Siberian Mathematical Journal, Vol. 53, *No.* 3, *pp.* , 2012
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PRONORMALITY OF HALL SUBGROUPS IN FINITE SIMPLE GROUPS

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Abstract: We prove that the Hall subgroups of finite simple groups are pronormal. Thus we obtain an affirmative answer to Problem 17.45(a) of the *Kourovka Notebook*.

Keywords: Hall subgroup, pronormal subgroup, simple group

Introduction

According to the definition by P. Hall, a subgroup H of a group G is called *pronormal*, if for every $g \in G$ the subgroups H and H^g are conjugate in $\langle H, H^g \rangle$. The classical examples of pronormal subgroups are

- normal subgroups;
- maximal subgroups;
- Sylow subgroups of finite groups;
- Carter subgroups (i.e., nilpotent selfnormalizing subgroups) of finite solvable groups;
- Hall subgroups (i.e. subgroups whose order and index are coprime) of finite solvable groups.

The pronormality of subgroups in the last three cases follows from the conjugacy of Sylow, Carter, and Hall subgroups in finite groups in corresponding classes. In [1, Theorem 9.2] the first author proved that Carter subgroups in finite groups are conjugate. As a corollary it follows that Carter subgroups of finite groups are pronormal.

In contrast with Carter subgroups, Hall subgroups in finite groups can be nonconjugate. The goal of the authors is to find the classes of finite groups with pronormal Hall subgroups. In the present paper the following result is obtained.

Theorem 1. *The Hall subgroups of finite simple groups are pronormal.*

The theorem gives an affirmative answer to Problem 17.45(a) from the *Kourovka Notebook* [2], and it was announced by the authors in [3, Theorem 7.9]. This result is supposed to use for studying the problem, whether C_π is inherited by overgroups of π-Hall subgroups [2, Problem 17.44(a); 4, Conjecture 3; 5, Problems 2 and 3] (all definitions are given below).

1. Notation, Conventions, and Preliminary Results

The notation of the paper is standard.

If G is a finite group, H is a subgroup of G, and x is an element of G; then by $Z(G)$, $O_\infty(G)$, $N_G(H)$, $C_G(H)$, and $C_G(x)$ we denote the center of G, the solvable radical of G, the normalizer of H in G, the centralizer of H in G, and the centralizer of x in G respectively. Given groups A and B, by $A \times B$ and $A \circ B$ we denote the direct product and the central product respectively. If A and B are subgroups of G, then by $\langle A, B \rangle$ and $[A, B]$ the subgroup generated by $A \cup B$ and the mutual commutant of A and B are denoted.

We often use the notation of [6]. In particular, by $A : B$, $A^{\cdot}B$, and $A\,.\,B$ we denote the split, nonsplit, and arbitrary extensions of A by B respectively. Given a group G and a subgroup S of the

The authors were supported by the Russian Foundation for Basic Research (Grants 10–01–00391, 11–01–00456, and 11–01–91158) and the Federal Target Program “Scientific and Educational Personnel of Innovation Russia” for 2009–2013 (State Contract 14.740.11.0346),

Novosibirsk. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 53, No. 3, pp. 527–542, May–June, 2012. Original article submitted July 28, 2011.

symmetric group Sym_n, we denote the permutation wreath product of G and S by $G \wr S$ (here n and the embedding of S into Sym_n assumed known).

We write $H \,\mathrm{prn}\, G$ if H is a pronormal subgroup of G.

Throughout π denotes a set of primes. A natural number n with $\pi(n) \subseteq \pi$, is called a *π-number*, while a group G with $\pi(G) \subseteq \pi$ is called a *π-group*. The symbol n_π is used for the maximal π-number dividing n. A subgroup H of G is called a *π-Hall subgroup*, if $\pi(H) \subseteq \pi$ and $\pi(|G : H|) \subseteq \pi'$. The set of all π-Hall subgroups of G we denote by $\mathrm{Hall}_\pi(G)$. A Hall subgroup is a π-Hall subgroup for some π.

According to [7] we say that G *satisfies* E_π (or briefly $G \in E_\pi$), if G possesses a π-Hall subgroup. If, moreover, every two π-Hall subgroups are conjugate, then we say that G *satisfies* C_π ($G \in C_\pi$). If, in addition, each π-subgroup of G lies in a π-Hall subgroup, then we say that G *satisfies* D_π ($G \in D_\pi$). A group satisfying E_π (C_π, D_π) we also call an *E_π-(C_π-, D_π-)group*.

