1. Can a group G be a union of its
    - two proper subgroups?
    - three proper subgroups?

2. Let H and K be finite subgroups of G and let $g \in G$. Prove that
$$|HgK| = \frac{|H| \cdot |K|}{|K \cap (g^{-1}Hg)|}.$$

3. Let A and B be finite subsets of G and $e \in B$, or $e \in A$. Prove that $ABA \subseteq A$ if and only if A is a subgroup of G and B is a subset of A.

4. A permutation σ on the set of symbols $\Omega = \{1, \ldots, n\}$ is called a *cycle* if there exist $i_1, \ldots, i_k \in \Omega$ such that
$$i_1\sigma = i_2,\ i_2\sigma = i_3,\ \ldots,\ i_k\sigma = i_1,$$
and $j\sigma = j$ for every $j \in \Omega \setminus \{i_1, \ldots, i_k\}$. In this case, the set $\{i_1, \ldots, i_k\}$ is called the *orbit* of σ, and we write $\sigma = (i_1, i_2, \ldots, i_k)$. Two cycles σ_1 and σ_2 are called *independent* if the intersection of their orbits is the empty set. If the cycles σ_1 and σ_2 are independent then $\sigma_1\sigma_2 = \sigma_2\sigma_1$. Every permutation σ can be expressed as a product of independent cycles. This expression is unique up to the order of the factors. A cycle of length 2 is called a *transposition*. The *decrement* of a permutation σ is the difference between the number of permutable symbols in Ω and the number of independent cycles of σ. The decrement of σ is denoted by $d(\sigma)$. Prove that every permutation σ can be decomposed into a product of transpositions, and the minimal possible number of transpositions in such a decomposition is equal to $d(\sigma)$.

5. Prove that, for every permutation σ, there exist permutations σ_1 and σ_2 such that $\sigma = \sigma_1\sigma_2$ and $\sigma_1^2 = \sigma_2^2 = 1$.

6. In the symmetric group S_n find a permutation χ such that $\chi^2 = (1, 2, \ldots, n)$. Prove that such a permutation is unique (if it exists).

Additional problems.

1. Let G be a (not necessary finite) group such that there exist subgroups $H_1, \ldots, H_s \leq G$ and elements $x_1, \ldots, x_s \in G$ with
$$G = H_1x_1 \cup \ldots \cup H_rx_r \cup x_{r+1}H_{r+1} \cup \ldots \cup x_sH_s.$$
Prove that there exists i such that $G = H_iy_1 \cup \ldots \cup H_iy_n$.

2. Let x, y be elements of a (not necessary finite) group G satisfying identities $1 = x^3 = y^3 = (xy)^3 = (xy^{-1})^3$. Prove that x, y lie in a subgroup H of order at most 27.

3. Assume that p is a prime and H is a subgroup of S_p satisfying
    (a) for every $i, j \in \{1, \ldots, p\}$ there exists $h \in H$ such that $ih = j$;
    (b) H contains a transposition.

   Prove that $H = S_p$.