The proper definition and Wielandt-Hartley's theorem for submaximal $\mathfrak{X}$-subgroups

To the 110th anniversary of Helmut Wielandt

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Abstract

A nonempty class $\mathfrak{X}$ of finite groups is called complete if it is closed under taking subgroups, homomorphic images and extensions. We deal with a classical problem of determining $\mathfrak{X}$-maximal subgroups. We consider two definitions of submaximal $\mathfrak{X}$-subgroups suggested by Wielandt and discuss which one better suits our task. We prove that these definitions are not equivalent yet Wielandt-Hartley's theorem holds true for either definition of $\mathfrak{X}$-submaximality. We also give some applications of the strong version of Wielandt-Hartley's theorem.

Keywords: finite nonsolvable group, complete class, maximal $\mathfrak{X}$-subgroups, submaximal $\mathfrak{X}$-subgroups, subnormal subgroups

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1 Introduction

In this article we focus on the relationship between two definitions of submaximal $\mathfrak{X}$-subgroups of a finite group[1] given by H. Wielandt: the first one appeared in his lectures [19] delivered at Tübingen in 1963–64, and the second one was presented in his talk [20] at the celebrated Santa Cruz conference on finite groups in 1979. We will show that these definitions are not equivalent, yet Wielandt-Hartley's theorem for submaximal $\mathfrak{X}$-subgroups is true for either definition of submaximality. In its strong version this theorem was announced by Wielandt in [20], but the proof was never published. As a demonstration of possible applications of the strong Wielandt-Hartley's theorem, we prove a sufficient condition for conjugacy of submaximal $\mathfrak{X}$-subgroups in terms of projections into the factors of subnormal series, obtain a characterization of submaximal $\mathfrak{X}$-subgroups in direct products and also find a new criterion for subnormality. The last section of the paper contains several short historical remarks.

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[1] We consider finite groups only, and from now on the term "group" means a "finite group".

We begin with the context where the notion of a submaximal $\mathfrak{X}$-subgroup arises. It is well known that one of the central topics in group theory is a study of subgroups of a given group. Apart from arbitrary subgroups, one can be interested in some special types of subgroups defined by their arithmetic or group-theoretic properties, or in other words, by belonging to the corresponding class $\mathfrak{X}$ of groups (abelian, nilpotent, solvable, p-groups for a prime p, π-groups for a set of primes π, etc.). This task is exceptionally difficult to achieve in its general setting, and it is universally accepted that attention can be restricted to maximal subgroups, i.e. subgroups which are maximal by inclusion

- among proper subgroups, if we are interested in all subgroups; these subgroups are simply called *maximal*,
- or among $\mathfrak{X}$-subgroups, i.e. subgroups from a class $\mathfrak{X}$; in this case we are talking about *maximal $\mathfrak{X}$-subgroups* or *$\mathfrak{X}$-maximal subgroups*.

Following Wielandt [19, 20], we consider maximal $\mathfrak{X}$-subgroups only in the case of a so-called complete class $\mathfrak{X}$. A nonempty class $\mathfrak{X}$ of finite groups is said to be *complete* (“*vollständig*” in Wielandt’s terms [19, Definition 11.3]), if it is closed under taking subgroups, homomorphic images and extensions, where the latter means that $G \in \mathfrak{X}$ whenever G contains a normal $\mathfrak{X}$-subgroup A and $G/A \in \mathfrak{X}$. Solvable groups, π-groups, and solvable π-groups, where π is a set of primes, are examples of complete classes. From now on the symbol $\mathfrak{X}$ will always mean a fixed complete class.

While studying a group G, it is natural to deal with the factors of its composition series, i.e. a subnormal[2] series

$$G = G_0 \geq G_1 \geq \cdots \geq G_n = 1, \tag{1}$$

whose factors $G^i = G_{i-1}/G_i$ are simple groups. Recall that the classical Jordan-Hölder theorem implies that the set of composition factors is a group invariant, meaning that up to reordering and isomorphism it does not depend on the series (1). The strategy of “reduction to simple groups”, which appeared at the dawn of group theory in works of Galois and Jordan, became truly effective after the classification of finite simple groups (CFSG) had been finished.

Applying this approach to our problem, define for a subgroup H of a group G the *projections*

$$H^i = (H \cap G_{i-1})G_i/G_i$$

of H on the factors G^i of a subnormal series (1). Wielandt [19, (12.1)(b)] noticed an obvious fact that if all projections H^i of H are maximal $\mathfrak{X}$-subgroups of G^i, then H is a maximal $\mathfrak{X}$-subgroup of G. It is a far less trivial question if the converse of this statement holds.

[2]Recall that the relation of normality between a group and its subgroup is not transitive in the sense that if H is normal in G (we write $H \trianglelefteq G$) and K is normal in H, then K is not necessarily normal in G. A subgroup H of G is *subnormal* (we write $H \trianglelefteq\trianglelefteq G$) if there exists a series $G = G_0 \geq G_1 \geq \cdots \geq G_n = H$, where $G_i \trianglelefteq G_{i-1}$ for all $i = 1, \ldots, n$. In other words, subnormality is the transitive closure of normality.

As a positive example, consider the class $\mathfrak{X}$ of all p-groups for some prime p. Suppose that H is a maximal $\mathfrak{X}$-subgroup of G. By the Sylow theorem, H is a Sylow p-subgroup of G, i.e. its order is equal to the highest power of p dividing the order of G. It can be easily shown that if A is a normal subgroup of G, then $H \cap A$ is a Sylow p-subgroup of A and HA/A is a Sylow p-subgroup of G/A. As a consequence, H^i is a Sylow p-subgroup of G^i for each i.

Does something similar hold for an arbitrary complete class $\mathfrak{X}$? More precisely, if $\mathfrak{X}$ is a complete class, A is a normal and H is an $\mathfrak{X}$-maximal subgroups of G, is it true that

(a) HA/A is a maximal $\mathfrak{X}$-subgroup of G/A, and

(b) $H \cap A$ is a maximal $\mathfrak{X}$-subgroup of A?

Wielandt showed that the answers to Questions (a) and (b) in general are negative.

In [19, (14.2)], it was demonstrated that there is a generic counterexample to Question (a) for every complete class $\mathfrak{X}$ with the following property: there is a group with nonconjugate maximal $\mathfrak{X}$-subgroups. Indeed, let N be such a group, and B an arbitrary group. If $G = N \wr B$ is the regular wreath product of N and B with the base subgroup $A = N^{|B|}$, then each (maximal or not) $\mathfrak{X}$-subgroup of $B = G/A$ is an image of some maximal $\mathfrak{X}$-subgroup of G.

