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ON THOMPSON'S CONJECTURE

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Abstract. For a finite group G denote by $N(G)$ the set of conjugacy class sizes of G. In 1980s J. G. Thompson posed the following conjecture: if L is a finite nonabelian simple group, G is a finite group with trivial center and $N(G) = N(L)$, then L and G are isomorphic. Here we prove Thompson's conjecture when L is one of the groups A_{10} and $L_4(4)$. This is the first time when Thompson's conjecture is checked for groups with connected prime graph.

Keywords: finite group, simple group, conjugacy class size, prime graph of a group.

1. Introduction

For a finite group G denote by $N(G)$ the set of conjugacy class sizes of G. In 1980s J. G. Thompson posed the following conjecture (see Question 12.38 in [1]).

Thompson's Conjecture. *If L is a finite nonabelian simple group, G is a finite group with trivial center and $N(G) = N(L)$, then L and G are isomorphic.*

The prime graph $GK(G)$ of a finite group G is defined as follows. The vertex set of $GK(G)$ is the set $\pi(G)$ of all prime divisors of the order of G. Two distinct primes $p, q \in \pi(G)$ considered as vertices of the graph are adjacent by edge if and only if there is an element of order pq in G. K. W. Gruenberg and O. Kegel introduced this graph (it is also called the Gruenberg — Kegel graph) in the middle of the 1970s and

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gave a characterization of finite groups with disconnected prime graph (we denote the number of connected components of $GK(G)$ by $s(G)$). J. S. Williams [2] and A. S. Kondratiev [3] obtained the classification of simple groups with $s(G) > 1$ and described the components of their prime graphs. Using these deep results G. Y. Chen [4] established that Thompson's conjecture is valid for every finite simple group L with $s(L) > 2$. Later he announced that the conjecture is true for all finite simple groups with disconnected prime graph. Although this assertion has not been proved in general yet, it was verified for most of such groups.

Contrary to the previous case, Thompson's conjecture has not been checked for any finite simple group L with connected prime graph. Here we investigate these groups. In particular, we prove that Thompson's conjecture holds true for the smallest (by order) nonabelian simple group with connected prime graph (that is the alternating group A_{10}) and for the smallest nonabelian simple group of Lie type with connected prime graph (that is the linear group $L_4(4)$).

Theorem. *Let $L = A_{10}$ or $L = L_4(4)$. If G is a finite group with trivial center and $N(G) = N(L)$, then $L \simeq G$.*

2. Preliminaries

Given an element x of group G, denote by x^G the set $\{x^g \mid g \in G\}$, that is the conjugacy class of G containing x, and by $|x^G|$ its size. The set $\{x_1, \dots, x_s\}$ of elements of G is called a complete set of representatives of conjugacy classes of G, if $G = \bigcup_{i=1}^{s} x_i^G$ and $x_i^G \cap x_j^G = \emptyset$ for $i \neq j$.

Lemma 1. *If G is a finite group and $\{x_1, \dots, x_s\}$ is a complete set of representatives of conjugacy classes of G, then $G = \langle x_1, \dots, x_s \rangle$.*

Proof. Let H be a proper subgroup of G. Since $|G : N_G(H)| = |\{H^g \mid g \in G\}|$, the cardinality of the set $\bigcup_{g\in G} H^g$ is less than $|G|$. Therefore this set is a proper subset of G. If $K = \langle x_1, \dots, x_s \rangle$ is a proper subgroup of G, then $\bigcup_{g\in G} K^g$ is a proper subset of G, which is impossible, since $\{x_1, \dots, x_s\}$ is a complete set of representatives of conjugacy classes of G. The lemma is proved.

Lemma 2. *If G is a finite group with trivial center and a prime $p \in \pi(G)$, then there exists element x of G such that p divides $|x^G|$.*

Proof. Assume that there is no element x of G such that p divides $|x^G|$. Let $\{x_1, \dots, x_s\}$ be a complete set of representatives of conjugacy classes of G. Then for every $i = 1, \dots, s$ the centralizer $C_G(x_i)$ includes a Sylow p-subgroup P_i. Fix some Sylow p-subgroup P of G. Since all Sylow subgroups of G are conjugate, there are elements $g_1, \dots, g_s$ of G such that for every $i = 1, \dots, s$ we have $P = P_i^{g_i}$. Put $y_i = x_i^{g_i}$ for $i = 1, \dots, s$. Then $P \leqslant C_G(y_i)$ for all $i = 1, \dots, s$. On the other hand, the set $\{y_1, \dots, y_s\}$ is a complete set of representatives of conjugacy classes of G and so $G = \langle y_1, \dots, y_s \rangle$ by Lemma 1. Thus P lies in the center of G; a contradiction. The lemma is proved.

