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ON RECOGNITION OF ALL FINITE NONABELIAN SIMPLE GROUPS WITH ORDERS HAVING PRIME DIVISORS AT MOST 13

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Abstract: The spectrum of a group is the set of its element orders. We say that the problem of recognition by spectrum is solved for a finite group if we know the number of pairwise nonisomorphic finite groups with the same spectrum as the group under study. In this article the problem of recognition by spectrum is completely solved for every finite nonabelian simple group with orders having prime divisors at most 13.

Keywords: recognition by spectrum, finite simple group, group of Lie type

Introduction

Let G be a finite group. Denote by $\pi(G)$ the set of all prime divisors of the order of G. In the present article we consider the finite nonabelian simple groups G with the property $\pi(G) \subseteq \{2, 3, 5, 7, 11, 13\}$. We denote the set of all these groups by $\mathscr{S}_{13}$. Since there exist only finitely many finite nonabelian simple groups G with the same $\pi(G)$ (see, for example, the remark after Lemma 2 in [1]), the set $\mathscr{S}_{13}$ contains finitely many isomorphic classes of groups. Using the classification of finite simple groups it is not hard to obtain a full list of all groups in $\mathscr{S}_{13}$. There are 55 such groups, all listed in Table 1 of this article.

The *spectrum* $\omega(G)$ of a finite group G is the set of all element orders of G. In other words, a natural number n is in $\omega(G)$ if and only if there is an element of order n in G. For an arbitrary subset ω of the set of natural numbers denote by $h(\omega)$ the number of pairwise nonisomorphic finite groups G such that $\omega(G) = \omega$. We say that for a finite group G the recognition problem is solved if we know the value of $h(\omega(G))$ (for brevity, $h(G)$). More precisely, G is said to be *recognizable by spectrum* (briefly, recognizable) if $h(G) = 1$, *almost recognizable* if $1 < h(G) < \infty$, and *nonrecognizable* if $h(G) = \infty$. Since a finite group with a nontrivial normal soluble subgroup is nonrecognizable (see [2, Lemma 1]), each recognizable or almost recognizable group is an extension of the direct product M of nonabelian simple groups by some subgroup of $\mathrm{Out}(M)$. So, of prime interest is the recognition problem for simple and almost simple groups (recall that G is almost simple if $S \leq G \leq \mathrm{Aut}(S)$ for some nonabelian simple group S). In the middle of the 1980s Shi found the first examples of recognizable finite simple groups (see [3, 4]). In 1994 Shi and Brandl obtained an infinite series of recognizable simple linear groups $L_2(q)$, $q \neq 9$ (see [5, 6]). At present the recognition problem is solved for many finite nonabelian simple and almost simple groups. The freshest list of those groups is available in [7]. We denote the set of groups in this list by $\mathscr{R}$. In particular, $\mathscr{R}$ contains all finite nonabelian simple groups with prime divisors at most 11 (see [2]). Moreover, the comparison between $\mathscr{R}$ and $\mathscr{S}_{13}$ shows that there are only two groups lying in $\mathscr{S}_{13} \setminus \mathscr{R}$; namely, the groups $L_6(3)$ and $U_4(5)$.

The main purpose of this article is to prove that the groups $L_6(3)$ and $U_4(5)$ are almost recognizable, and thus demonstrating that the recognition problem is solved for all groups in $\mathscr{S}_{13}$.

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Theorem 1. *Let G be the finite simple group $L_6(3)$ and let H be a finite group with $\omega(H) = \omega(G)$. Then $H \simeq G$ or $H \simeq G\langle\gamma\rangle$, where γ is the graph automorphism of G of order 2. In particular, $h(G) = 2$.*

Theorem 2. *Let G be the finite simple group $U_4(5)$ and let H be a finite group with $\omega(H) = \omega(G)$. Then $H \simeq G$ or $H \simeq G\langle\gamma\rangle$, where γ is the field automorphism of G of order 2. In particular, $h(G) = 2$.*

Corollary. *The recognition problem is solved for every group G in $\mathscr{S}_{13}$. The values of $h(G)$ are listed in the last column of Table 1.*