There are examples where the intersection of a maximal $\mathfrak{X}$-subgroup with a normal subgroup A is not $\mathfrak{X}$-maximal in A (cf. Question (b)). Such examples can be found even among almost simple groups[3] (see, e.g., [19, p. 27] and [5, Tables 6 and 11]). However, in contrast to the situation with homomorphic images, not every $\mathfrak{X}$-subgroup of a normal subgroup A can be represented as an intersection of A and some maximal $\mathfrak{X}$-subgroup. The relevant constraint here is the theorem proved by Wielandt [19, Hauptsatz 13.2] and B. Hartley [8, Lemmas 2 and 3] independently, so we further refer to this result and its variations as Wielandt-Hartley's theorems.

We write $\mathrm{m}_{\mathfrak{X}}(G)$ for the set of all maximal $\mathfrak{X}$-subgroups of G. Recall that if P and Q are subgroups of a group G then the *normalizer*

$$N_Q(P) = \{x \in Q \mid x^{-1}Px = P\}$$

of P in Q is the largest subgroup of Q which normalizes P.

Theorem 1 (Wielandt-Hartley's theorem for normal subgroups)**.** *Let G be a finite group and let $\mathfrak{X}$ be a complete class. If A is a normal subgroup of G, then for every $H \in \mathrm{m}_{\mathfrak{X}}(G)$ the quotient $N_A(H \cap A)/(H \cap A)$ contains no nontrivial $\mathfrak{X}$-subgroups.*

All known proofs of Theorem 1 and its special cases (see Section 5) use the Schreier conjecture asserting solvability of the outer automorphism group $\mathrm{Out}(S)$ of every simple group S, i.e. of the quotient of the automorphism group

[3]Recall that a finite group G is called *almost simple* if its *socle*, that is the subgroup generated by all (nontrivial) minimal normal subgroups, is a nonabelian simple group.

Aut(S) of S by the group Inn(S) of inner automorphisms. Recall that the validity of the Schreier conjecture follows from CFSG.

Theorem 1 prompted Wielandt to introduce a new concept: submaximal $\mathfrak{X}$-subgroups.

Definition 1. [19, Definition 15.1] Let $\mathfrak{X}$ be a complete class of finite groups. A subgroup H of a finite group G is called a *(strongly) submaximal $\mathfrak{X}$-subgroup* or *$\mathfrak{X}$-submaximal in the sense of* [19] (we write $H \in \mathrm{sm}^{\circ}_{\mathfrak{X}}(G)$) if there exists an embedding

$$\phi : G \hookrightarrow G^*$$

of a group G in some finite group G^* such that

$$G^{\phi} \trianglelefteq G^* \text{ and } H^{\phi} = X \cap G^{\phi} \text{ for some } X \in \mathrm{m}_{\mathfrak{X}}(G^*).$$

Less formally, $H \in \mathrm{sm}^{\circ}_{\mathfrak{X}}(G)$ if there is a group G^* and its subgroup $X \in \mathrm{m}_{\mathfrak{X}}(G^*)$ such that $G \trianglelefteq G^*$ and $H = G \cap X$.

Since we can take $G^* = G$, it is clear that $\mathrm{m}_{\mathfrak{X}}(G) \subseteq \mathrm{sm}^{\circ}_{\mathfrak{X}}(G)$.

Theorem 1 can now be reformulated in this new language as follows.

Theorem 2 (Wielandt-Hartley's theorem for strongly submaximal $\mathfrak{X}$-subgroups)**.** *Let G be a finite group and let $\mathfrak{X}$ be a complete class. If $H \in \mathrm{sm}^{\circ}_{\mathfrak{X}}(G)$, then $N_G(H)/H$ contains no nontrivial $\mathfrak{X}$-subgroups.*

Fifteen years later, at the Santa Cruz conference [20], Wielandt suggested a program for studying maximal $\mathfrak{X}$-subgroups by projecting them into the factors of a composition series. For that purpose, he came to a different, though close to original, definition of $\mathfrak{X}$-submaximality. He expected to find a generalization of a maximal $\mathfrak{X}$-subgroup that would "preserve as many properties" of Sylow p-subgroups and Hall π-subgroups as possible, "for example, compatibility with normal subgroups" [20, p. 170].

Definition 2. [20, p. 170] Let $\mathfrak{X}$ be a complete class of finite groups. A subgroup H of G is called a *submaximal $\mathfrak{X}$-subgroup* or *$\mathfrak{X}$-submaximal in the sense of* [20] (we write $H \in \mathrm{sm}_{\mathfrak{X}}(G)$) if there exists an embedding

$$\phi : G \hookrightarrow G^*$$

of a group G in some finite group G^* such that

$$G^{\phi} \trianglelefteq\trianglelefteq G^* \text{ and } H^{\phi} = X \cap G^{\phi} \text{ for some } X \in \mathrm{m}_{\mathfrak{X}}(G^*).$$

Comparing Definitions 1 and 2, one can see that the only difference lies in the requirements on the embedding of G into G^*: in the first case G embeds as a normal subgroup, while in the second case it embeds as a subnormal subgroup. Since a normal subgroup is also subnormal,

$$\mathrm{m}_{\mathfrak{X}}(G) \subseteq \mathrm{sm}^{\circ}_{\mathfrak{X}}(G) \subseteq \mathrm{sm}_{\mathfrak{X}}(G). \tag{2}$$

Definition 2 is *a priori* more general and intuitively more complicated than Definition 1. But it fits the goal of Wielandt's program better, because $\mathfrak{X}$-submaximal (in the sense of [20]) subgroups have the obvious inductive property resembling properties of Sylow subgroups:

$$\text{if } H \in \mathrm{sm}_{\mathfrak{X}}(G) \text{ and } N \trianglelefteq\trianglelefteq G, \text{ then } H \cap N \in \mathrm{sm}_{\mathfrak{X}}(N). \tag{3}$$

Hence Definition 2 exactly satisfies the requirements that Wielandt posed on the "proper" ("richtig" [20, p. 170]) generalization of maximal $\mathfrak{X}$-subgroups.

