$, $\alpha\in\mathrm{k}^*\backslash(\mathrm{k}^*)^2$, for u, $v\in W$, $F(u)=F(v)=0$, and $(u,v)=1$, is contained in $P(SO^\varepsilon(V))$ but not in $\Omega^\varepsilon(V)$. The stabilizer $P(\Omega^\varepsilon(V))$ of C in $\Omega^\varepsilon(V)$ is isomorphic to $\Omega_m^{\varepsilon'}(q)\cdot 2$. Thus, we arrive at the following analog of Proposition 3 for the group $\Omega^\varepsilon(V)$.

Proposition 3′. In the group $\Omega_m^\varepsilon(q)$ with q odd, the stabilizer of a one-dimensional subspace C anisotropic with respect to a nondegenerate quadratic form F has the form $\Omega_{m-1}^{\varepsilon'}(q)\cdot 2$, where ε' is empty, if $m=2t$, and equals $+1$ or -1 if $m=2t+1$ and C is a plus point or a minus point, respectively. The index of the stabilizer in $\Omega_m^\varepsilon(q)$ is equal to the index of the stabilizer of C in $GO_m^\varepsilon(q)$, indicated in Proposition 3.

Consider the group $GO_m(3)$, $m=2t+1$. We assume that $F(a)=1$, where a is a vector of the standard basis. It is obviously a plus point and the subspace $C=\mathrm{k}c$, where $c=u_1-v_1$, is a minus point. Denote by $P_3=P_3(GO_m(3))$ the stabilizer of C in $GO_m(3)$. By Proposition 3, we have $P_3=GO^-(W)\cdot 2$, where $W=C^\perp$.

Set $\Delta_C=\{X=\mathrm{k}x\mid(c,x)=0,\ F(x)=-1\}$, $\Gamma_C=\{Y=\mathrm{k}y\mid Y\neq C,\ (c,y)\neq 0,\ F(y)=-1\}$. Since, by Witt's lemma, $GO(V)$ acts transitively on the set of minus points, P_3 acts transitively both on Δ_C and Γ_C.

Let $x=\alpha c+w$, where $\alpha\in\mathrm{k}$, $w\in W$, and $X=\mathrm{k}x\in\Delta_C$. Then $(c,x)=2\alpha F(c)+(c,w)=0$. Since $(c,w)=0$, we have $\alpha=0$ and $x=w$. Furthermore, $F(w)=F(x)=F(c)=-1$. Therefore, the stabilizer D_2 of X in P_3 is isomorphic to the direct product of a group of order 2 and the stabilizer of an anisotropic point $X=\mathrm{k}w$ of W. By Proposition 3, $D_2\simeq(GO_{m-2}(3)\times 2)\times 2$ and $n_2=|\Delta_c|=3^{t-1}(3^t+1)/2$.

Similarly, let $y = \alpha c + w$, $\alpha \in \mathrm{k}$, $w \in W$, and $Y = \mathrm{k}y \in \Gamma_C$. The coefficient α is nonzero since $(c, y) \neq 0$. Hence, $F(c) = F(y) = \alpha^2 F(c) + F(w)$ and, therefore, $F(w) = 0$. Thus, the stabilizer D_3 of Y in P_3 is isomorphic to the direct product of a group of order 2 and the stabilizer of an anisotropic point w of W in $GO^-(W)$. By Proposition 1, $D_3 \simeq (3^{m-3} \cdot (GO^-_{m-3}(3) \times 2)) \times 2$ and $n_3 = |\Gamma_C| = 2(3^t + 1)(3^{t-1} - 1)/2 = (3^t + 1)(3^{t-1} - 1)$. Therefore, $|GO_m(3) \colon P_3| = 1 + |\Gamma_C| + |\Delta_C|$.

