Brauer's theorem.

1. Let G be a dihedral group of order $2n$ (see Seminar 3, Problem 3), where n is odd. Prove that all involutions of G are conjugate. Is the same true for n even?

2. Let G be a finite group and let $x \in G$ be an arbitrary element. Denote by $C_G(x) = \{g \in G \mid x^g = x\}$ the centralizer of x in G and by $x^G = \{x^g \mid g \in G\}$ the conjugacy class of x in G. Prove that
$$|C_G(x)| \cdot |x^G| = |G|.$$

3. Let s, t be two involutions in a finite group G. Prove that either s and t are conjugate in G or there exists an involution $z \in G$ commuting with both s and t.

4. Given a group G, define a graph Γ as follows: the vertices of Γ are the involutions of G. Two involutions are joined by an edge if and only if they commute. Prove that the distance between two nonconjugate involutions in Γ is at most 2.

5. Denote by c the maximum of the orders of centralizers of involution in a group G. Prove that if G contains two nonconjugate involutions then $|G| < c^3$.