If Definition 1 was equivalent to Definition 2, it would also be "richtig." However, it is not the case. In Section 2, we provide a series of almost simple groups G with socles isomorphic to the orthogonal groups $\mathrm{P}\Omega^+_{4n}(q)$ such that $\mathrm{sm}^\circ_{\mathfrak{X}}(G) \neq \mathrm{sm}_{\mathfrak{X}}(G)$ for suitable classes $\mathfrak{X}$. Since Definitions 1 and 2 are not equivalent, in what follows we refer to $\mathfrak{X}$-subgroups from Definition 1 as *strongly* submaximal.

Now, it is natural to ask if submaximal $\mathfrak{X}$-subgroups inherit main properties of strongly submaximal $\mathfrak{X}$-subgroups. In [20, 5.4(a)], Wielandt announced the following theorem.

Theorem 3 (Wielandt-Hartley's theorem for submaximal $\mathfrak{X}$-subgroups)**.** *Let G be a finite group and let $\mathfrak{X}$ be a complete class. If $H \in \mathrm{sm}_{\mathfrak{X}}(G)$, then $N_G(H)/H$ contains no nontrivial $\mathfrak{X}$-subgroups.*

As in the case of Theorems 1 and 2, this result can be reformulated without using the notion of a submaximal $\mathfrak{X}$-subgroup.

Theorem 4 (Wielandt-Hartley's theorem for subnormal subgroups)**.** *Let G be a finite group and let $\mathfrak{X}$ be a complete class. If A is a subnormal subgroup of G, then for every $H \in \mathrm{m}_{\mathfrak{X}}(G)$ the quotient $N_A(H \cap A)/(H \cap A)$ contains no nontrivial $\mathfrak{X}$-subgroups.*

As is easily seen, the only difference of the latter assertion from Theorem 1 is that A is a subnormal (not necessarily normal) subgroup of G.

In group theory properties of subnormal subgroups can be often extracted from the corresponding properties of normal subgroups by means of straightforward induction. However, some statements are indeed harder to prove for subnormal subgroups. That is the case for the classical theorem of Wielandt [17, Statements 7 and 9], [15, Ch. 2, (3.23)] which asserts that in a finite group a subgroup generated by subnormal subgroups is also subnormal. The same difficulty arises in the case of Theorem 4.

As far as we know, a proof of Theorem 4 (and Theorem 3) never appeared. In Section 3, we fill a gap by proving this theorem.

It is worth mentioning that L. A. Shemetkov [14, Theorem 7] proved an important special case of Theorem 4. Namely, he showed that *if H is a maximal π-subgroup of a finite group G, and A is a subnormal subgroup of G which is not a π'-group, then* $H \cap A \neq 1$. In [4, Proposition 8], W. Guo and D. Revin generalized this result to an arbitrary complete class $\mathfrak{X}$. We use the latter in the proof of Theorem 4.

We establish several applications of Wielandt-Hartley's theorem in this strong version. To begin with, Theorem 3 and inductive property (3) allow us to prove the sufficient condition for conjugacy of submaximal $\mathfrak{X}$-subgroups, which, as Theorem 3, was announced in [20]. Recall that if a group G with a subnormal series (1) contains a subgroup H, then the projection of H on $G^i = G_{i-1}/G_i$, $i = 1, \dots, n$, is denoted by H^i.

Corollary 1. *Suppose that a group G possesses a subnormal series* (1) *and $H, K \in \mathrm{sm}_{\mathfrak{X}}(G)$ satisfy*

$$H^i = K^i \text{ for all } i = 1, \dots, n.$$

Then H and K are conjugate in the subgroup $\langle H, K\rangle$.

Since $\mathrm{m}_{\mathfrak{X}}(G) \subseteq \mathrm{sm}_{\mathfrak{X}}(G)$, the same assertion holds for $\mathfrak{X}$-maximal subgroups.

In [4, pp. 30–31], it was noticed that Theorem 3 implies a characterization of submaximal $\mathfrak{X}$-subgroups in direct products:

Corollary 2. *Let $G = G_1 \times \dots \times G_n$ be a direct product of its subgroups G_i, $i = 1, \dots, n$. Then for every complete class $\mathfrak{X}$,*

$$\mathrm{sm}_{\mathfrak{X}}(G) = \{\langle H_1, \dots, H_n\rangle \mid H_i \in \mathrm{sm}_{\mathfrak{X}}(G_i), i = 1, \dots, n\}.$$

This corollary, helpful in inductive arguments, is an analogue of the well-known property of maximal $\mathfrak{X}$-subgroups [4, Proposition 10].

In his talk in 1979, Wielandt posed a problem of reversing Theorem 4 for classes of π-groups [20, p. 171, Problem (i)]: Must a subgroup A be subnormal in G if the order of $N_A(H \cap A)/(H \cap A)$ is not divisible by any number in π for all sets of primes π and all maximal π-subgroups H of G? In 1991, P. Kleidman obtained a positive answer with the help of CFSG [11]. Combining Theorem 4 with Kleidman's result, we come to the following criterion of subnormality.

Corollary 3. *A subgroup A of a group G is subnormal if and only if for every complete class $\mathfrak{X}$ and every maximal $\mathfrak{X}$-subgroup H in G, the quotient $N_A(H \cap A)/(H \cap A)$ contains no nontrivial $\mathfrak{X}$-subgroups.*

To sum up, this article contributes to Wielandt's program of studying maximal $\mathfrak{X}$-subgroups, which Wielandt himself deemed to be a development of the Hölder program. We would also like to mention some recent progress in this direction made in [4], [5], and [6], where, in particular, the authors suggested an inductive algorithm of finding maximal $\mathfrak{X}$-subgroups in a finite group provided all submaximal $\mathfrak{X}$-subgroups in finite simple groups are known for a given class $\mathfrak{X}$.

2 Examples of submaximal but not strongly submaximal $\mathfrak{X}$-subgroups

As mentioned in the introduction, we will find desired examples among almost simple groups. Recall that a group G is *almost simple* with socle S, if

$$S \leq G \leq \mathrm{Aut}(S),$$

where a nonabelian simple group S is identified with the group $\mathrm{Inn}(S)$ of its inner automorphisms.

The following lemma refines the definition of strong submaximality for almost simple groups.