Lemma 3. *If G and H are finite groups with trivial center and $N(G) = N(H)$, then $\pi(G) = \pi(H)$.*

Proof. This is a direct consequence of Lemma 2.

Lemma 4. *Suppose that G is a finite group with trivial center and p is a prime from $\pi(G)$ such that p^2 does not divide $|x^G|$ for all x in G. Then a Sylow p-subgroup of G is elementary abelian.*

Proof. Fix some Sylow p-subgroup P of G. Arguing as in the proof of Lemma 2, we can choose a complete set $\{x_1, \dots, x_s\}$ of representatives of conjugacy classes of G such that $C_G(x_i)$ includes a subgroup M_i of P with $|P : M_i| \leqslant p$ for every $i = 1, \dots, s$. Since the Frattini subgroup $\Phi(P)$ is the intersection of maximal subgroups of P, it lies in $C_G(x_i)$ for every $i = 1, \dots, s$. Thus, $\Phi(P)$ lies in the center of G and so it is trivial. The lemma is proved.

Lemma 5. *Let K be a normal subgroup of a finite group G, and $\overline{G} = G/K$. If $\overline{x}$ is the image of an element x of G in $\overline{G}$, then $|\overline{x}^{\overline{G}}|$ divides $|x^G|$. Moreover, if $(|x|, |K|) = 1$, then $C_{\overline{G}}(\overline{x}) = C_G(x)K/K$.*

Proof. The preimage of $C_{\overline{G}}(\overline{x})$ in G includes $C_G(x)$. It follows that $|\overline{x}^{\overline{G}}|$ divides $|x^G|$. The rest is true by [5, Theorem 1.6.2]. The lemma is proved.

Given natural numbers $n_1, \dots, n_s$, denote by $(n_1, \dots, n_s)$ its greatest common divisor and by $\mathrm{lcm}(n_1, \dots, n_s)$ its least common multiple. For a finite group G denote by $\omega(G)$ the spectrum of G, that is the set of element orders of G, and by $\mu(G)$ the set of maximal by divisibility elements of $\omega(G)$. Obviously, the set $\omega(G)$ and the prime graph $GK(G)$ are uniquely determined by $\mu(G)$. In naming simple groups we use the notation from [6].

3. Case $L = A_{10}$

We start with some properties of $L = A_{10}$.

Lemma 6 ([6]). *Let $L = A_{10}$. Then the following hold.*

1. $|L| = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$.
2. $\mu(L) = \{8, 9, 10, 12, 15, 21\}$.
3. $N(L)$ *consists of* $n_1 = 1$, $n_2 = 3^3 \cdot 5^2 \cdot 7$, $n_3 = 2 \cdot 3^2 \cdot 5 \cdot 7$, $n_4 = 2^2 \cdot 3^3 \cdot 5^2 \cdot 7$, $n_5 = 2^2 \cdot 3^4 \cdot 5^2 \cdot 7$, $n_6 = 2^4 \cdot 3 \cdot 5$, $n_7 = 2^4 \cdot 3 \cdot 5^2 \cdot 7$, $n_8 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7$, $n_9 = 2^4 \cdot 3^4 \cdot 5^2 \cdot 7$, $n_{10} = 2^5 \cdot 3^3 \cdot 7$, $n_{11} = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7$, $n_{12} = 2^5 \cdot 3^4 \cdot 5 \cdot 7$, $n_{13} = 2^7 \cdot 5^2 \cdot 7$, $n_{14} = 2^7 \cdot 3^2 \cdot 5^2 \cdot 7$, $n_{15} = 2^7 \cdot 3^3 \cdot 5 \cdot 7$, $n_{16} = 2^7 \cdot 3^3 \cdot 5^2$, $n_{17} = 2^7 \cdot 3^4 \cdot 7$.

Let G be a finite group with trivial center and $N(G) = N(L)$. We will prove that $G \simeq L$.

Since every number from $N(G)$ divides the order of G, Lemma 6 implies that $|L|$ divides $|G|$. By Lemma 3 we have $\pi(G) = \pi(L) = \{2, 3, 5, 7\}$.