Since $m \geq 7$, the arguments used in proving Propositions 1′ and 3′ are valid for $D_2(\Omega_m(3))$ and $D_3(\Omega_m(3))$, whose indices in $\bar{P}_3 = P_3(\Omega(V))$ are equal to n_2 and n_3, respectively. Moreover, $D_2(\Omega_m(3)) \simeq (\Omega_{m-2}(3) \cdot 2) \cdot 2$ and $D_3(\Omega_m(3)) \simeq (3^{m-3} \cdot (\Omega^-_{m-3}(3) \times 2)) \cdot 2$.

Proposition 4. The rank of the permutation representation of $\Omega_m(3)$, $m = 2t + 1$, on the set of minus points with respect to a nondegenerate quadratic form F is equal to 3. If n is a degree and n_2, n_3 are nontrivial subdegrees of the representation, then

$n = 3^t(3^t - 1)/2,$

$n_2 = 3^{t-1}(3^t + 1)/2,$

$n_3 = (3^t + 1)(3^{t-1} - 1).$

If H is a point stabilizer and D_2, D_3 are two-point stabilizers in this representation, then

$H = \Omega^-_{m-1}(3) \cdot 2,$

$D_2 = (\Omega_{m-2}(3) \cdot 2) \cdot 2,$

$D_3 = (3^{m-3} \cdot (\Omega^-_{m-3}(3) \times 2)) \cdot 2.$

Now let $m = 2t$ and consider the group $GO^+_m(3)$. The subspace $C = \mathrm{k}c$, where $c = u_1 - v_1$, is anisotropic. Denote by $P_4 = P_4(GO^+(V))$ the stabilizer of C in $GO^+(V)$. By Proposition 3, $P_4 = GO(W) \times 2$ where $W = C^\perp$. Choose the vectors $u_2, v_2, \ldots, u_t, v_t, \bar{c}$, where $\bar{c} = u_1 + v_1$, as a standard basis for W.

By Witt's lemma, $GO^+(V)$ acts transitively on the set of one-dimensional anisotropic subspaces $X = \mathrm{k}x$ such that $F(x) = F(c) = -1$. Therefore, P_4 acts transitively on the sets Δ_C and Γ_C, defined by the following equalities:

$\Delta_C = \{X = \mathrm{k}x | (c, x) = 0,\ F(x) = -1\},$

$\Gamma_C = \{Y = \mathrm{k}y | Y \neq C,\ (c, y) \neq 0,\ F(y) = -1\}.$

As in the proof of Proposition 4, we can show that if $x = \alpha c + w$, $\alpha \in \mathrm{k}$, $w \in W$, and $X = \mathrm{k}x \in \Delta_C$, then $x = w$ and $F(w) = F(c) = -1$. Since $\bar{C} = \mathrm{k}\bar{c}$ is a plus point for the restriction of F to W, and $F(\bar{c}) = 1$, it follows that $X = \mathrm{k}w$ is a minus point. Therefore, $D_2 \simeq (GO^-_{m-2}(3) \times 2) \times 2$, $n_2 = |\Delta_C| = 3^{t-1}(3^{t-1} - 1)/2$.

Arguing as in the proof of Proposition 4, we conclude that the stabilizer D_3 of $Y = \mathrm{k}y$ in P_4 is isomorphic to the direct product of a group of order 2 and the stabilizer of an isotropic point w from W in $GO(W)$, i.e., $D_3 \simeq (3^{m-3} \cdot (GO_{m-3}(3) \times 2)) \times 2$, $n_3 = |\Gamma_C| = 3^{2t-2} - 1$. Hence, $|GO^+_m(3) \colon P_4| = 1 + |\Gamma_C| + |\Delta_C|$.

Using Propositions 1′ and 3′, we obtain the following:

Proposition 5. The rank of the permutation representation of the group $\Omega^+_m(3)$, $m = 2t$, on the set of one-dimensional anisotropic subspaces spanned by vectors with equal values of a nondegenerate quadratic form F is equal to 3.