Lemma 1. *Let G be an almost simple group with socle $S \notin \mathfrak{X}$. Then $H \in \mathrm{sm}^{\circ}_{\mathfrak{X}}(G)$ if and only if there exists a group G^* such that $G \trianglelefteq G^* \leq \mathrm{Aut}(S)$ and $H = K \cap G$ for some $K \in \mathrm{m}_{\mathfrak{X}}(G^*)$.*

Proof. The "if" part follows from the definition of a strongly submaximal $\mathfrak{X}$-subgroup.

To show the converse, take a group G^* of the smallest order such that $G \trianglelefteq G^*$ and $H = K \cap G$ for some $K \in \mathrm{m}_{\mathfrak{X}}(G^*)$. It is clear that $G^* = GK$ and, in particular, $G^*/G \in \mathfrak{X}$.

We claim that G^* does not contain any nontrivial normal $\mathfrak{X}$-subgroups. Indeed, suppose that $1 \neq U \in \mathfrak{X}$ is a normal subgroup in G^*. If $G \cap U \neq 1$, then $G \cap U$ being a normal subgroup of G contains the unique minimal normal subgroup S of G, contrary to the assumption that $S \notin \mathfrak{X}$. Therefore, $G \cap U = 1$, so $G \simeq \overline{G} \trianglelefteq \overline{G^*}$, where $^- : G^* \to G^*/U$ denotes the canonical epimorphism. Moreover, $K \in \mathrm{m}_{\mathfrak{X}}(G^*)$ implies $U \leq K$, and hence $\overline{K} \in \mathrm{m}_{\mathfrak{X}}\left(\overline{G^*}\right)$. Dedekind's lemma (see, e.g., [15, Ch. 1, Theorem 3.14]) yields

$$HU = (G \cap K)U = GU \cap K.$$

It follows that $\overline{H} = \overline{G} \cap \overline{K}$, so we arrive at a contradiction with the minimality of G^*.

Since S is a characteristic subgroup of G, both S and its centralizer $C_{G^*}(S)$ are normal in G^*. The group G is almost simple, so $C_{G^*}(S) \cap G = C_G(S) = 1$. Hence $C_{G^*}(S)$ can be isomorphically embedded into an $\mathfrak{X}$-group G^*/G, so it is an $\mathfrak{X}$-group itself. As G^* contains no nontrivial normal $\mathfrak{X}$-subgroups, $C_{G^*}(S) = 1$. Thus, G^* is isomorphic to a subgroup of $\mathrm{Aut}(S)$. □

For the rest of this section, we fix the following notation.

Let S be a simple group $D_{2n}(q) \simeq \mathrm{P}\Omega^+_{4n}(q)$, where q is an odd prime and $n > 2$. Set $A = \mathrm{Aut}(S)$. Denote by $\widehat{S}$ the group of inner-diagonal automorphisms of S, see [2, 7.1, 8.4.7, 12.2] or [3, Definitions 2.5.10 and 1.15]. Then $S \leq \widehat{S} \leq \mathrm{Aut}(S)$.

Let $\Pi = \{r_1, \dots, r_{2n}\}$ be a fundamental root system of type D_{2n}. Its Dynkin diagram is indicated in Fig. 1. The symmetry of the graph in Fig. 1 corresponding to the transposition of roots r_{2n-1} and r_{2n} induces a so-called graph automorphism $\gamma \in A$ of order 2, see [2, Proposition 12.2.3]. Since q is a prime, $A = \langle \widehat{S}, \gamma \rangle$ (see [2, Theorem 12.5.1] or [3, Theorem 2.5.12]). It is known that A/S is isomorphic to the dihedral group of order 8 [3, Theorem 2.5.12(j)] and contains the normal subgroup $\widehat{S}/S$ which is isomorphic to the elementary abelian group of order 4 (see [2, 8.4.7 and 8.6] or [3, Theorem 2.5.12(c)]).

Let P_0, P_1 and P_2 denote parabolic subgroups of S containing the same Borel subgroup and corresponding to the following sets of roots (see [2, 8.2.2

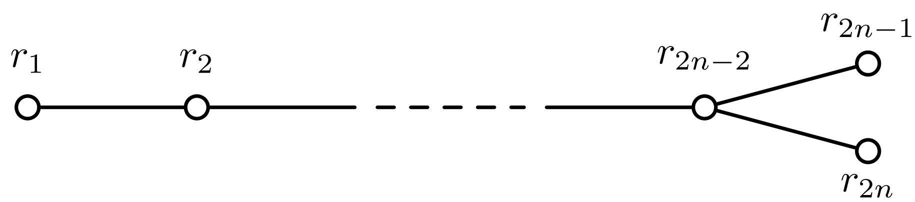

Figure 1: D_{2n} root system

and 8.3]): $\Pi \setminus \{r_{2n-1}, r_{2n}\}$, $\Pi \setminus \{r_{2n-1}\}$ and $\Pi \setminus \{r_{2n}\}$. The next lemma states some properties of these subgroups.

Lemma 2. *In the above notation, the following hold.*

(i) $N_S(P_i) = P_i$ *for every* $i = 0, 1, 2$.

(ii) P_i *is a maximal subgroup of* S *for* $i = 1, 2$, *and if* $P_0 < P < S$, *then* $P \in \{P_1, P_2\}$.

(iii) $P_1^\gamma = P_2$ *and* $P_2^\gamma = P_1$.

(iv) $P_0^\gamma = P_0$.

(v) *There exists an abelian subgroup* $\widehat{T}$ *of* $\widehat{S}$ *such that* $\widehat{S} = \widehat{T}S$ *and* $\widehat{T}$ *normalizes* P_i *for* $i = 0, 1, 2$. *In particular,* $N_{\widehat{S}}(P_i) = \widehat{T}P_i$.

(vi) *Subgroups* P_1 *and* P_2 *are not conjugate in* S.

Proof. See [2, Theorem 8.3.3] for (i) and [2, Theorems 8.3.2 and 8.3.4] for (ii). Statements (iii) and (iv) follow from the definitions of subgroups P_i and automorphism γ, see [2, Proposition 12.2.3]. It follows from [3, Proposition 2.6.9] that the Cartan subgroup $\widehat{T}$ is an abelian subgroup of $\widehat{S}$ such that $\widehat{S} = \widehat{T}S$ and $\widehat{T}$ normalizes P_i for $i = 0, 1, 2$. Now, if $g \in N_{\widehat{S}}(P_i)$ then $g = xy$ for $x \in \widehat{T}$ and $y \in S$. Hence

$$P_i = P_i^g = P_i^{xy} = P_i^y,$$

so $y \in N_S(P_i) = P_i$ by (i). Therefore, $g \in \widehat{T}P_i$ and $N_{\widehat{S}}(P_i) \leq \widehat{T}P_i$. The reverse inclusion is clear, so (v) is proved. Statement (vi) follows from [2, Theorem 8.3.3] or [3, Theorem 2.6.5(c)]. □

Let G denote a subgroup of $\widehat{S}$ containing S such that G/S has order 2 and it is not normal in the dihedral group A/S (there are exactly two such subgroups; they are normal in $\widehat{S}$ and are permuted by γ, see [3, Theorem 2.5.12(j)]). Observe that G is subnormal in A.