§ 1. Preliminaries

The set $\omega(H)$ of a finite group H is closed under divisibility and determined uniquely from the set $\mu(H)$ of those elements in $\omega(H)$ that are maximal under the divisibility relation. Furthermore, the set $\omega(H)$ determines the Gruenberg–Kegel graph (or prime graph) $GK(H)$ whose vertices are all prime divisors of the order of H and two primes p and q are adjacent if H has an element of order $p \cdot q$. Denote by $s(H)$ the number of connected components of $GK(H)$ and by $\pi_i(H)$, $i = 1, \dots, s(H)$, the ith connected component of $GK(H)$. If H has an even order then put $2 \in \pi_1(H)$. Denote by $\mu_i(H)$ ($\omega_i(H)$) the set of numbers $n \in \mu(H)$ ($n \in \omega(H)$) such that every prime divisor of n belongs to $\pi_i(H)$.

Lemma 1.1 (Gruenberg–Kegel Theorem). *If H is a finite group with disconnected graph $GK(H)$ then one of the following conditions holds:*

(a) *$s(H) = 2$ and H is a Frobenius group;*

(b) *$s(H) = 2$ and $H = ABC$ where A and AB are normal subgroups of H; AB and BC are Frobenius groups with kernels A and B and complements B and C respectively;*

(c) *there exists a nonabelian simple group S such that $S \le \overline{H} = H/K \le \mathrm{Aut}(S)$ for some nilpotent normal $\pi_1(H)$-subgroup K of H and the group $\overline{H}/S$ is a $\pi_1(H)$-subgroup; moreover, the graph $GK(S)$ is disconnected, $s(S) \ge s(H)$ and for every i, $2 \le i \le s(H)$, there is j, $2 \le j \le s(S)$, such that $\omega_i(H) = \omega_j(S)$.*

Proof. See [8].

Lemma 1.2. *Let S be a finite simple group with disconnected graph $GK(S)$. Then $|\mu_i(S)| = 1$ for $2 \le i \le s(S)$. Denote by $n_i = n_i(S)$ the only element of $\mu_i(S)$, $i \ge 2$. Then S, $\pi_1(S)$, and $n_i(S)$, $2 \le i \le s(S)$, are so as indicated in Tables 2a–2c in [7].*

Proof. See [8–10].

Remark. The Gruenberg–Kegel Theorem (Lemma 1.1) and the classification of all finite nonabelian simple groups with disconnected prime graph (Lemma 1.2) are used in most papers devoted to the recognition problem. As a matter of fact, there are only two groups with connected prime graph in $\mathscr{R}$. Namely, the alternating groups A_{10} with $h(A_{10}) = \infty$ and A_{16} with $h(A_{16}) = 1$. Note that the group $U_4(5)$, considered in the present paper, also has a connected prime graph. Since it turns out that $h(U_4(5)) = 2$, this group is a first example of an almost recognizable simple group with connected prime graph.

Lemma 1.3. *Let G be a Frobenius group with kernel F and complement C. Then*

(a) *F is nilpotent. If U is a subgroup of order pq in C, where p and q are primes (not necessary distinct), then U is cyclic. In particular, for every odd prime p a Sylow p-subgroup of C is cyclic.*

(b) *Let H be a finite group, $K \lhd H$, and $H/K = G$. If C is cyclic, $(|F|, |K|) = 1$, and F does not lie in $KC_H(K)/K$ then $p|C| \in \omega(H)$ for some prime divisor p of K.*

Proof. See [1, Lemma 1] and [11, Lemma 1].

Following [12], denote by $A \cdot B$ ($A : B$) an extension (a split extension) of a group A by a group B, and by A^m the direct product of m isomorphic copies of A. Furthermore, we denote the cyclic group of order n simply by n whenever this does not lead to confusion. For example, the record $2^2 : 3$ implies the split extension of the elementary abelian 2-group of order 4 by the cyclic group of order 3.