If n is a degree and n_2, n_3 are nontrivial subdegrees of the representation, then

$n = 3^{t-1}(3^t - 1)/2,$

$n_2 = 3^{t-1}(3^{t-1} - 1)/2,$

$n_3 = 3^{2t-2} - 1.$

If H is a point stabilizer and D_2, D_3 are two-point stabilizers in this representation, then

$H = \Omega_{m-1}(3) \cdot 2,$

$D_2 = (\Omega^-_{m-2}(3) \cdot 2) \cdot 2,$

$D_3 = (3^{m-3} \cdot (\Omega_{m-3}(3) \times 2.$

We turn to the case where $q = 2^s$. Recall that we can assume $m = 2t$.

LEMMA 5. The index of the stabilizer of a one-dimensional anisotropic subspace in $\Omega_m^\varepsilon(q)$, where $q = 2^s$ and $m = 2t$, is equal to $q^{t-1}(q^t - \varepsilon)$.

Proof. In a field of characteristic 2 we have $(\mathrm{k}^*)^2 = \mathrm{k}^*$. Therefore, by Witt's lemma, $GO_m^\varepsilon(q)$ acts transitively on the set of all anisotropic subspaces. Thus, if C is a one-dimensional anisotropic subspace spanned by a vector c and P is its stabilizer in $GO_m^\varepsilon(q)$, then $|GO_m^\varepsilon(q) : P| = |C^{GO_m^\varepsilon(q)}| = (q^{2t} - 1)/(q - 1) - (q^t - \varepsilon)(q^{t-1} + \varepsilon)/(q-1) = q^{t-1}(q^t - \varepsilon)$. Denote by $\bar{P}$ the stabilizer of C in $\Omega_m^\varepsilon(q)$. The reflection r_c does not belong to $\Omega_m^\varepsilon(q)$ but belongs to P. Hence, $|\Omega_m^\varepsilon(q) : \bar{P}| = |GO_m^\varepsilon(q) : P| = q^{t-1}(q^t - \varepsilon)$. The lemma is proved.

Now we consider $GO_m^+(2)$ in more detail. Denote by c the sum of vectors u_1 and v_1 from a standard basis. Then $C = \mathrm{k}c$ is a one-dimensional anisotropic subspace of V. The group $P_5 = P_5(GO^+(V))$ is its stabilizer in $GO^+(V)$. Let $c, u_2, v_2, \dots, u_t, v_t, u$ be a new basis for V. Here $u = u_1$. Let W be the subspace spanned by $u_2, v_2, \dots, u_t, v_t$. Then $C^\perp = C + W$. The equalities $0 = ((\alpha c + w), c) = ((\alpha c + w)\varphi, c\varphi) = ((\alpha c + w)\varphi, c)$, where $\alpha \in \mathrm{k}$, $w \in W$, and $\varphi \in P_5$, show that $(\alpha c + w)\varphi \in C^\perp$. On the other hand, $C = \ker f|_{C^\perp} = (C^\perp)^\perp \cap C^\perp$. Thus, P_5 coincides with the stabilizer of $C^\perp$ and, therefore, contains a subgroup isomorphic to $GO_{m-1}(2)$.