Lemma 7. *A Sylow 7-subgroup S of G is of order 7. For every nontrivial element x of S the equality $|x^G| = n_{16} = 2^7 \cdot 3^3 \cdot 5^2$ holds.*

Proof. Using Lemma 4 and (3) of Lemma 6 we derive that S is elementary abelian. Assume that 7^2 divides $|G|$. Since $N(G) = N(L)$, the centralizer of every element of G contains an element of order 7. Consider an element y of G such that $|y^G| = n_{17} = 2^7 \cdot 3^4 \cdot 7$.

Suppose that 7 does not divide $|y|$. Let x be an element of $C_G(y)$ having order 7. Then $C_G(xy) = C_G(x) \cap C_G(y)$ and so $\mathrm{lcm}(|x^G|, |y^G|)$ divides $|(xy)^G|$. Since S is abelian, $C_G(x)$ includes S up to conjugacy. Hence 7 does not divide $|x^G|$. It follows that $|x^G|$ is equal to $n_6 = 2^4 \cdot 3 \cdot 5$ or $n_{16} = 2^7 \cdot 3^3 \cdot 5^2$. In both cases, $2^7 \cdot 3^4 \cdot 5 \cdot 7$ divides $|(xy)^G|$, which is impossible.

Suppose that 7 divides $|y|$. Let $|y| = 7t$. Since S is elementary abelian, the numbers 7 and t are coprime. Put $u = y^7$ and $v = y^t$. Then $y = uv$ and $C_G(y) = C_G(u) \cap C_G(v)$. Therefore, $|v^G|$ divides $|y^G| = 2^7 \cdot 3^4 \cdot 7$. On the other hand, the element v is of order 7 and so $|v^G|$ is equal to $n_6 = 2^4 \cdot 3 \cdot 5$ or $n_{16} = 2^7 \cdot 3^3 \cdot 5^2$; a contradiction.

Thus S has order 7.

If the second assertion of the lemma is false, there is an element x of S such that $|x^G| = n_6 = 2^4 \cdot 3 \cdot 5$. Let P be a Sylow 3-subgroup of G such that $M = C_G(x) \cap P$ has index 3 in P. Since M is maximal, there exists a nontrivial element y in the intersection of the center $Z(P)$ of P and M. Since $y \in Z(P)$, we have $|y^G| = n_{13} = 2^7 \cdot 5^2 \cdot 7$ by (3) of Lemma 6, which is impossible, beqause $x \in C_G(y)$.

The lemma is proved.

Lemma 8. *Suppose that $q \in \{2, 3, 5\}$, Q is a Sylow q-subgroup of G, and $Z(Q)$ is its center. Then the order of the centralizer $C_G(y)$ of every element y of $Z(Q)$ is not divisible by 7. If $q = 5$, then either $|Q| = 5^2$ or $|Z(Q)| = 5$.*

Proof. If y lies in $Z(Q)$, then q does not divide $|y^G|$, so by Lemma 6 we have $|y^G| = n_2 = 3^3 \cdot 5^2 \cdot 7$ for $q = 2$, $|y^G| = n_{13} = 2^7 \cdot 5^2 \cdot 7$ for $q = 3$, and $|y^G|$ is equal to $n_{10} = 2^5 \cdot 3^3 \cdot 7$ or $n_{17} = 2^7 \cdot 3^4 \cdot 7$ for $q = 5$. In all cases 7 does not divide $C_G(y)$ by Lemma 7.

Let $q = 5$. By Lemma 7 there exists an element x of order 7 such that $|Q : C_Q(x)| = 5^2$. If $|Q| > 5^2$, then $C = C_Q(x) \neq 1$. On the other hand, $Z(Q) \cap C = 1$. If $|Z(Q)| > 5$, then $Q = \langle C, Z(Q)\rangle$. However, in that case $1 \neq Z(C) \leqslant Z(Q)$; a contradiction.

Lemma 9. $O_{2,2'}(G) = O_2(G)$. *In particular, G is insoluble.*

Proof. Put $K = O_2(G)$, $\overline{G} = G/K$, and denote by $\overline{x}$ and by $\overline{H}$ the images of an element x and a subgroup H of G in $\overline{G}$. If the statement of the lemma is false, then there is $r \in \{3, 5, 7\}$ such that $O_r(\overline{G}) \neq 1$.