Table 1
Finite nonabelian simple groups with orders having prime divisors at most 13

G	Order of G	Out(G)	$s(G)$	$h(G)$
A_5	$2^2 \cdot 3 \cdot 5$	2	3	1
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	3	1
A_6	$2^3 \cdot 3^2 \cdot 5$	2^2	3	∞
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	3	1
$L_2(11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$	2	3	1
$L_2(13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	2	3	1
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	3	1
$L_3(3)$	$2^4 \cdot 3^3 \cdot 13$	2	2	∞
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2	2	∞
$L_2(25)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	2^2	3	1
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	1	3	1
$L_2(27)$	$2^2 \cdot 3^3 \cdot 7 \cdot 13$	6	3	1
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	2	1
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	D_{12}	4	1
$U_4(2)$	$2^6 \cdot 3^4 \cdot 5$	2	2	∞
Sz(8)	$2^6 \cdot 5 \cdot 7 \cdot 13$	3	4	1
$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	2^2	3	1
$U_3(4)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13$	4	2	1
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	2	2	1
$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	S_3	2	∞
A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2	2	1
$L_2(64)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	6	3	1
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	2	4	1
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2	2	∞
$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1	2	2
A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2	1	∞
$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	D_8	2	1
$G_2(3)$	$2^6 \cdot 3^6 \cdot 7 \cdot 13$	2	3	1
$S_4(5)$	$2^6 \cdot 3^2 \cdot 5^4 \cdot 13$	2	2	∞
$L_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 13$	2^2	2	1
$U_5(2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11$	2	2	∞
${}^2F_4(2)'$	$2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$	2	2	1
A_{11}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	2	2	1
$L_3(9)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$	2^2	2	2
HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	2	3	1
$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2	2	∞
$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	S_3	2	2
${}^3D_4(2)$	$2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$	3	2	1
A_{12}	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$	2	2	1
$G_2(4)$	$2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$	2	2	1
M^cL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	2	2	1
$S_4(8)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$	6	2	∞

Table 1 (continuation)

G	Order of G	Out(G)	$s(G)$	$h(G)$
A_{13}	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2	3	1
$S_6(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	2	2	2
$O_7(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	2	2	2
$U_6(2)$	$2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	S_3	3	1
$U_4(5)$	$2^7 \cdot 3^4 \cdot 5^6 \cdot 7 \cdot 13$	2^2	1	2
A_{14}	$2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$	2	2	1
$L_5(3)$	$2^9 \cdot 3^{10} \cdot 5 \cdot 11^2 \cdot 13$	2	2	1
Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2	3	1
A_{15}	$2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	2	2	1
$O_8^+(3)$	$2^{12} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13$	S_4	2	2
A_{16}	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	2	1	1
Fi_{22}	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2	2	1
$L_6(3)$	$2^{11} \cdot 3^{15} \cdot 5 \cdot 7 \cdot 11^2 \cdot 13^2$	2^2	2	2

§ 2. Proof of Theorem 1

Let G be the finite simple group $L_6(3)$ and let H be the finite group with $\omega(H) = \omega(G)$. We have $\mu(G) = \mu(H) = \{182, 121, 120, 104, 80, 78, 36\}$. Thus $s(G) = 2$ and we apply Lemma 1.1. A result of [13] allows us to eliminate the cases (a) and (b) of the lemma. So we come to the following situation. There exists a nonabelian simple group S such that $S \le \overline{H} = H/K \le \mathrm{Aut}(S)$ for some nilpotent normal $\pi_1(H)$-subgroup K of H and the group $\overline{H}/S$ is a $\pi_1(H)$-subgroup; moreover, the graph $GK(S)$ is disconnected, $s(S) \ge s(H)$ and there is j, $2 \le j \le s(S)$, such that $\omega_2(H) = \omega_j(S)$. Obviously, $\pi(S) \subseteq \pi(H) = \pi(G)$. So $S \subseteq \mathscr{S}_{13}$. On the other hand, $n_2(H) = n_2(G) = 121$. Therefore, one of the connected components of $GK(S)$ must be $\{121\}$. There are only two groups in $\mathscr{S}_{13}$ satisfying this condition: $L_5(3)$ and G itself. Let $S \simeq L_5(3)$. Since 7 does not divide the order of $\mathrm{Aut}(S)$, a Sylow 7-subgroup P of H lies in K. The nilpotency of K implies that a subgroup P is normal in H. Since $21 = 7 \cdot 3 \notin \omega(H)$, the group $P : R$, where R is the Sylow 3-subgroup of H/K, is the Frobenius group with kernel P and complement R. By Lemma 1.3 R must be cyclic. But a Sylow 3-subgroup of $S \le H/K$ is obviously not cyclic; a contradiction.