Now consider the action on u of φ from P_5. We have $u\varphi = \beta(\varphi)u + \alpha(\varphi)c + w(\varphi)$. Since $(c, u) = 1$, $\beta = 1$. We show that $w(\varphi)$ is uniquely determined by the restriction of φ to $C^\perp$. Choose $\varphi_1, \varphi_2 \in P_5$ such that $\varphi_1|_{C^\perp} = \varphi_2|_{C^\perp} = \psi$. For every $w \in W$, there exist a vector $w_0 \in W$ and a scalar $\gamma \in \mathrm{k}$ such that $(\gamma c + w_0)\psi = w$. Therefore, $\gamma = (\gamma c + w_0, u) = ((\gamma c + w_0)\varphi_1, u\varphi_1) = (w, u + \alpha(\varphi_1)c + w(\varphi_1)) = (w, w(\varphi_1))$. Similarly, $\gamma = (w, w(\varphi_2))$. Summing the two equalities, we obtain $0 = (w, w(\varphi_1) + w(\varphi_2))$. Since w is arbitrary, $w(\varphi_1) + w(\varphi_2) \in \ker f|_W = 0$. Hence, $w(\varphi_1) = w(\varphi_2)$. Since $0 = F(u) = F(u\varphi) = F(w(\varphi)) + \alpha^2(\varphi) + \alpha(f) = F(w(\varphi))$, $\alpha(\varphi)$ does not depend on $\varphi|_{C^\perp}$ and can assume any value from k. If $\varphi|_{C^\perp} = \psi$, then, in the basis chosen, the matrix φ has the form $\left[\begin{array}{cc|c} \multicolumn{2}{c|}{} & 0 \\ \multicolumn{2}{c|}{[\psi]} & \vdots \\ \multicolumn{2}{c|}{} & 0 \\ \hline \alpha & w(\psi) & 1 \end{array}\right]$, where $\psi \in GO(C^\perp)$. Since all elements (except the first one) in the first row of $[\psi]$ are equal to 0, the reflection r_c, whose matrix in this basis is equal to

$$\begin{bmatrix} 1 & & & & \\ 0 & 1 & & 0 & \\ 0 & 0 & & & \\ \vdots & \vdots & \ddots & & \\ 0 & & & \ddots & \\ 1 & 0 & \dots & 0 & 1 \end{bmatrix},$$

commutes with all elements of P_5. Therefore, it generates a normal subgroup of order 2, with the factor group isomorphic to $GO_{m-1}(2)$. On the other hand, by Lemma 5, P_5 contains a normal subgroup $\bar{P}_5 = P_5(\Omega_m^+(2))$ of index 2, which does not contain r_c. Thus, $P_5 = \langle r_c \rangle \times \bar{P}_5 = 2 \times GO_{m-1}(2)$, and the stabilizer $\bar{P}_5$ of C in $\Omega_m^+(2)$ is isomorphic to $GO_{m-1}(2)$.

The group P_5 acts transitively on the sets $\Delta_C = \{X = \mathrm{k}x | F(x) = 1,\ (x, c) = 0,\ X \neq C\}$ and $\Gamma_C = \{Y = \mathrm{k}y | F(y) = 1,\ (y, c) = 1\}$.

Let $x = \beta u + \alpha c + w$, α, $\beta \in \mathrm{k}$, $w \in W$, and $X = \mathrm{k}x \in \Delta_C$. Since $(x, c) = 0$, it follows that $\beta = 0$ and

$x = \alpha c + w$. Therefore, Δ_C coincides with the set of all one-dimensional anisotropic subspaces of $C^\perp$.

Consider the factor space $C^\perp/\ker(f|_{C^\perp})$. Each element of this factor space is a coset $C + w$ consisting of two vectors $c + w$ and w of the space $C^\perp$. If $F(w) = 1$, then $F(c + w) = 0$, and vice versa. Thus, with each nonzero vector of $C^\perp/C$ we can naturally associate the unique anisotropic vector in $C^\perp \backslash C$. On the other hand, there exists an isomorphism $i\colon GO(C^\perp) \to \mathrm{Sp}(C^\perp/C)$. In the group $GO(C^\perp)$, the stabilizer of an anisotropic subspace distinct from C coincides, therefore, with the preimage of the stabilizer of a one-dimensional subspace in $\mathrm{Sp}(C^\perp/C)$. Moreover, r_c also stabilizes $X \in \Delta_C$. The group D_2 is isomorphic to $(2^{m-3} \cdot GO_{m-3}(2)) \times 2$ and $D_2(\Omega_m^+(2))$ is isomorphic to $2^{m-3} \cdot GO_{m-3}(2)$.