If $P = O_7(\overline{G}) \neq 1$, then $|P| = 7$. Let y be an element of the center $Z(Q)$ of a Sylow 5-subgroup Q. Since 5 does not divide $6 = 7 - 1$, the subgroup $P\langle\overline{y}\rangle$ is cyclic. Hence 7 divides $|C_{\overline{G}}(\overline{y})|$. Since $(5, |K|) = 1$, Lemma 5 implies that 7 divides $|C_G(y)|$, which is impossible by Lemma 8. Thus, $O_7(\overline{G}) = 1$.

Let $q \in \{3, 5\}$, and Q be a Sylow q-subgroup of $\overline{G}$. If $O_q(\overline{G}) \neq 1$, then $V = Z(O_q(\overline{G}))$ is a nontrivial normal subgroup of $\overline{G}$. If x is an element of order 7 in G, then $V = C_V(\overline{x}) \times [V, \overline{x}]$. Lemma 7 implies that $|\overline{x}^{\overline{G}}|$ is a divisor of $2^7 \cdot 3^3 \cdot 5^2$, hence the index of $C_V(\overline{x})$ in V is at most 5^2 for $q = 5$, and 3^3 for $q = 3$. Therefore, $|[V, \overline{x}]| \leqslant q^3$. Since $n = 6$ is the least number such that 7 divides $q^n - 1$, the subgroup $[V, \overline{x}]\langle\overline{x}\rangle$ must be abelian. Thus, $[V, \overline{x}] = 1$ and $V = C_V(\overline{x})$. On the other hand, the center $Z(Q)$ of a Sylow subgroup Q has a nontrivial intersection with V. The element $\overline{z}$ of order q from this intersection commutes with $\overline{x}$. Since $(|K|, q) = 1$, there exists a preimage z of $\overline{z}$ in G such that z lies in the center of a Sylow q-subgroup of G. By Lemma 5, the centralizer of z also contains an element of order 7, that contradicts Lemma 8. Therefore, $O_q(\overline{G}) = 1$, and the lemma is proved.

Lemma 10. $G \simeq L$.

Proof. We preserve the notation $K = O_2(G)$ and $\overline{G} = G/K$ from the previous lemma. By that lemma $M \leqslant \overline{G} \leqslant \operatorname{Aut} M$, where $M = S_1 \times \ldots \times S_k$ is a direct product of finite nonabelian simple groups $S_1, \ldots, S_k$. Obviously, $\pi(S_i) \subseteq \pi(L)$ for $i = 1, \ldots, k$. Up to isomorphism, there are finitely many finite nonabelian simple groups S with $\pi(S) \subseteq \pi(L)$. Using the classification of finite simple groups one can list them all, similarly to [7]. It follows that [6] contains the information about all necessary groups, and we will use it without further reference.

Suppose that there is an element $x \in \overline{G} \setminus M$ of order 7. Put $P = S_1^x$. Since P is simple, every its natural projection P_i on S_i, $i = 1, \ldots, k$, is either trivial or isomorphic to S_1. On the other hand, since P is normal in M, every subgroup P_i, $i = 1, \ldots, k$ is also normal. Hence $P_i = 1$ or $P_i = S_i$. Therefore, there is the unique $j \in \{1, \ldots, k\}$ such that $S_1^x = S_j$. Since $(|\operatorname{Out} S|, 7) = 1$ for all finite nonabelian simple groups S with $\pi(S) \subseteq \pi(L)$, we have $S_1 = S_j \neq S_1$. Therefore, $k \geqslant 7$. The order of every nonabelian simple group is divided by at least three primes, and 7 does not divide $|M|$, so 5 divides the order of every S_i, $i = 1, \ldots, k$. Lemma 8 implies that the center $Z(Q)$ of a Sylow 5-subgroup Q of $\overline{G}$ has the order 5, and so lies in M. Put $N = N_{\overline{G}}(Q)$. By Frattini argument, $N/(N \cap M) \simeq \overline{G}/M$. Hence x normalizes $Z(Q)$ and so centralizes it; a contradiction.

Thus, 7 divides the order of M, and so divides the order of exactly one of the subgroup S_i. We denote this subgroup by S_1. The subgroup S_1 is a normal subgroup of $\overline{G}$, and we denote by $\widehat{G}$ and $\widehat{M}$ the factor groups $\overline{G}/S_1$ and M/S_1 respectively. Suppose that $k \geqslant 2$. Then a Sylow 5-subgroup of $\widehat{G}$ is nontrivial and its center Z has a nontrivial intersection with $\widehat{M}$. Consider a nontrivial element y of $T = S_2 \times \ldots \times S_k$ such that its image in $\widehat{G}$ lies in Z. Since y centralizes S_1, it lies in the center of a Sylow 5-subgroup of $\overline{G}$ and centralizes an element of order 7; a contradiction.