Thus $S \simeq G$, and we will write G instead of S. Let $K \ne 1$. Since K is nilpotent, inducting on the order of H, we may suppose that K is a p-group for some prime p. If we take a factor group $H/\Phi(K)$ instead of H, where $\Phi(K)$ is a Frattini subgroup of K, then the same induction argument allows us to consider K as an elementary abelian p-group. Assume that $p \ne 3$. The group G includes a Frobenius subgroup with kernel F of order $3^5 = 243$ and cyclic complement C of order $(3^5 - 1)/2 = 121$ (see the proof of Lemma 3 in [14]). Since $p \ne 3$ and G is simple, we have $(|F|, |K|) = 1$ and $C_H(K) = K$. By Lemma 1.3, H contains an element of order $121 \cdot p$; a contradiction.

We may thus suppose that K is a 3-group. The group H/K includes a subgroup L isomorphic to $S_6(3)$ which acts on K by conjugation. Inspection of the table of the Brauer 3-characters for L in [15] shows that an element $x \in L$ of order 7 has a fixed point in every absolutely irreducible module over a field of characteristic 3. Thus x centralizes some nontrivial element in K and hence $21 \in \omega(H)$; a contradiction.

We have $G \le H \le \mathrm{Aut}(G)$. The group $\mathrm{Out}(G)$ is an elementary abelian group of order 4. Let δ be a diagonal automorphism and let γ be a graph automorphism of G. Then $\mathrm{Aut}(G) = G\langle\delta, \gamma\rangle$. The action of δ and γ on G can be represented as follows: Let g denote the image in G of a matrix A in $SL_6(3)$. Then g^δ is the image of A^D, where $D = \mathrm{diag}\{1, 1, 1, 1, 1, -1\}$, and g^γ is the image of the matrix A^{-T}, the inverse transpose of A. Thus, $|\delta| = |\gamma| = 2$. Furthermore, since $D^{-T} = D$, we see that δ and γ commute and the automorphism $\tau = \gamma\delta$ is also of order 2.

Suppose that $H \not\leq G\langle\gamma\rangle$. Since D centralizes in $SL_6(3)$ a subgroup isomorphic to $SL_5(3)$; therefore, δ centralizes the image of this subgroup in G. Hence, if $\delta \in H$, there exists an element of order $121 \cdot 2$ in H, which is impossible. In $SL_6(3)$ we now consider the matrix

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Direct calculation shows that the matrix $B = A \cdot (A^{-T})^D$ is of order 24 and the cyclic subgroup generated by B has trivial intersection with the center of $SL_6(3)$. If g is the image of A in G then $(g\tau)^2 = gg^{\gamma\delta}$ is the image of B in G of order 24. Therefore, $g\tau$ is of order 48. Since $48 \notin \omega(H)$, we have $\tau \notin H$.

Thus, $H \leq G\langle\gamma\rangle$. Using GAP (see [16]), we obtain some matrix representatives A of all conjugacy classes in $SL_6(3)$. Calculation of the projective orders of AA^{-T} allows us to find the spectrum of $G\langle\gamma\rangle$. It turns out that $\omega(G) = \omega(G\langle\gamma\rangle)$. Hence, $H = G$ or $H = G\langle\gamma\rangle$, and so $h(G) = 2$. The theorem is proved.

§ 3. Proof of Theorem 2

Let G be the finite simple group $U_4(5)$ and let H be a finite group with $\omega(H) = \omega(G)$. We have $\mu(G) = \mu(H) = \{63, 60, 52, 24\}$. The prime graph $GK(G) = GK(H)$ has the following form:

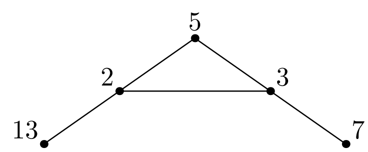

Fig. 1

Since this graph is connected, we cannot apply Lemma 1.1 directly. So the proof is more complicated and we divide it in several natural steps.

Lemma 3.1. *Let K be a maximal normal soluble subgroup of H. Then only one of the three primes $5, 7, 13$ can divide the order of K. In particular, H is insoluble.*

Proof. Assume firstly that each of the three primes 5, 7, 13 divides the order of K. Since K is soluble, it includes the soluble Hall $\{5, 7, 13\}$-subgroup R. Since every two distinct primes in $\{5, 7, 13\}$ are nonadjacent in $GK(H)$, the same is true for $GK(R)$. Then $s(R) = 3$ and R is insoluble by Lemma 1.1; a contradiction. Therefore, $K \neq H$ and H is insoluble.