Choose a vector y such that $Y = \mathrm{k}y \in \Gamma_C$. Denote by U the subspace spanned by c and y. Since $(c, y) = 1$, we have $V = U + U^\perp$. Obviously, U is a subspace of type $-$; hence, $U^\perp$ is also of type $-$. If D_3 is the stabilizer of c and y in $GO^+(V)$, then D_3 acts trivially on U. Hence, $D_3 = GO^-(U^\perp) = GO_{m-2}^-(2)$.

Now we will find $D_3(\Omega^+(V))$. By assumption, $m \geq 7$ and, therefore, $U^\perp$ contains isotropic vectors u_0 and v_0 such that $(u_0, v_0) = 1$. The reflection generated by the vector $u_0 + v_0$ lies in D_3 but does not lie in $\Omega_{m-2}^-(2)$. Hence, $D_3(\Omega^+(V)) = \Omega^-(U^\perp) = \Omega_{m-2}^-(2)$. In order to prove the next result, we need only observe that $|GO_m^+(2)\colon P_5| = |\Omega_m^+(2)\colon \bar{P}_5| = 1 + |\Gamma_C| + |\Delta_C|$.

Proposition 6. The rank of the permutation representation of $\Omega_m^+(2)$, $m = 2t$, on the set of one-dimensional subspaces anisotropic with respect to a nondegenerate quadratic form F is equal to 3.

If n is a degree and n_2, n_3 are nontrivial subdegrees of the representation, then

$n = 2^{t-1}(2^t - 1)$,

$n_2 = 2^{2t-2} - 1$,

$n_3 = 2^{t-1}(2^{t-1} - 1)$.

If H is a point stabilizer and D_2, D_3 are two-point stabilizers in this representation, then

$H = \Omega_{m-1}(2)$,

$D_2 = 2^{m-3} \cdot \Omega_{m-3}(2)$,

$D_3 = \Omega_{m-2}^-(2)$.

Proposition 7. Let $m \geq 7$ and let m be even if $q = 2^s$, $s \geq 1$. Let π be a nontrivial permutation representation of minimal degree for $G = \Omega_m^\varepsilon(q)$. Then, for $m = 8$, $\varepsilon = +$, and $q > 3$, π is similar to the representation of G on the cosets of $\bar{P}_1$ or $\bar{P}_2$; for odd m and $q = 3$, π is similar to the representation on the cosets of $\bar{P}_3$, and for even m, $\varepsilon = +$, and $q = 3$, to the representation on the cosets of $\bar{P}_4$; π is similar to the representation on the cosets of $\bar{P}_5$ if m is even, $\varepsilon = +$, $q = 2$, and to the representation on the cosets of $\bar{P}_1$ in the remaining cases.

Proof. Let H be the proper subgroup of least index in G. It was proved in [5] that H is reducible. If W is a minimal H-invariant subspace, then $H = G_W$. Let $\dim(W) = d$. Since W is minimal, only the following cases are possible: (a) $\ker(f|_W) = \ker(F|_W) = W$; (b) $\ker(f|_W) = 0$ and $\ker(F|_W) = W$; (c) $\ker(f|_W) = \ker(F|_W) = 0$. We consider them separately.

(a) Here W is isotropic. By Lemma 4, $|G\colon H| = i_d(V)$, except for the case where $m = 2t$, $\varepsilon = +$, and $d = t$, for which we have $|G\colon H| = i_t(V)/2$. Elementary calculations show that for $d \geq 2$, the inequalities $|G\colon H| \geq i_d(V)/2 \geq i_1(V) = |G\colon \bar{P}_1|$ hold. The equality is realized only if $m = 8$, $\varepsilon = +$, and $d = 4$, in which case $|G\colon H| = |G\colon \bar{P}_1| = |G\colon \bar{P}_2|$ and H is a conjugate of $\bar{P}_2$.

If $d = 1$, then H is a conjugate of $\bar{P}_1$. Since for odd m and $q = 3$ we have $|G\colon \bar{P}_1| > |G\colon \bar{P}_3|$, it follows that $|G\colon H| > |G\colon \bar{P}_3|$. Similarly, $|G\colon H| > |G\colon P_4|$ and $|G\colon H| > |G\colon \bar{P}_5|$ for suitable m, ε, and q. Case (a) is completed.

