1. Given a group G and subgroups $M_i \trianglelefteq G$ for $1 \leqslant i \leqslant r$, prove that the following statements are equivalent:

   (a) every element $g \in G$ can be uniquely written in the form $g = m_1 m_2 \ldots m_r$ with $m_i \in M_i$;

   (b) the identity element $1 \in G$ can be uniquely written in the form $1 = m_1 m_2 \ldots m_r$ with $m_i \in M_i$;

   (c) some element $g \in G$ can be uniquely written in the form $g = m_1 m_2 \ldots m_r$ with $m_i \in M_i$.

2. Suppose that $G = HK$, where H and K are subgroups of G. Show that also $G = H^x K^y$ for all elements $x, y \in G$. Deduce that if $G = HH^x$ for a subgroup H and an element $x \in G$ then $H = G$.

3. Let H be a subgroup of prime index p in a finite group G. Suppose that no prime smaller than p divides $|G|$. Prove that $H \trianglelefteq G$.

Additional problems.

1. If $G = H^{x_1} \ldots H^{x_k}$ for a subgroup H and $H^{x_i} H^{x_j} = H^{x_j} H^{x_i}$ for all i, j, then $H = G$.