Let p, q, r be distinct primes in $\{5, 7, 13\}$ given in arbitrary order. Assume that two of them, for definiteness p and q, divide $|K|$, whereas r does not. Consider the Hall $\{p, q\}$-subgroup T in K. By the Frattini argument $H = KN_H(T)$. Therefore, the normalizer $N = N_H(T)$ contains an element of order r, which acts fixed-point-freely on T. Lemma 1.3 implies that T is nilpotent. Hence $p \cdot q \in \omega(T) \subseteq \omega(H)$; a contradiction.

Lemma 3.2. *There exists a finite simple group $S \in \mathscr{S}_{13}$ such that $S \leq \overline{H} = H/K \leq \mathrm{Aut}(S)$.*

Proof. If we denote by $L = S_1 \times \cdots \times S_m$ the socle of $\overline{H}$, where S_i are nonabelian simple groups; then $\overline{H} \leq \mathrm{Aut}(L)$. It is obvious that every S_i is $\mathscr{S}_{13}$. So we need only to prove $m = 1$.

Suppose that $m \geq 2$. By Lemma 3.1 there exists a prime $p \in \{7, 13\}$ that divides the order of $\overline{H}$. Assume that p divides $|L|$. If $p = 7$ then there exists an element of order $7 \cdot 2$ in L, which is impossible, since $14 \notin \omega(H)$. If $p = 13$ then either $13 \cdot 3 \in \omega(H)$ or every S_i is a Suzuki group $Sz(8)$ and so $13 \cdot 5 \in \omega(H)$. In both cases we have a contradiction.

Thus, we may assume that p divides only the order of $\mathrm{Out}(L)$. Let $\varphi \in \overline{H}$ be an automorphism of L of order p and $P = S_1^{\varphi}$. Since P is simple, its every natural projection P_i to S_i, $i = 1, \ldots, m$, is either trivial or isomorphic to S_1. On the other hand, since P is normal in L, the same is true for every P_i, $i = 1, \dots, m$. Hence $P_i = 1$ or $P_i = S_i$. Therefore, there exists unique $j \in \{1, \dots, m\}$ such that $S_1^{\varphi} = S_j$. If $j \neq 1$ then there arises a φ-orbit Δ of length p which consists of subgroups isomorphic to S_1. Without loss of generality we may assume that $\Delta = \{S_1, S_2, \dots, S_p\}$. Let a_1 be an element of order t in S_1 and $a_i = a_1^{\varphi^i}$. Let g be the element of L, whose projections g_i to S_i are defined as follows: $g_i = a_i$ for $i = 1, \dots, p$ and $g_i = 1$ otherwise. Then g is of order t, and an element $g\varphi \in \overline{H}$ is of order $t \cdot p$. If $p = 7$ then we may take a_1 of order $t = 2$. But $7 \cdot 2 \notin \omega(H)$; a contradiction. If $p = 13$ then either we take a_1 of order 3 or $S_1 \simeq Sz(8)$ and we take a_1 of order 5. In both cases we obtain a contradiction. Thus, $S_1^{\varphi} = S_1$, and the same is true for every S_i, $i = 1, \dots, m$. Since $\varphi \neq 1$; therefore, φ acts nontrivially on some S_k. Then φ induces an outer automorphism of S_k of order p. Since $S_k \in \mathscr{S}_{13}$, it is impossible (see the third column of Table 1). Thus, $m = 1$. The lemma is proved.

Lemma 3.3. *$S \simeq G$.*

Proof. We consider all possibilities for the group S consecutively:

A. $S \simeq A_5, A_6, L_2(7), L_2(9), L_2(8), U_3(3), U_4(2), U_5(2), L_3(3)$. Since $\mathrm{Out}(S)$ is not divided by $5, 7, 13$ and only one of these primes divides the order of S, we have a contradiction by Lemma 3.1.

B. $S \simeq L_2(11), M_{11}, M_{12}, M_{22}, HS, M^cL, U_6(2), L_5(3), \mathrm{Suz}, Fi_{22}, L_6(3)$ and A_n, with $n = 11, \dots, 16$. Since $11 \in \omega(S) \setminus \omega(G)$, we have a contradiction.

