1. Prove that the sign map $sgn : S_n \to \langle \{1, -1\}, \cdot \rangle$ is a group homomorphism.

2. Prove that the determinant map $\det : GL_n(F) \to F^*$ is a group homomorphism, where F is a field and F^* is its multiplicative group.

3. Prove that every cyclic subgroup of a homomorphic image of a group G is the image of a cyclic subgroup of G.

Additional problems.

1. Find $Z(S_n)$, $Z(GL_n(F))$, $Z(SL_n(F))$, $Z(D_{2n})$, where D_{2n} is a dihedral group of order $2n$.

2. Let $H \trianglelefteq G$ and let $K_1, K_2 \leq G$. Suppose that K_1H/H is conjugate with K_2H/H in G/H. Prove that K_1H is conjugate with K_2H in G.

3. Assume that $G \leq S_4$ is generated by (1234) and (24). Prove that

   - $G \cong D_8$, a dihedral group of order 8;
   - G is isomorphic to the group of all symmetries of a square;
   - $Z(G) = \langle (13)(24) \rangle$;
   - every element of $G/Z(G)$ is an involution, in particular $G/Z(G)$ is abelian;
   - G does not posses an abelian subgroup A such that $AZ(G)/Z(G) = G/Z(G)$ (cf. Problem 5).