It is easily seen that the class $\mathfrak{X}$ of all finite groups with nonabelian composition factors having order less than $|S|$ is complete. Since A/S is solvable, the set $\mathrm{m}_{\mathfrak{X}}(G)$ coincides with the set of subgroups of G maximal among those that do not contain S. The following proposition shows that there are submaximal $\mathfrak{X}$-subgroups which are not strongly submaximal.

Proposition 1. *If $H = N_G(P_0)$, then $H \in \mathrm{sm}_{\mathfrak{X}}(G) \setminus \mathrm{sm}^{\circ}_{\mathfrak{X}}(G)$.*

Proof. First, we prove that $H \in \mathrm{sm}_{\mathfrak{X}}(G)$. It suffices to show that $N_A(P_0)$ is maximal in A, because in this case $N_A(P_0)$ is $\mathfrak{X}$-maximal in A and

$$H = N_G(P_0) = G \cap N_A(P_0) \in \mathrm{sm}_{\mathfrak{X}}(G).$$

If $N_A(P_0)$ is not maximal in A, then there is a subgroup M with $N_A(P_0) < M < A$. By Statements (iv) and (v) of Lemma 2, we have $A = \langle \widehat{S}, \gamma \rangle \leq N_A(P_0)S$. Therefore,

$$N_A(P_0)/(N_A(P_0) \cap S) \simeq N_A(P_0)S/S = A/S = MS/S \simeq M/(M \cap S).$$

Hence, the equality $N_A(P_0) \cap S = M \cap S$ does not hold. By Lemma 2(i),

$$P_0 = N_S(P_0) = N_A(P_0) \cap S < M \cap S.$$

As $M \cap S < S$, Lemma 2(ii) yields $M \cap S = P_i$ for $i = 1$ or $i = 2$. Since $M \cap S \trianglelefteq M$, we obtain

$$M \leq N_A(P_i) \text{ and } A = MS = N_A(P_i)S.$$

This implies that the conjugacy class of P_i is fixed by A, contrary to the fact that $\gamma \in A$ permutes the conjugacy classes of P_1 and P_2 in S (see Statements (iii) and (vi) of Lemma 2). Thus, $N_A(P_0)$ is maximal in A, and we are done.

Let us show that $H \notin \mathrm{sm}^{\circ}_{\mathfrak{X}}(G)$. Suppose the contrary. Then by Lemma 1, there is a subgroup G^* of A such that $G \trianglelefteq G^*$ and $H = K \cap G$ for some $K \in \mathrm{m}_{\mathfrak{X}}(G^*)$, i.e. for some maximal subgroup K of G^* that does not contain S. Since G^* normalizes G, a subgroup G^*/S of the dihedral group A/S lies in the normalizer $N_{A/S}(G/S)$, which is equal to $\widehat{S}/S$. Therefore, $G^* \leq \widehat{S}$.

We claim that $K = N_{G^*}(P_0)$. As K is maximal in G^*, it suffices to show that K normalizes P_0. Indeed, K normalizes $K \cap S$ and, by Lemma 2(i),

$$K \cap S = K \cap G \cap S = H \cap S = N_G(P_0) \cap S = N_S(P_0) = P_0.$$

If $\widehat{T}$ is a subgroup from Lemma 2(v), then for all $i = 0, 1, 2$,

$$N_{\widehat{S}}(P_i) = \widehat{T}P_i.$$

By Dedekind's lemma,

$$N_{G^*}(P_i) = G^* \cap N_{\widehat{S}}(P_i) = G^* \cap \widehat{T}P_i = (G^* \cap \widehat{T})P_i.$$

Since

$$K = N_{G^*}(P_0) = (G^* \cap \widehat{T})P_0 \leq (G^* \cap \widehat{T})P_1 = N_{G^*}(P_1),$$

the maximality of K implies $N_{G^*}(P_0) = N_{G^*}(P_1)$. Lemma 2(ii) yields

$$P_0 = N_S(P_0) = S \cap N_{G^*}(P_0) = S \cap N_{G^*}(P_1) = N_S(P_1) = P_1,$$

contrary to the fact that $P_0 \neq P_1$. □

3 The proof of Wielandt-Hartley's theorem for submaximal $\mathfrak{X}$-subgroups

Given a group G, the set of prime divisors of the order of G is denoted by $\pi(G)$. If π is an arbitrary set of primes, then G is called a *π-group* provided $\pi(G) \subseteq \pi$, and a *π'-group* whenever $\pi(G) \cap \pi = \varnothing$.

Given a class $\mathfrak{X}$, set $\pi(\mathfrak{X}) = \bigcup_{X \in \mathfrak{X}} \pi(X)$. Denote by $O_{\mathfrak{X}}(G)$ the largest normal $\mathfrak{X}$-subgroup of G. If $\mathfrak{X}$ is a class of p-groups, π-groups or π'-groups, then we denote the subgroup $O_{\mathfrak{X}}(G)$ by $O_p(G)$, $O_\pi(G)$ or $O_{\pi'}(G)$ respectively.

Observe that for every $p \in \pi(\mathfrak{X})$ the group of order p lies in $\mathfrak{X}$. Therefore, a group contains no nontrivial $\mathfrak{X}$-subgroups if and only if it is a π'-group for $\pi = \pi(\mathfrak{X})$. Thus, Theorem 4 and hence Theorem 3 are equivalent to the following proposition.

Proposition 2. *Let $\mathfrak{X}$ be a complete class of finite groups and $\pi = \pi(\mathfrak{X})$. Let G be a finite group and let $H \in \mathrm{m}_{\mathfrak{X}}(G)$. If A is subnormal in G, then $N_A(H \cap A)/(H \cap A)$ is a π'-group.*

Proof. Suppose the contrary, and let G be a counterexample of minimal order. If $A = 1$ or $A = G$, then we immediately get a contradiction, so $1 < A < G$ and, in particular, G is not a simple group.