C.1. $S \simeq L_2(49)$. In this case $25 \in \omega(S) \setminus \omega(G)$; a contradiction.

C.2. $S \simeq L_2(64), S_4(8)$. We have $65 \in \omega(S) \setminus \omega(G)$, which is impossible.

C.3. $S \simeq {}^2F_4(2)', L_3(9)$. In this case $16 \in \omega(S) \setminus \omega(G)$; a contradiction.

C.4. $S \simeq {}^3D_4(2), S_6(3), O_7(3), S_4(7), L_2(27)$. Since $14 \in \omega(S) \setminus \omega(G)$, we obtain a contradiction.

D. $S \simeq L_3(4), U_3(5), J_2, S_6(2), U_4(3), O_8^+(2)$ and A_n, where $n = 7, \dots, 10$. Since 13 does not divide $|\mathrm{Aut}(S)|$, we have $13 \in \omega(K)$. On the other hand, each of these groups includes a subgroup isomorphic to $L_2(7)$ and so it includes a Frobenius group $F \simeq 7 : 3$. Consider the factor group $\widetilde{H} = H/O_{13'}(K)$. We have $P = O_{13}(\widetilde{H}) \neq 1$. So F acts on P faithfully and its kernel of order 7 acts fixed-point-freely on P. By Lemma 1.3, there exists an element of order $13 \cdot 3$ in H; a contradiction.

E. $S \simeq L_2(25), U_3(4), S_4(5), L_3(3)$. Since $7 \in \omega(K)$ and S includes a Frobenius subgroup $5 : 2$, we have $7 \cdot 2 \in \omega(H)$; a contradiction.

F. $S \simeq L_2(13), G_2(3)$. Since 5 does not divide $|\mathrm{Aut}(S)|$, it divides the order of K. Since $3 \cdot 7 \notin \omega(\mathrm{Aut}(S))$; therefore, $\omega(K)$ contains 7 or 9. Then Lemma 3.1 implies that $9 \in \omega(K)$. Let T be the Hall $\{3, 5\}$-subgroup in K. Since 13 does not divide $|K|$, there exists an element of order 13 in $N_H(T)$ which acts fixed-point-freely on T. Hence T is nilpotent. So $9 \cdot 5$ belongs to $\omega(H)$; a contradiction.

G. $S \simeq G_2(4)$. Since there are no elements of order 9 in $\mathrm{Aut}(S)$; therefore, $|K|$ is divided by 3. On the other hand, S includes a Frobenius subgroup with kernel of order 13 and cyclic complement of order 6. Therefore, $3 \cdot 6$ belongs to $\omega(H)$; a contradiction.

H. $S \simeq Sz(8) \simeq {}^2B_2(8)$. Using [12], we find that $\mu(S) = \{4, 5, 7, 13\}$, $|\mathrm{Out}(S)| = 3$, and $\mu(\mathrm{Aut}(S)) = \{7, 12, 13, 15\}$. Since S includes the Frobenius subgroups $2^6 : 7$ and $13 : 4$, there are no elements of order $5, 7, 13$ in K. Furthermore, $4 \cdot 5, 9 \cdot 7 \in \omega(H)$, hence $4, 9 \in \omega(K)$. If $\overline{H} \simeq \mathrm{Aut}(S)$ then $\overline{H}$ includes the Frobenius group $13 : 12$ and there exists an element of order 36 in H; a contradiction. Thus, $\overline{H} = S$ and K is a $\{2, 3\}$-group.

Consider the factor group $\widetilde{H} = H/O_2(K)$ and the subgroup $\widetilde{K} = K/O_2(K)$. Assume that $O_{3,2}(\widetilde{K}) \neq O_3(\widetilde{K})$. Notice first of all that the nontrivial abelian group $P = Z(O_2(\widetilde{K}/O_3(\widetilde{K})))$ acts on $O_3(\widetilde{K})$ faithfully. Furthermore, if $C = C_{\widetilde{H}/O_3(\widetilde{K})}(P)$ contains an element of order 13 then $C = H/O_3(\widetilde{K})$. Hence $7 \cdot 2 \in \omega(H)$; a contradiction. Let X be a subgroup of order 13 in $\widetilde{H}/O_3(\widetilde{K})$. A subgroup $F = [P, X] : X$ is a Frobenius group with kernel $[P, X]$ and complement X. The group F acts on $O_3(\widetilde{K})$ faithfully. So there exists an element of order $3 \cdot 13$ in H; a contradiction. Thus, $O_{3,2}(\widetilde{K}) = O_3(\widetilde{K})$, and $O_2(K)$ is a Sylow 2-subgroup of K.