Thus, M is a nonabelian simple group with Sylow 7-subgroup of order 7, and $2^7 \cdot 3^4 \cdot 5^2$ divides $|\mathrm{M}|$. Using [6, 7] again, we obtain that, up to isomorphism, there are only two such groups: A_{10} and $O_8^+(2)$.

Let x be an element of order 7 in G and $\overline{x}$ be its image in $\overline{G}$. If $M \simeq O_8^+(2)$, then $|\overline{x}^{\overline{G}}|$ is a multiple of 3^5, and so is $|x^G|$ by Lemma 5. This contradicts (3) of Lemma 6. Thus, M is isomorphic to A_{10}. Since $|\overline{x}^{\overline{G}}| = 2^7 \cdot 3^3 \cdot 5^2 = |x^G|$, the element x centralizes K. If $K \neq 1$, then x centralizes an element from the center of a Sylow 2-subgroup of G, which is impossible by Lemma 8. Therefore, $\overline{G} = G$. If $G \simeq S_{10}$, then there is an element x in G such that $|x^G| = 3^2 \cdot 5$; a contradiction.

The lemma and the theorem in case $L = A_{10}$ are proved.

4. Case $L = L_4(4)$

We start with some properties of $L = L_4(4)$.

Lemma 11 ([6]). *Let $L = L_4(4)$. Then the following hold.*

1. $|L| = 2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$.
2. $\mu(L) = \{12, 30, 63, 85\}$.
3. *$N(L)$ consists of* $n_1 = 1$, $n_2 = 3^2 \cdot 5 \cdot 7 \cdot 17$, $n_3 = 2^2 \cdot 3^3 \cdot 5 \cdot 7 \cdot 17$, $n_4 = 2^4 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 17$, $n_5 = 2^6 \cdot 5 \cdot 17$, $n_6 = 2^6 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 17$, $n_7 = 2^6 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$, $n_8 = 2^8 \cdot 3 \cdot 7 \cdot 17$, $n_9 = 2^8 \cdot 3^2 \cdot 5 \cdot 7 \cdot 17$, $n_{10} = 2^8 \cdot 3^3 \cdot 7$, $n_{11} = 2^8 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 17$, $n_{12} = 2^8 \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$, $n_{13} = 2^{10} \cdot 3 \cdot 5 \cdot 7 \cdot 17$, $n_{14} = 2^{10} \cdot 3^2 \cdot 7 \cdot 17$, $n_{15} = 2^{10} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 17$, $n_{16} = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 17$, $n_{17} = 2^{12} \cdot 3^2 \cdot 5 \cdot 7 \cdot 17$, $n_{18} = 2^{12} \cdot 3^2 \cdot 5^2 \cdot 17$, $n_{19} = 2^{12} \cdot 3^3 \cdot 7 \cdot 17$, $n_{20} = 2^{12} \cdot 3^4 \cdot 5 \cdot 7$.

Let G be a finite group with trivial center and $N(G) = N(L)$.

Since every number from $N(G)$ divides the order of G, Lemma 11 implies that $|L|$ divides $|G|$. By Lemma 3 we have $\pi(G) = \pi(L) = \{2, 3, 5, 7, 17\}$.

Lemma 12. *If $p \in \{7, 17\}$, then a Sylow p-subgroup S of G is of order p. There is no element of order $7 \cdot 17$ in G.*

Proof. The proof is quite similar to that of Lemma 7. First of all, Lemma 4 implies that S is elementary abelian. If $|x| = p$, then $|x^G| = n_5 = 2^6 \cdot 5 \cdot 17$ or $|x^G| = n_{18} = 2^{12} \cdot 3^2 \cdot 5^2 \cdot 17$ for $p = 7$, and $|x^G| = n_{10} = 2^8 \cdot 3^3 \cdot 7$ or $|x^G| = n_{20} = 2^{12} \cdot 3^4 \cdot 5 \cdot 7$ for $p = 17$.

Assume that $p = 7$ and $|S| \geqslant p^2$. Consider an element y of G such that $|y^G| = n_{20} = 2^{12} \cdot 3^4 \cdot 5 \cdot 7$.

