1. Let G be a finite Abelian group. Prove that

   (a) if $|G| \neq 1$ then G has an element of prime order;

   (b) for every prime divisor r of $|G|$, there exists an element $g \in G$ of order r;

   (c) if m divides $|G|$ then G has a subgroup of order m;

   (d) if $|G| = mn$, where $(m, n) = 1$, then G has exactly one subgroup of order m;

   (e) if $|G| = mn$, where $(m, n) = 1$, A is the subgroup of order m of G, and m_1 divides m, then every subgroup of G of order m_1 is contained in A.

2. Let $\mathbb{Q}$ be the additive group of the field of rationals.

   (a) Is there a nontrivial group homomorphism $\varphi : \mathbb{Q} \to G$ such that $\mathbb{Q}\varphi$ is finite?

   (b) Is $\mathbb{Q} = A + B$ for some proper subgroups A and B of $\mathbb{Q}$?

   (c) Does $\mathbb{Q}$ contain a maximal subgroup?

   (d) Prove that, for every positive integer n, the factor group $\mathbb{Q}/\mathbb{Z}$ contains exactly one subgroup of order n.

   (e) Does there exist a surjective homomorphism $\mathbb{Q} \to \mathbb{Z}$?

Additional problems.

1. Prove that

$$Q_8 = \left\{ \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\},$$

   where $i^2 = -1$, is a subgroup of $SL_2(\mathbb{C})$. Prove that Q_8 is not Abelian. Are Q_8 and D_8 isomorphic? Find all normal subgroups of Q_8. The group Q_8 is called the *quaternion group*.

2. Let F be a field of finite order q. Prove that the minimal p such that

$$\underbrace{1 + 1 + \ldots + 1}_{p \text{ times}} = 0$$

   is prime. Prove that F has a subfield isomorphic to $\mathbb{Z}_p$. Deduce that $q = p^n$ for some positive integer n.

3. Let F be a finite field of order q. The groups $GL_n(F)$ and $SL_n(F)$ are usually denoted by $GL_n(q)$ and $SL_n(q)$, respectively.

   (a) Find $|GL_2(q)|$, $|SL_2(q)|$, $|GL_2(q)/Z(GL_2(q))|$, $|SL_2(q)/Z(SL_2(q))|$.

   (b) Prove that $GL_2(2) \simeq S_3$, $GL_2(3)/Z(GL_2(3)) \simeq S_4$, $SL_2(3)/Z(SL_2(3)) \simeq A_4$.

   (c) Find $|GL_n(q)|$, $|SL_n(q)|$, $|GL_n(q)/Z(GL_n(q))|$, $|SL_n(q)/Z(SL_n(q))|$.

4. Let A and B be subsets of a group G and let $g \in G$. Prove that

(a) $(A \cup B)^g = A^g \cup B^g$;

(b) $(A \cap B)^g = A^g \cap B^g$;

(c) $(A \setminus B)^g = A^g \setminus B^g$;

(d) $(AB)^g = A^g B^g$;

(e) $(A^{-1})^g = (A^g)^{-1}$;

(f) $\langle A \rangle^g = \langle A^g \rangle$;

(g) $(N_B(A))^g = N_{B^g}(A^g)$;

(h) $(C_B(A))^g = C_{B^g}(A^g)$;

(i) if $A \leq B \leq G$ then $|B^g : A^g| = |B : A|$.