Lemma 3. $O_{\mathfrak{X}}(G) = 1$ *and* $O_{\pi'}(G) = 1$. *In particular, G contains no abelian minimal subnormal subgroups.*

Proof. Suppose that $K = O_{\mathfrak{X}}(G) > 1$. Let $\overline{} : G \to G/K$ denote the canonical epimorphism. Clearly, $K \leq H$, so $\overline{H}$ is a maximal $\mathfrak{X}$-subgroup of $\overline{G}$. Therefore,

$$N_{\overline{AK}}(\overline{AK} \cap \overline{H})/(\overline{AK} \cap \overline{H}) \simeq N_{AK}(AK \cap H)/(AK \cap H),$$

where by the minimality of G, the quotient on the left-hand side is a π'-group. In virtue of Dedekind's lemma,

$$N_{AK}(AK \cap H)/(AK \cap H) = N_{AK}((A \cap H)K)/((A \cap H)K).$$

Since K is normal in G, we have $KN_A(A \cap H) \leq N_{AK}((A \cap H)K)$. Thus,

$$N_{AK}((A \cap H)K)/((A \cap H)K) \geq KN_A(A \cap H)/((A \cap H)K),$$

and the group on the right-hand side is also a π'-group. Its order is equal to

$$|KN_A(A \cap H) : K(A \cap H)| = \frac{|K||N_A(A \cap H)||A \cap H \cap K|}{|K \cap N_A(A \cap H)||K||A \cap H|} =$$
$$= |N_A(A \cap H) : A \cap H||A \cap K : N_{A \cap K}(A \cap H)|.$$

Since the second factor is an integer, it follows that $N_A(A \cap H)/(A \cap H)$ is a π'-group, contrary to the choice of G. Therefore, $O_{\mathfrak{X}}(G) = 1$.

Suppose that $O_{\pi'}(G) > 1$, and denote by K a minimal normal subgroup of G contained in $O_{\pi'}(G)$. Then $K > 1$ and K normalizes A by [9, Theorem 2.6].

Let P be a Sylow p-subgroup of AK for some $p \in \pi$. The subgroup A is normal in AK and K is a π'-group, so $P \leq A$. Since this holds for every $p \in \pi$, we derive that all Sylow p-subgroups and hence all π-subgroups of AK lie in A. Therefore, $AK \cap H = A \cap H$. As K is normal in G, we have

$$N_A(A \cap H) \leq N_{AK}((A \cap H)K) = N_{AK}((AK \cap H)K) = N_{AK}(AK \cap HK),$$

where the last equality holds by Dedekind's lemma.

Let $\overline{} : G \to G/K$ be the canonical epimorphism. In view of [4, Proposition 4], $\overline{H}$ is a maximal $\mathfrak{X}$-subgroup of $\overline{G}$. By the minimality of G, the quotient $N_{\overline{A}}(\overline{A} \cap \overline{H})/(\overline{A} \cap \overline{H})$ is a π'-group. Since $KN_A(A \cap H)/(K(A \cap H))$ is a subgroup of

$$N_{AK}(AK \cap HK)/(AK \cap HK) \simeq N_{\overline{A}}(\overline{A} \cap \overline{H})/(\overline{A} \cap \overline{H}),$$

$KN_A(A \cap H)/K(A \cap H)$ is a π'-group. As $K \cap H = 1$,

$$|KN_A(A \cap H) : K(A \cap H)||K \cap N_A(A \cap H)| = |N_A(A \cap H) : A \cap H|,$$

and hence $N_A(A \cap H)/(A \cap H)$ is also a π'-group, a contradiction. Thus, $O_{\pi'}(G) = 1$.

If p is a prime, then $O_p(G)$ lies either in $O_{\pi'}(G)$ or $O_{\mathfrak{X}}(G)$. It follows that $O_p(G) = 1$ for every prime p. As a consequence, all minimal subnormal subgroups of G are nonabelian. □

Lemma 4. *If S is a minimal subnormal subgroup of G and $S \nleq A$, then $[S, A] = 1$.*

Proof. It follows from [9, Lemma 9.17] that the normal closure $M = S^G$ of S in G is a minimal normal subgroup of G. By [9, Theorem 2.6], M normalizes A, so A is normal in $N = \langle M, A \rangle$. The normal closure S^N of S in N is a minimal normal subgroup of N and does not lie in A. Hence $S^N \cap A = 1$ and $[S, A] \leq [S^N, A] = 1$, as required. □

Lemma 5. *Let $M = S_1 \times \cdots \times S_n$ be a subnormal subgroup of G, where S_i, $i = 1, \ldots, n$, are simple groups. Then*

$$M \cap H = (S_1 \cap H) \times \cdots \times (S_n \cap H).$$

Proof. Note that $M \cap H \geq S_i \cap H$ for all $i = 1, \ldots, n$, so

$$M \cap H \geq (S_1 \cap H) \ldots (S_n \cap H).$$

By [4, Proposition 10], there are $L_i \in \mathrm{sm}_{\mathfrak{X}}(S_i)$, $i = 1, \ldots, n$, such that $M \cap H = L_1 \times \cdots \times L_n$. Clearly $L_i \leq S_i \cap H$ for $i = 1, \ldots, n$. It follows that

$$M \cap H \leq (S_1 \cap H) \ldots (S_n \cap H),$$

and we are done. □

We proceed with the proof of Proposition 2. Set $X = \langle N_A(A \cap H), H\rangle$ and $L = A \cap X$. Observe that H is a maximal $\mathfrak{X}$-subgroup of X, L is subnormal in X, and $L \cap H = A \cap H$. Furthermore,

$$N_L(L \cap H) = N_{A\cap X}(A \cap H) = N_A(A \cap H).$$

If $X < G$, then by the minimality of G the quotient $N_L(L \cap H)/(L \cap H)$ is a π'-group. It follows that $N_A(A\cap H)/(A\cap H)$ is also a π'-group, a contradiction. Hence we may assume that $G = \langle N_A(A \cap H), H\rangle$.