(b) Obviously, this situation is possible only if $q = 2^s$. Hence we can assume that $m = 2t$. Let u and v be two linearly independent vectors in W, $F(u) = \alpha$, $F(v) = \beta$. Since $(k^*)^2 = k^*$, there exists $\lambda \in k^*$ such that $\lambda^2 = \alpha\beta^{-1}$. Therefore, $F(u + \lambda v) = \alpha + \alpha\beta^{-1} \cdot \beta = 2\alpha = 0$, which contradicts the assumption that $\ker(F|_W) = 0$. Thus, the dimension of W is equal to 1. By Lemma 5, $|G : H| = q^{t-1}(q^t - \varepsilon) > (q^t - \varepsilon)(q^{t-1} + \varepsilon)/(q-1) = |G: \bar{P}_1|$, except for the case where $\varepsilon = +1$ and $q = 2$, for which $|G: H| = |G: \bar{P}_5|$ and H is a conjugate of G in $\bar{P}_5$.

(c) Here $V = W + W^{\perp}$, and so $H \leq GO^{\varepsilon_1}(W) \times GO^{\varepsilon_2}(W^{\perp})$. Elementary calculations show that if $d \geq 2$, we have $|G: H| > |G: \bar{P}_3|$ for $q = 3$ and m odd; $|G: H| > |G: \bar{P}_4|$ for $m = 2t$, $\varepsilon = +$, and $q = 3$; $|G: H| > |G: \bar{P}_5|$ for m even, $\varepsilon = +$, $q = 2$; and $|G: H| > |G: \bar{P}_1|$ in the other cases.

Let $d = 1$, then q is odd. If $m = 2t$ and V is a space of type ε, then, by Proposition 3′, $|G : H| = q^{t-1}(q^t - \varepsilon) > (q^t - \varepsilon)(q^{t-1} + \varepsilon)/(q-1) = |G: \bar{P}_1|$. This inequality is violated only if $q = 3$, $\varepsilon = +1$, where H is a conjugate of $\bar{P}_4$, and $|G: H| = |G: \bar{P}_4| > |G: \bar{P}_1|$. If $m = 2t + 1$ and W is a point of type ε, then, by Proposition 3′, $|G: H| = q^t(q^t + \varepsilon)/2 > (q^{2t} - 1)/(q-1) = |G: \bar{P}_1|$, except for the case where $q = 3$, $\varepsilon = -1$. In this last case H is a conjugate of $\bar{P}_3$, and $|G: H| = |G: \bar{P}_3| > |G: \bar{P}_1|$. The proposition is proved.

Every maximal subgroup of the group $\Omega_m^{\varepsilon}(q)$ contains its center, which is of order 2 and is generated by $-E$, if $m = 2t$ and $((q^t - \varepsilon), 4) = 4$, and is trivial in other cases. Therefore, Proposition 7, together with information about permutation representations of $\Omega_m^{\varepsilon}(q)$ on the cosets of $\bar{P}_i$, $i = 1, \ldots, 5$, implies the following:

THEOREM. For the simple non-Abelian groups $G = O_m^{\varepsilon}(q)$, $m \geq 7$, where m is even for $q = 2^s$, the parameters n, n_2, n_3, H, D_2, and D_3 of minimal permutation representations are contained in the following list:

in the case where $m = 8$, $\varepsilon = +$, and $q > 3$, we have

$n = (q^4 - 1)(q^3 + 1)/(q - 1)$, $n_2 = q(q^3 - 1)(q^2 + 1)/(q - 1)$, $n_3 = q^6$;

for $(q^3 - 1, 4) = 4$, $q = p^s$, where p is an odd prime,

$H = p^{6s} \cdot ((\Omega_6^+(q) \times (q-1)/4) \cdot 2)$,

$D_2 = p^{5s} \cdot ((p^{4s} \cdot ((\Omega_4^+(q) \times (q-1)/2) \cdot 2) \times (q-1)/4) \cdot 2)$,