We now denote by $\widetilde{H}$ and by $\widetilde{K}$ the factor groups $H/O_3(K)$ and $K/O_3(K)$ respectively. Assume that $O_{2,3}(\widetilde{K}) \neq O_2(\widetilde{K})$. Using the same arguments as in previous paragraph and an element of order 7 instead of an element of order 13, we arrive at a contradiction.

$O_3(K)$ is a Sylow 3-subgroup of K. Hence, Thus, K is a direct product of a Sylow 2-subgroup and a Sylow 3-subgroup. Therefore, K contains an element of order 36, which is impossible. The lemma is proved.

Thus, $S \simeq G$ and we will write G instead of S.

Lemma 3.4. $G \leq H \leq \mathrm{Aut}(G)$.

Proof. Let $K \neq 1$. There exists a prime p such that $O^p(K) \neq K$. Denote by $\widetilde{H}$ the factor group $H/O^p(K)$. The normal subgroup $\widetilde{K} = K/O^p(K)$ is a nontrivial p-group. Denote by $\widehat{H}$ the factor group $\widetilde{H}/\Phi(\widetilde{K})$ and by $\widehat{K}$, the factor group $\widetilde{K}/\Phi(\widetilde{K})$, where $\Phi(\widetilde{K})$ is a Frattini subgroup of $\widetilde{K}$. Since $H/K \simeq \widetilde{H}/\Phi(\widetilde{K})$, it is sufficient to show that $\omega(\widehat{H}) \not\subseteq \omega(H)$. So we assume that $H = \widehat{H}$ and K is nontrivial elementary abelian p-group for some prime p.

Suppose that $C = C_H(K) \not\leq K$. Since CK is a normal subgroup of H, the group CK/K includes G. Then H contains an element of order $13 \cdot p$. Hence $p = 2$ and $2 \cdot 7 \in \omega(H)$; a contradiction. Thus we may assume that $C \leq K$ and G acts on K faithfully.

The group G includes a subgroup $L_2(25)$, which includes a Frobenius subgroup with kernel isomorphic to 5^2 and cyclic complement of order 12. If $p = 3, 7, 13$ then $p \cdot 12 \in \omega(H)$, which is impossible.

Let $p = 2$ or 5. The group G includes a subgroup L isomorphic to $U_3(5)$, which acts on K by conjugation. Inspection of the tables of the Brauer p-characters of the group L in [15] shows that the element $x \in L$ of order 7 has a fixed point in every absolutely irreducible module over a field of characteristic p (that is, 2 or 5). Thus, x centralizes some nontrivial element in K and hence $p \cdot 7 \in \omega(H)$; a contradiction. The lemma is proved.

Remark. The case $p = 5$ can be eliminated by another way, since there is a Frobenius subgroup $2^4 : 5$ in $S_4(5) \leq G$.

Lemma 3.5. *$H = G$ or $H = G\langle\gamma\rangle$, where γ is the field automorphism of G of order* 2.

Proof. We have $G \leq H \leq \mathrm{Aut}(G)$. Notice first of all that G is a centralizer in $L_4(25)$ of the automorphism $\sigma = \theta\gamma$, where θ is the graph automorphism and γ is the field automorphism of $L_4(25)$. This implies that the action of θ and the action of γ on G coincide. We fix the notation γ for the automorphism of G induced by this action.

The group $\mathrm{Out}(G)$ is an elementary abelian group of order 4. Let δ be the diagonal automorphism and let γ be the field automorphism of G. Then $\mathrm{Aut}(G) = G\langle\delta, \gamma\rangle$. The action of δ and γ on G can be represented as follows. Let g denote the image in G of A in $SU_4(5)$. Then g^δ is the image of A^D, where $D = \mathrm{diag}\{1, 1, 1, -1\}$, and g^γ is the image of A^{-T}, i.e., the inverse transpose of A (see the remark in the previous paragraph). Thus, $|\delta| = |\gamma| = 2$. Furthermore, since $D^{-T} = D$, therefore δ and γ commute and the automorphism $\tau = \gamma\delta$ is also of order 2.