Let M be a minimal normal subgroup of G. Then $M = S_1 \times \cdots \times S_n$, where S_i, $i = 1, \ldots, n$, are nonabelian simple groups. Since $A \cap M$ is subnormal in M, it is a (possibly empty) product of some S_i, $i = 1, \ldots, n$. Without loss of generality, we may assume that $A \cap M = S_1 \times \cdots \times S_k$, $0 \leq k \leq n$. Applying Lemma 5 to M and $A \cap M$, we obtain

$$M \cap H = (S_1 \cap H) \times \cdots \times (S_n \cap H) \text{ and } A \cap M \cap H = (S_1 \cap H) \times \cdots \times (S_k \cap H).$$

Since $M = (A \cap M) \times S_{k+1} \times \cdots \times S_n$, it follows that

$$M \cap H = (A \cap M \cap H) \times (S_{k+1} \cap H) \times \cdots \times (S_n \cap H).$$

Clearly, $N_A(A \cap H)$ normalizes $A \cap M \cap H$. For every $i \in \{k + 1, \ldots, n\}$, the subgroup A centralizes S_i due to Lemma 4, so $N_A(A \cap H)$ centralizes $S_i \cap H$. Consequently, $N_A(A\cap H)$ normalizes all factors constituting $M \cap H$, and therefore it normalizes $M \cap H$ itself. Obviously, H also normalizes $M \cap H$. Since $G = \langle N_A(A \cap H), H\rangle$, we derive that $M \cap H$ is normal in G.

Now, $M \cap H$ is a normal $\mathfrak{X}$-subgroup of G, so $M \cap H = 1$ by Lemma 3. It follows from [4, Proposition 8] that M is a π'-group, hence $M = 1$ again by Lemma 3. Thus, G must be a simple group which gives us a final contradiction. □

4 Applications

In this section we prove Corollaries 1–3 from Introduction and thus show how one can apply the strong version of Wielandt-Hartley's theorem.

Given a complete class $\mathfrak{X}$, a group G is said to be *$\mathfrak{X}$-separable*, if G possesses a subnormal series (1) such that each factor G^i either belongs to $\mathfrak{X}$ or contains no nontrivial $\mathfrak{X}$-subgroups. Clearly, a subgroup of an $\mathfrak{X}$-separable group is also $\mathfrak{X}$-separable.

Lemma 6. [19, 12.10] *Let G be an $\mathfrak{X}$-separable group. Then all maximal $\mathfrak{X}$-subgroups of G are conjugate.*

We note that Lemma 6 is equivalent to Chunikhin's lemma on π-separable groups [16, Ch. 5, Theorem 3.7].

Proof of Corollary 1. We use induction on the length n of a series (1). Base $n = 0$ is clear.

Property (3) yields $H\cap G_1 \in \mathrm{sm}_{\mathfrak{X}}(G_1)$ and $K\cap G_1 \in \mathrm{sm}_{\mathfrak{X}}(G_1)$. By induction hypothesis, subgroups $H\cap G_1$ and $K\cap G_1$ are conjugate in $\langle H\cap G_1, K\cap G_1\rangle \leq \langle H,K\rangle\cap G_1$. Without loss of generality, we may assume that $H\cap G_1 = K\cap G_1$. Equality $H^1 = K^1$ implies $HG_1 = KG_1$. Set

$$G^* = HG_1 = KG_1 \quad \text{and} \quad T = H\cap G_1 = K\cap G_1.$$

Then $H, K \leq N_{G^*}(T)$ and $N_{G^*}(T) = HN_{G_1}(T)$. Moreover,

$$N_{G_1}(T) = G_1 \cap N_{G^*}(T) \trianglelefteq N_{G^*}(T) = HN_{G_1}(T).$$

Therefore, $N_{G^*}(T)/N_{G_1}(T)$ is isomorphic to a quotient group of H, so it lies in $\mathfrak{X}$. Next, $T = H\cap G_1 \in \mathrm{sm}_{\mathfrak{X}}(G_1)$, and Theorem 3 yields that the group $N_{G_1}(T)/T$ contains no nontrivial $\mathfrak{X}$-subgroups. Now, $N_{G^*}(T)$ is $\mathfrak{X}$-separable because it possesses a (sub)normal series

$$N_{G^*}(T) \geq N_{G_1}(T) \geq T \geq 1,$$

where every factor either lies in $\mathfrak{X}$ or contains no nontrivial $\mathfrak{X}$-subgroups. Projections of H and K on factors of that series are maximal $\mathfrak{X}$-subgroups in respective factors, hence $H, K \in \mathrm{m}_{\mathfrak{X}}(N_{G^*}(T))$. A subgroup $J = \langle H, K\rangle$ of $N_{G^*}(T)$ is also $\mathfrak{X}$-separable, and $H, K \in \mathrm{m}_{\mathfrak{X}}(J)$. By Lemma 6, H and K are conjugate in J.

Inclusion (2) and Corollary 1 immediately imply

Corollary 4. *Suppose that the group G possesses a subnormal series* (1) *and $H, K \in \mathrm{m}_{\mathfrak{X}}(G)$ are such that*

$$H^i = K^i \text{ for all } i = 1, \dots, n.$$

Then H and K are conjugate in the subgroup $\langle H, K\rangle$.

The classical Kaloujnine-Krasner theorem [10] provides a source of groups G possessing subgroups H and K such that the projections H^i and K^i on the sections G^i of a subnormal series (1) coincide, while H and K are not even isomorphic. In contrast, Corollaries 1 and 4 mean that, up to conjugation, every submaximal and, in particular, every maximal $\mathfrak{X}$-subgroup is uniquely determined by its projections on the sections of a subnormal series. In the case where all terms of a series (1) are normal in G, Corollary 4 is proved in [19, Hauptsatz 14.1] and in [16, Ch. 5, (3.21) and (3.21)′]. To realize the effectiveness of the notion of submaximal $\mathfrak{X}$-subgroups and the power of inductive property (3), one can compare the proof of Corollary 1 with the proof of a weaker statement [16, Ch. 5, 3.21]. Property (3) and Theorem 3 allow us to use the induction hypothesis straightaway, which makes the reasoning much simpler and shorter.

