1. (The Frattini Argument) Assume that G acts transitively on a set X and H is a transitive subgroup of G. Denote by G_x the stabilizer in G of $x \in X$. Prove that $G = HG_x$.

2. If G acts on a set X, then the Frattini subgroup $\Phi(G)$ does not act transitively on X.

3. If $P \in \mathrm{Syl}_p(\Phi(G))$, then P is normal in G. In particular, $\Phi(G)$ is the direct product of its Sylow subgroups.

4. Assume that a finite group G acts transitively on Ω, $\alpha \in \Omega$, and G_α is the stabilizer of α in G. Let P be a Sylow p-subgroup of G_α. Denote by $\mathrm{Fix}(P)$ the set

$$\{\,\omega \in \Omega \mid \forall x \in P,\ \omega x = \omega\,\}.$$

   Prove that $N_G(P)$ acts transitively on $\mathrm{Fix}(P)$.

5. Assume that G is a transitive subgroup of S_p, p is prime, and G contains a transposition. Prove that $G = S_p$.

6. If A is an abelian transitive subgroup of S_n, then $|A| = n$. In particular, $C_{S_n}(A) = A$.

7. An abelian group G is said to be *elementary abelian*, if there exists a prime p such that $g^p = 1$ for every $g \in G$. Prove that

   (a) G is elementary abelian of order p^n if and only if $G \simeq \mathbb{Z}_p^n$, where $\mathbb{Z}_p^n$ is a direct product of n cyclic groups of order p;

   (b) there exist transitive elementary abelian subgroups of S_9, S_{25}, and S_{p^2}, where p is prime;

   (c) if G is elementary abelian of order p^n, then $\mathrm{Aut}(G) \simeq \mathrm{GL}_n(p)$.