Suppose that 7 does not divide $|y|$. Let x be an element of $C_G(y)$ having order 7. Then $C_G(xy) = C_G(x) \cap C_G(y)$ and so $\mathrm{lcm}(|x^G|, |y^G|)$ divides $|(xy)^G|$. Since S is abelian, $C_G(x)$ includes S up to conjugacy. Hence 7 does not divide $|x^G|$. It follows that $|x^G|$ is equal to $n_5 = 2^6 \cdot 5 \cdot 17$ or $n_{18} = 2^{12} \cdot 3^2 \cdot 5^2 \cdot 17$. In both cases, $2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$ divides $|(xy)^G|$, which is impossible by (3) of Lemma 11.

Suppose that 7 divides $|y|$. Let $|y| = 7t$. Since S is elementary abelian, the numbers 7 and t are coprime. Put $u = y^7$ and $v = y^t$. Then $y = uv$ and $C_G(y) = C_G(u) \cap C_G(v)$. Therefore, $|v^G|$ divides $|y^G| = 2^{12} \cdot 3^4 \cdot 5 \cdot 7$. On the other hand, the element v is of order 7 and so $|v^G|$ is equal to $n_5 = 2^6 \cdot 5 \cdot 17$ or $n_{18} = 2^{12} \cdot 3^2 \cdot 5^2 \cdot 17$; a contradiction.

Assume that $p = 17$ and $|S| \geqslant p^2$. Considering an element y with $|y^G| = n_{18} = 2^{12} \cdot 3^2 \cdot 5^2 \cdot 17$, we obtain a contradiction as in the case $p = 7$. Therefore, S is of order p in both cases. Thus, the centralizer of every element of order 7 does not contain an element of order 17. The lemma is proved.

Lemma 13. *Suppose that $q \in \{2, 3, 5\}$, Q is a Sylow q-subgroup of G, and $Z(Q)$ is its center. Then for every element $y \in Z(Q)$, the order of the centralizer $C_G(y)$ is coprime to 7 for $q = 5$, coprime to 17 for $q = 3$, and coprime to $7 \cdot 17$ for $q = 2$.*

Proof. If y lies in $Z(Q)$, then q does not divide $|y^G|$, so by (3) Lemma 11 we have $|y^G| = n_2 = 3^2 \cdot 5 \cdot 7 \cdot 17$ for $q = 2$, $|y^G| = n_5 = 2^6 \cdot 5 \cdot 17$ for $q = 3$, and $|y^G|$ is equal to one of the numbers $n_8 = 2^8 \cdot 3 \cdot 7 \cdot 17$, $n_{10} = 2^8 \cdot 3^3 \cdot 7$, $n_{14} = 2^{10} \cdot 3^2 \cdot 7 \cdot 17$, or $n_{19} = 2^{12} \cdot 3^3 \cdot 7 \cdot 17$ for $q = 5$. Thus, the centralizer of y has the desired order.

Lemma 14. *$O_{2,2'}(G) = O_2(G)$. In particular, G is insoluble.*

Proof. Put $K = O_2(G)$, $\overline{G} = G/K$, and denote by $\overline{x}$ and by $\overline{H}$ the images of an element x and a subgroup H of G in $\overline{G}$. If the statement of the lemma is false, then there is $p \in \pi(L) \setminus \{2\}$ such that $O_p(\overline{G}) \neq 1$.

Suppose that $O_p(\overline{G}) \neq 1$ for p equal to 7 or 17. Then $\overline{G}$ includes a Hall $\{7, 17\}$-subgroup of order $7 \cdot 17$. However, this subgroup must be cyclic, which contradicts Lemma 12.

Let P be a Sylow 3-subgroup of $\overline{G}$. If $O_3(\overline{G}) \neq 1$, then $V = Z(O_3(\overline{G}))$ is a nontrivial normal subgroup of $\overline{G}$. Let $\overline{x}$ be an element of order 17 in $\overline{G}$. Since $|\overline{x}^{\overline{G}}|$ is a divisor of $2^8 \cdot 3^3 \cdot 7$ or $2^{12} \cdot 3^4 \cdot 5 \cdot 7$, we may assume that $|V : C_V(\overline{x})| \leqslant 3^4$. Therefore, the order of $[V, \overline{x}]$ is at most 3^4. It follows that $[V, \overline{x}] = 1$, and so $V = C_V(\overline{x})$. If $\overline{z}$ is a nontrivial element of $Z(P) \cap V$, then $|C_{\overline{G}}(\overline{z})|$ is a multiple of 17. By Lemma 5 the same is true for its preimage z in G such that z lies in the center of a Sylow 3-subgroup of G. This contradicts Lemma 13. Therefore, $O_3(\overline{G}) = 1$.