$D_3 = (\Omega_6^+(q) \times (q-1)/4) \cdot 2$

or

$H = p^{6s} \cdot (SL_4(q) \cdot (q-1)/4)$,

$D_2 = p^{5s} \cdot (p^{4s} \cdot (GL_2(q) \times (SL_2(q) \cdot (q-1)/4)))$,

$D_3 = SL_4(q) \cdot (q-1)/4$;

for $(q^3 - 1, 4) = 2$, $q = p^s$, where p is an odd prime,

$H = p^{6s} \cdot ((\Omega_6^+(q) \times (q-1)/2) \cdot 2)$,

$D_2 = p^{5s} \cdot ((p^{4s} \cdot (\Omega_4^+(q) \times (q-1)/2) \cdot 2) \times (q-1)/2) \cdot 2)$,

$D_3 = (\Omega_6^+(q) \times (q-1)/2) \cdot 2$

or

$H = p^{6s} \cdot (SL_4(q) \cdot (q-1)/2)$,

$D_2 = p^{5s} \cdot (p^{4s} \cdot (GL_2(q) \times (SL_2(q) \cdot (q-1)/2)))$,

$D_3 = SL_4(q) \cdot (q-1)/2$;

for $q = 2^s$, $s \geq 2$,

$H = 2^{6s} \cdot (\Omega_6^+(q) \times (q-1))$,

$D_2 = 2^{5s} \cdot ((2^{4s} \cdot (\Omega_4^+(q) \times (q-1)) \times (q-1)))$,

$D_3 = \Omega_6^+(q) \times (q-1)$

or

$H = 2^{6s} \cdot GL_4(q)$,

$D_2 = 2^{5s} \cdot (2^{4s} \cdot (GL_2(q) \times GL_2(q)))$,

$D_3 = GL_4(q)$;

in the case where $m = 2t$, $\varepsilon = +$, and $q = 2$,

$n = 2^{t-1}(2^t - 1)$, $n_2 = 2^{2t-2} - 1$, $n_3 = 2^{t-1}(2^{t-1} - 1)$,

$H = \Omega_{m-1}(2)$,

$D_2 = 2^{m-3} \cdot \Omega_{m-3}(2)$,

$D_3 = \Omega^-_{m-2}(2)$;

in the case where $m = 2t + 1$ and $q = 3$,

$n = 3^t(3^t - 1)/2$, $n_2 = 3^{t-1}(3^t + 1)/2$, $n_3 = (3^t + 1)(3^{t-1} - 1)$,

$H = \Omega^-_{m-1}(3) \cdot 2$,

$D_2 = (\Omega_{m-2}(3) \cdot 2) \cdot 2$,

$D_3 = (3^{m-3} \cdot (\Omega^-_{m-3}(3) \times 2)) \cdot 2$;

in the case where $m = 2t$, $\varepsilon = +$, and $q = 3$,

$n = 3^{t-1}(3^t - 1)/2$, $n_2 = 3^{t-1}(3^{t-1} - 1)/2$, $n_3 = 3^{2t-2} - 1$;

for t odd,

$H = \Omega_{m-1}(3) \cdot 2$,

$D_2 = (\Omega^-_{m-2}(3) \cdot 2) \cdot 2$,

$D_3 = (3^{m-3} \cdot (\Omega_{m-3}(3) \times 2)) \cdot 2$;

for t even,

$H = \Omega_{m-1}(3)$,

$D_2 = \Omega^-_{m-2}(3) \cdot 2$,

$D_3 = 3^{m-3} \cdot (\Omega_{m-3}(3) \times 2)$;

in the case where $m = 2t + 1$, $q = p^s \neq 3$, and p is an odd prime,

$n = (q^{2t} - 1)/(q - 1)$, $n_2 = (q^{2t-1} - q)/(q - 1)$, $n_3 = q^{2t-1}$,

$H = p^{s(m-2)} \cdot ((\Omega_{m-2}(q) \times (q - 1)/2) \cdot 2)$,

$D_2 = p^{s(m-3)}(((p^{s(m-4)} \cdot ((\Omega_{m-4}(q) \times (q - 1)/2) \cdot 2)) \times (q - 1)/2) \cdot 2)$,

