1. Prove that a subgroup of index 2 is normal.

2. If A, B are normal subgroups of G, then

   (a) $A \cap B$ is a normal subgroup of G;

   (b) AB is a normal subgroup of G.

3. Prove that the factor group $\mathbb{Z}/n\mathbb{Z}$ is a cyclic group of order n.

4. Denote by $\mathbb{C}^*$ the multiplicative group of the field of complex numbers $\mathbb{C}$. Let $\mathbb{R}_{>0}$ be the group of all positive real numbers under multiplication and $U = \{z \in \mathbb{C} \mid z \cdot \bar{z} = 1\}$ be the group of all complex numbers of module 1 under multiplication. Construct surjective homomorphisms from $\mathbb{C}^*$ onto $\mathbb{R}_{>0}$ and onto U, and find the kernels of these homomorphisms.

5. Find all normal subgroups and all (up to isomorphism) homomorphic images of symmetric groups S_3 and S_4.

6. The set $Z(G) = \{z \in G \mid zg = gz \text{ for all } g \in G\}$ is called the *center* of a group G. Prove that

   (a) $Z(G)$ is a characteristic subgroup of G;

   (b) if $G/Z(G)$ is cyclic, then G is abelian.

7. For a subset M of G, set $M^G := \{m^g \mid m \in M, g \in G\}$. Suppose that M is invariant, i. e., $M = M^G$. Prove that $\langle M^G \rangle$ is normal in G.

8. Prove that the Frattini subgroup $\Phi(G)$ is a normal and, moreover, a characteristic subgroup of G.

9. Let H be a normal subgroup of G and $x \in G$ is such that $(|x|, |G : H|) = 1$. Prove that $x \in H$.

Additional problems

1. Let $A \leq B \leq G$.

   (a) If A is characteristic in B and B is characteristic in G, then A is characteristic in G.

   (b) If A is characteristic in B and $B \trianglelefteq G$, then $A \trianglelefteq G$.

   (c) The condition $A \trianglelefteq B \trianglelefteq G$ does not imply that $A \trianglelefteq G$ (consider the dihedral group of order 8 or the alternating group A_4 for examples).

2. In the dihedral group G of order 8 find subgroups A, B such that A is normal in G, B is normal in A, and B is not normal in G.

3. Find all normal subgroups of a dihedral group.