Proof of Corollary 2. First, let $H \in \mathrm{sm}_{\mathfrak{X}}(G)$. Property (3) yields $H\cap G_i \in \mathrm{sm}_{\mathfrak{X}}(G_i)$ for all $i = 1, \dots, n$. Let $\rho_i : G \to G_i$ be the coordinate projection mapping and $H_i = H^{\rho_i}$. Since $H\cap G_i \trianglelefteq H$ and ρ_i acts identically on G_i,

$$H\cap G_i = (H\cap G_i)^{\rho_i} \trianglelefteq H^{\rho_i} = H_i.$$

Therefore, $H_i \le N_{G_i}(H \cap G_i)$. Theorem 3 implies that $N_{G_i}(H \cap G_i)/(H \cap G_i)$ contains no nontrivial $\mathfrak{X}$-subgroups. Consequently, the image of H_i in $N_{G_i}(H \cap G_i)/(H \cap G_i)$ is trivial and $H_i = H \cap G_i$. It follows that

$$\langle H \cap G_1, \dots, H \cap G_n \rangle \le H \le \langle H_1, \dots, H_n \rangle = \langle H \cap G_1, \dots, H \cap G_n \rangle,$$

and $H = \langle H \cap G_1, \dots, H \cap G_n \rangle$, as desired.

To prove the converse, take arbitrary $H_i \in \mathrm{sm}_{\mathfrak{X}}(G_i)$ for all $i = 1, \dots, n$. We may assume that for every i a group G_i^* exists, in which G_i is subnormal and $H_i = K_i \cap G_i$ for a suitable $K_i \in \mathrm{m}_{\mathfrak{X}}(G_i^*)$. It is easy to see that

$$K = \langle K_1, \dots, K_n \rangle = K_1 \times \cdots \times K_n \in \mathrm{m}_{\mathfrak{X}}(G^*), \text{ where } G^* = G_1^* \times \cdots \times G_n^*.$$

Moreover, $H_i = K_i \cap G_i \trianglelefteq\trianglelefteq K_i \trianglelefteq K$, whence $H_i \trianglelefteq\trianglelefteq K \cap G$ and

$$H_i = H_i^{\rho_i} \trianglelefteq\trianglelefteq (K \cap G)^{\rho_i},$$

where again $\rho_i : G^* \to G_i^*$ is the coordinate projection mapping. Theorem 3 implies that $N_{(K\cap G)^{\rho_i}}(H_i) = H_i$, which is possible only if $H_i = (K \cap G)^{\rho_i}$. Thus,

$$\langle H_i \mid i = 1, \dots, n \rangle \le K \cap G \le \langle (K \cap G)^{\rho_i} \mid i = 1, \dots, n \rangle = \langle H_i \mid i = 1, \dots, n \rangle.$$

In view of $K \in \mathrm{m}_{\mathfrak{X}}(G^*)$ and $G \trianglelefteq\trianglelefteq G^*$, we obtain

$$\langle H_i \mid i = 1, \dots, n \rangle = K \cap G \in \mathrm{sm}_{\mathfrak{X}}(G),$$

and this completes the proof. □

Proof of Corollary 3. It suffices to establish the following

Proposition 3. *Let A be a subgroup of a finite group G. Then the following statements are equivalent.*

(i) $A \trianglelefteq\trianglelefteq G$.

(ii) $N_A(H \cap A)/(H \cap A)$ *contains no nontrivial $\mathfrak{X}$-subgroups for every complete class $\mathfrak{X}$ and every $H \in \mathrm{m}_{\mathfrak{X}}(G)$.*

(iii) $H \cap A$ *is a Sylow p-subgroup of A for all primes p and every Sylow p-subgroup H of G.*

Proof. (i)⇒(ii) is the statement of Theorem 4.
(ii)⇒(iii). Let p be a prime and let H be a Sylow p-subgroup of G. Applying (ii) to the situation where $\mathfrak{X}$ is the class of all p-groups, we obtain that $H \cap A$ is a Sylow p-subgroup of $N_A(H \cap A)$. Now it follows from the well-known consequence of the Sylow theorem (see, e.g., [15, Ch. 2, (2.5)]) that $H \cap A$ is a Sylow p-subgroup of A.
(iii)⇒(i) is the main result of [11]. □

5 Concluding remarks

Wielandt-Hartley's theorem for normal subgroups (Theorem 1) is an invaluable tool for studying subgroup structure of finite groups. It can be found in Suzuki's classic book [16, Ch. 5, (3.20) and (3.20)′]. Although Wielandt proved this theorem in his lectures [19] delivered at Tübingen in 1963–64, various particular cases of that result were independently proved by different authors without mentioning Wielandt. One of the reasons was that the lectures were first published only in 1994, when the collection of Wielandt's mathematical works appeared.

The first published proof (1971) of that theorem is by Hartley [8, Lemmas 2 and 3]. It was obtained for the case when $\mathfrak{X}$ is a class of π-groups. A similar result was proved by Shemetkov [13] in 1972. Both Hartley and Shemetkov used this version of the theorem as a technical instrument for studying the well-known D_π-problem: Is it true that the class of groups with all maximal π-subgroups being conjugate is closed under extensions? This problem was posed by Wielandt at the XIII International Congress of Mathematicians in Edinburgh in 1958 [18] and traces back to P. Hall's theorem [7, Theorem D5].

Another special case of Theorem 1 can be obtained by fixing a nonabelian simple group S and considering the complete class $\mathfrak{X}$ of all finite groups with all composition factors having order less than $|S|$. Then given an almost simple group G with socle S, maximal $\mathfrak{X}$-subgroups are exactly maximal subgroups of G not containing S. Wielandt-Hartley's theorem for such a group G implies the following statement: *If G is a finite almost simple group with socle S and M is a maximal subgroup of G, then* $S \cap M \neq 1$. It was proved by R. A. Wilson in [21] while studying novel subgroups in almost simple groups (see also [1, Section 1.3.1]); and by M. W. Liebeck, C. E. Praeger and J. Saxl in course of the proof of the O'Nan-Scott theorem for primitive permutation groups [12, pp. 395–396].

As mentioned in Introduction, Wielandt-Hartley's theorem for submaximal $\mathfrak{X}$-subgroups (Theorem 3) was announced by Wielandt at Santa Cruz conference on finite groups in 1979 [20, 5.4(a)]. At this meeting, one of the most important in the history of the classification of finite simple groups, Wielandt gave a talk entitled "Zusammengesetzte Gruppen: Hölder Programm heute." Concerning the subject, Wielandt anticipated (see [20, p. 171]) that the theorem about submaximal $\mathfrak{X}$-subgroups (in the sense of Definition 2 instead of Definition 1) would be harder to prove and admitted that this proof had not been already written in details. The present article provides the proof and explains why Theorem 3 is indeed stronger than Theorem 2: because submaximal $\mathfrak{X}$-subgroups are not always strongly submaximal.

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