1. Prove that

   (a) A_5 is a simple group of order 60;

   (b) a nonabelian group of order less than 60 cannot be simple;

   (c) all simple groups of order 60 are isomorphic.

2. (Note: problems 2–6 give a description of Sylow subgroups in symmetric groups.) Given integers t and p, define $t_p = p^\alpha$ if p^α divides t and $p^{\alpha+1}$ does not divide t. Assume that n is natural and p is prime. Prove that

$$(n!)_p = p^{\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+\dots}.$$

3. Consider the cyclic subgroup $Z = \langle(1, 2, \dots, p)\rangle$ of S_p. Prove that

   (a) $|S_p : Z|_p = 1$, so $Z \in \mathrm{Syl}_p(S_p)$;

   (b) $C_{S_p}(Z) = Z$.

4. Consider the cyclic subgroup

$$Z = \langle(1, 2, \dots, p)(p + 1, p + 2, \dots, 2p) \dots ((p - 1)p + 1,(p - 1)p + 2, \dots, p^2)\rangle$$

   of S_{p^2}. Let $C = C_{S_{p^2}}(Z)$. Denote by X the set

$$\big\{\{1, 2, \dots, p\}, \{p + 1, p + 2, \dots, 2p\}, \dots, \{(p - 1)p + 1,(p - 1)p + 2, \dots, p^2\}\big\}$$

   of subsets of $\{1, \dots, p^2\}$. Prove that:

   (a) C permutes the elements of X. In particular, there exists a homomorphism $\varphi : C \to S(X) \simeq S_p$;

   (b) $\ker \varphi = A = \langle(1, 2, \dots, p),(p+1, p+2, \dots, 2p), \dots,((p-1)p+1,(p-1)p+2, \dots, p^2)\rangle$ is abelian;

   (c) there exists a subgroup B of C such that $B \simeq S_p$ and $B \cap A = \{e\}$;

   (d) $C\varphi = B\varphi = S(X) \simeq S_p$. In particular φ is surjective and $C = AB = A \rtimes B$;

   (e) $|S_{p^2} : C|_p = 1$. In particular, for every Sylow p-subgroup P of B, the product AP is a Sylow p-subgroup of S_{p^2}.

5. Consider the cyclic subgroup

$$Z = \langle(1, 2, \dots, p)(p + 1, p + 2, \dots, 2p) \dots ((p - 1)p^{n-1} + 1,(p - 1)p^{n-1} + 2, \dots, p^n)\rangle$$

   of S_{p^n}. Let $C = C_{S_{p^n}}(Z)$. Denote by X the set

$$\big\{\{1, 2, \dots, p\}, \{p + 1, p + 2, \dots, 2p\}, \dots, \{(p - 1)p^{n-1} + 1,(p - 1)p^{n-1} + 2, \dots, p^n\}\big\}$$

   of subsets of $\{1, \dots, p^n\}$. Prove that:

   (a) C permutes the elements of X. In particular, there exists a homomorphism $\varphi : C \to S(X) \simeq S_{p^{n-1}}$;

(b) $\ker\varphi = A = \langle (1,2,\ldots,p), (p+1,p+2,\ldots,2p), \ldots, ((p-1)p^{n-1}+1, (p-1)p^{n-1}+2,\ldots,p^n)\rangle$ is abelian;

(c) there exists a subgroup B of C such that $B \simeq S_{p^{n-1}}$ and $B \cap A = \{e\}$;

(d) $C\varphi = B\varphi = S(X) \simeq S_{p^{n-1}}$. In particular, φ is surjective and $C = AB = A \rtimes B$;

(e) $|S_{p^n} : C|_p = 1$. In particular, for every Sylow p-subgroup P of B, the product AP is a Sylow p-subgroup of S_{p^n}.

Thus, a Sylow p-subgroup P of S_{p^n} can be written as $P = A_{n-1}A_{n-2}\ldots A_1$, where A_i is elementary abelian of order p^i and, for every i, the product $A_{n-1}A_{n-2}\ldots A_i$ is a normal subgroup of P.

6. Write

$$n = \alpha_0 + \alpha_1 p + \ldots + \alpha_k p^k,$$

where $0 \leqslant \alpha_i < p$ for all i. (This is called the *p-adic expansion* of n.) Consider a subgroup

$$L = \underbrace{S_p \times \ldots \times S_p}_{\alpha_1 \text{ times}} \times \underbrace{S_{p^2} \times \ldots \times S_{p^2}}_{\alpha_2 \text{ times}} \times \ldots \times \underbrace{S_{p^k} \times \ldots \times S_{p^k}}_{\alpha_k \text{ times}}$$

of S_n. Prove that $|S_n : L|_p = 1$. Deduce the structure of a Sylow p-subgroup of S_n.

7. Find all Sylow subgroups of

(a) S_5;

(b) S_{10}.