Assume that $O_5(\overline{G}) \neq 1$ and consider an element $\overline{x}$ of order 7 in $\overline{G}$ instead of an element of order 17. Arguing as in the previous paragraph we come to an element

z of the center of a Sylow 5-subgroup of G such that $|z^G|$ is a multiple of 7. This is impossible by Lemma 13. Therefore, $O_5(\overline{G}) = 1$, and the lemma is proved.

Lemma 15. $G \simeq L$.

Proof. Lemma 15 implies that $M \leqslant \overline{G} \leqslant \operatorname{Aut} M$, where $M = S_1 \times \ldots \times S_k$ is a direct product of nonabelian simple groups $S_1, \ldots, S_k$. Since G cannot include a Hall $\{7, 17\}$-subgroup, the numbers 7 and 17 divide the order of exactly one of these groups, and we assume that they divide S_1. Therefore, S_1 is a normal subgroup of $\overline{G}$, and we denote by $\widehat{G}$ and $\widehat{M}$ the factor groups $\overline{G}/S_1$ and M/S_1 respectively. Suppose that $k \geqslant 2$. Then a Sylow 5-subgroup of $\widehat{G}$ is nontrivial and its center Z has a nontrivial intersection with $\widehat{M}$. Consider a nontrivial element y of $T = S_2 \times \ldots \times S_k$ such that its image in $\widehat{G}$ lies in Z. Since y centralizes S_1, it lies in the center of a Sylow 5-subgroup of $\overline{G}$ and centralizes an element of order 7; a contradiction.

Thus, $M = S_1$ and $\overline{G}$ is almost simple. Consider all nonabelian simple group S such that $\pi(S) \subseteq \pi(L)$, the number $3^4 \cdot 5^2$ divides $|\operatorname{Aut} S|$, and Sylow p-subgroups of S are of order p for $p \in \{7, 17\}$. Using [6, 7] one can check that only groups $L = L_4(4)$ and $S_8(2)$ satisfy these conditions. If $\overline{G} \simeq S_8(2) \simeq \operatorname{Aut}(S_8(2))$, then there exists an element $\overline{x}$ of order 17 with $|\overline{x}^{\overline{G}}| = 2^{16} \cdot 3^5 \cdot 5^2 \cdot 7$, which is impossible by Lemma 5 and (3) of Lemma 11.

Thus, $L \leqslant \overline{G} \leqslant \operatorname{Aut} L$. Therefore, the number $2^{12} \cdot 3^4 \cdot 5 \cdot 7$ divides $|\overline{x}^{\overline{G}}|$ for an element $\overline{x}$ of $\overline{G}$ of order 17. By Lemma 5 the same is true for the preimage x of $\overline{x}$ in G. By (3) of Lemma 11 we have $|x^G| = |\overline{x}^{\overline{G}}| = 2^{12} \cdot 3^4 \cdot 5 \cdot 7$. Therefore, x centralizes $K = O_2(G)$. If $K \neq 1$, then x centralizes a nontrivial element from the center of a Sylow 2-subgroup of G, that contadicts Lemma 13.

Therefore, $L \leqslant G \leqslant L\langle \phi, \tau \rangle$, where ϕ is an involutory field automorphism, τ is an involutory graph automorphism of L, and $\phi\tau = \tau\phi$. Using [8, (19.1),(19.6),(19.9)] we obtain the information on the structure of centralizers of ϕ, τ and $\tau\phi$ in L. If $G = L\langle \phi \rangle$, then $|C_L(x)| = |C_G(x)|$ for every element x of order 17. So $|x^G| = 2|x^L|$, which is impossible by (3) of Lemma 11. If G is equal to $L\langle \tau \rangle$ or $L\langle \tau\phi \rangle$, then $|x^G| = 2|x^L|$ for every element x of order 7; a contradiction. Finally, if $G = \operatorname{Aut} L$, then for element x of order 17, we have $|C_G(x)| = 2|C_L(x)|$, but $|G : L| = 4$, and so again $|x^G| = 2|x^L|$, which is impossible.

The lemma and the theorem are proved.

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