Suppose that $H \not\leq G\langle\gamma\rangle$. Since D centralizes in $SU_4(5)$ a subgroup isomorphic to $SU_3(5)$; we see that δ centralizes the image of this subgroup in G. Hence, if $\delta \in H$ then there exists an element of order $7 \cdot 2$ in H, which is impossible. We now consider in $SU_4(5)$ the matrix

$$A = \begin{pmatrix} \lambda & \lambda^2 & 0 & 0 \\ \lambda^{10} & \lambda^{17} & 0 & 0 \\ 0 & 0 & \lambda & \lambda^2 \\ 0 & 0 & \lambda^{10} & \lambda^{17} \end{pmatrix},$$

where λ generates the multiplicative group of the field $\mathbf{F}_{5^2}$. Direct calculation shows that the matrix $B = A \cdot (A^{-T})^D$ is of order 20 and the cyclic subgroup generated by B has trivial intersection with the center of $SU_4(5)$. If g is the image of A in G then $(g\tau)^2 = gg^{\gamma\delta}$ is the image of B in G of order 20. Therefore, $g\tau$ is of order 40. Since $40 \notin \omega(H)$, we have $\tau \notin H$.

Thus, $H \leq G\langle\gamma\rangle$. Using GAP (see [16]), we obtain some matrix representatives A of all conjugacy classes in $SU_4(5)$. Calculation of the projective orders of the matrices AA^{-T} allows us to find the spectrum of $G\langle\gamma\rangle$. It turns out that $\omega(G) = \omega(G\langle\gamma\rangle)$. Hence, $H = G$ or $H = G\langle\gamma\rangle$, and $h(G) = 2$. The theorem is proved.

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References

1. *Mazurov V. D.*, "The set of orders of elements in a finite group," Algebra and Logic, **33**, No. 1, 49–56 (1994).
2. *Mazurov V. D.*, "Recognition of finite groups by a set of orders of their elements," Algebra and Logic, **37**, No. 6, 371–379 (1998).
3. *Shi W.*, "A characteristic property of $PSL_2(7)$," J. Austral. Math. Soc. Ser. A, **36**, No. 3, 354–356 (1984).
4. *Shi W.*, "A characteristic property of A_5," J. Southwest-China Teach. Univ., **3**, 11–14 (1986).
5. *Shi W.*, "A characteristic property of J_1 and $PSL_2(2^n)$," Adv. Math., **16**, 397–401 (1987).
6. *Brandl R. and Shi W.*, "The characterization of $PSL(2, q)$ by its element orders," J. Algebra, **163**, No. 1, 109–114 (1994).
7. *Mazurov V. D.*, "Characterizations of groups by arithmetic properties," Algebra Colloq., **11**, No. 1, 129–140 (2004).
8. *Williams J. S.*, "Prime graph components of finite groups," J. Algebra, **69**, No. 2, 487–513 (1981).
9. *Kondrat′ev A. S.*, "On prime graph components for finite simple groups," Mat. Sb., **180**, No. 6, 787–797 (1989).
10. *Kondrat′ev A. S. and Mazurov V. D.*, "Recognition of alternating groups of prime degree from their element orders," Siberian. Math. J., **41**, No. 2, 294–302 (2000).
11. *Mazurov V. D.*, "Characterizations of finite groups by sets of all orders of the elements," Algebra and Logic, **36**, No. 1, 23–32 (1997).
12. *Conway J. H., Curtis R. T., Norton S. P., Parker R. A., and Wilson R. A.*, Atlas of Finite Groups, Clarendon Press, Oxford (1985).
13. *Aleeva M. R.*, "On finite simple groups with the set of element orders as in a Frobenius or a double Frobenius group," Math. Notes, **73**, No. 3, 299–313 (2003).
14. *Zavarnitsin A. V.*, Element Orders in Coverings of the Groups $L_n(q)$ and Recognition of the Alternating Group A_{16} [in Russian], NIIDMI, Novosibirsk (2000).
15. *Jansen C., Lux K., Parker R. A., and Wilson R. A.*, An Atlas of Brauer Characters, Clarendon Press, Oxford (1995).
16. *The GAP Group*, GAP—Groups, Algorithms, and Programming. Version 4.4. 2004 (http:// www.gap-system.org).