$D_3 = (\Omega_{m-2}(q) \times (q - 1)/2) \cdot 2$;

in the case where $m = 2t$, $q = 2^s$, $s \geq 2$, and the pair (m, ε) is distinct from $(8, +)$,

$n = (q^t - \varepsilon)(q^{t-1} + \varepsilon)/(q - 1)$, $n_2 = (q^t - \varepsilon q)(q^{t-2} + \varepsilon)/(q - 1)$, $n_3 = q^{2t-2}$,

$H = 2^{s(m-2)} \cdot (\Omega^\varepsilon_{m-2}(q) \times (q - 1))$,

$D_2 = 2^{s(m-3)} \cdot ((2^{s(m-4)} \cdot (\Omega^\varepsilon_{m-4}(q) \times (q - 1)) \times (q - 1))$,

$D_3 = \Omega^\varepsilon_{m-2}(q) \times (q - 1)$;

in the case where $m = 2t$, $q = p^s$, p is an odd prime, the pair (m, ε) is distinct from $(8, +)$, and (q, ε) is distinct from $(3, +)$,

$n = (q^t - \varepsilon)(q^{t-1} + \varepsilon)/(q - 1)$, $n_2 = (q^t - \varepsilon q)(q^{t-2} + \varepsilon)/(q - 1)$, $n_3 = q^{2t-2}$;

for $((q^t - \varepsilon), 4) = 2$,

$H = p^{s(m-2)} \cdot ((\Omega^\varepsilon_{m-2}(q) \times (q - 1)/2) \cdot 2)$,

$D_2 = p^{s(m-3)} \cdot (((p^{s(m-4)} \cdot ((\Omega^\varepsilon_{m-4}(q) \times (q - 1)/2) \cdot 2) \times (q - 1)/2)) \cdot 2$,

$D_3 = (\Omega^\varepsilon_{m-2}(q) \times (q - 1)/2) \cdot 2$;

for $(q^t - \varepsilon, 4) = 4$ and $(q^{t-1} - \varepsilon, 4) = 2$,

$H = p^{s(m-2)} \cdot (\Omega^\varepsilon_{m-2}(q) \times (q - 1)/2)$,

$D_2 = p^{s(m-3)} \cdot ((p^{s(m-4)} \cdot ((\Omega^{\varepsilon}_{m-4}(q) \times (q-1)/2) \cdot 2) \times (q-1)/2)$,
$D_3 = \Omega^{\varepsilon}_{m-2}(q) \times (q-1)/2$;
for $(q^t - \varepsilon, 4) = 4$ and $(q^{t-1} - \varepsilon, 4) = 4$,
$H = p^{s(m-2)} \cdot ((\Omega^{\varepsilon}_{m-2}(q) \times (q-1)/4) \cdot 2)$,
$D_2 = p^{s(m-3)} \cdot (((p^{s(m-4)} \cdot ((\Omega^{\varepsilon}_{m-4}(q) \times (q-1)/2) \cdot 2) \times (q-1)/4) \cdot 2)$,
$D_3 = (\Omega^{\varepsilon}_{m-2}(q) \times (q-1)/4) \cdot 2$.

In every case, the rank of the minimal permutation representation of G is equal to 3.

In conclusion, we note that the degrees of minimal permutation representations of classical (including orthogonal) groups were obtained by Cooperstein in [5]. His paper, however, contained some inaccuracies. A refined list of minimal degrees appeared in [2, p. 175]. In that list, however, the minimal degree of the group $O^+_m(3)$, where m is an even number greater than 6, was indicated incorrectly, which is confirmed by the results of the present investigation.

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