A finite group possessing a (sub)normal series such that all factors of the series are either π- or π'-groups is called *π-separable*.

Lemma 2 [7, Lemma 1]. *Let A be a normal subgroup of a finite group G. If $G \in E_\pi$ and $H \in \mathrm{Hall}_\pi(G)$, then $A, G/A \in E_\pi$. Moreover, $H \cap A \in \mathrm{Hall}_\pi(A)$ and $HA/A \in \mathrm{Hall}_\pi(G/A)$.*

Lemma 3 [8; 7, Corollary D5.2]. *Every π-separable group satisfies D_π.*

Lemma 4. *Let H be a subgroup of G, $g \in G$, $y \in \langle H, H^g\rangle$. If the subgroups H^y and H^g are conjugate in $\langle H^y, H^g\rangle$, then H and H^g are conjugate in $\langle H, H^g\rangle$.*

Proof. Let $z \in \langle H^y, H^g\rangle$, and $H^{yz} = H^g$. Then $z \in \langle H, H^g\rangle$ since $\langle H^y, H^g\rangle \le \langle H, H^g\rangle$. Put $x = yz$. Then $x \in \langle H, H^g\rangle$ and $H^x = H^g$. $\square$

Lemma 5. *Let H be a subgroup of a finite group G. Assume that H includes a pronormal (for example, a Sylow) subgroup S of G. Then the following are equivalent:*

(1) *$H \,\mathrm{prn}\, G$;*

(2) *H and H^g are conjugate in $\langle H, H^g\rangle$ for each $g \in N_G(S)$.*

Proof. Clearly (1) $\Rightarrow$ (2). We prove that (2) $\Rightarrow$ (1). Assume (2). Choose an arbitrary $g \in G$. Notice that $S, S^g \le \langle H, H^g\rangle$. Since S is pronormal, there exists $y \in \langle S, S^g\rangle \le \langle H, H^g\rangle$ such that $S^{gy} = S$. In particular, $gy \in N_G(S)$. In view of (2), the subgroups H and H^{gy} are conjugate in $\langle H, H^{gy}\rangle$. Then $H^{y^{-1}}$ and H^g are conjugate in $\langle H^{y^{-1}}, H^g\rangle$. Now H and H^g are conjugate in $\langle H, H^g\rangle$ by Lemma 4. $\square$

Lemma 6. *Let $^- : G \to G_1$ be a homomorphism of groups, $H \le G$. If $H \,\mathrm{prn}\, G$, then $\overline{H} \,\mathrm{prn}\, \overline{G}$.*

Proof. Clear. $\square$

Lemma 7. *Let G be a finite group and let $G_1, \dots, G_n$ be normal subgroups of G such that $[G_i, G_j] = 1$ for $i \ne j$ and $G = G_1 \dots G_n$. Assume that for each $i = 1, \dots, n$ a pronormal subgroup H_i of G_i is chosen, and $H = \langle H_1, \dots, H_n\rangle$. Then $H \,\mathrm{prn}\, G$.*

Proof. Choose an arbitrary $g \in G$. Then $g = g_1 \dots g_n$ for some $g_1 \in G_1, \dots, g_n \in G_n$. Since H_i is pronormal in G_i for each $i = 1, \dots, n$, there exist $x_i \in \langle H_i, H_i^{g_i}\rangle$ such that $H_i^{x_i} = H_i^{g_i}$. Since $[G_i, G_j] = 1$ for $i \ne j$, we have $H_i^g = H_i^{g_i}$ for each $i = 1, \dots, n$. The same arguments imply $H_i^{x_i} = H_i^x$, where $x = x_1 \dots x_n$. Clearly,

$$x \in \left\langle H_i, H_i^{g_i} \mid i = 1, \dots, n\right\rangle = \left\langle H_i, H_i^g \mid i = 1, \dots, n\right\rangle = \langle H, H^g\rangle.$$

Further,

$$\begin{aligned} H^g = \left\langle H_i^g \mid i = 1, \dots, n\right\rangle &= \left\langle H_i^{g_i} \mid i = 1, \dots, n\right\rangle = \left\langle H_i^{x_i} \mid i = 1, \dots, n\right\rangle \\ &= \left\langle H_i^x \mid i = 1, \dots, n\right\rangle = H^x. \quad \square \end{aligned}$$

Lemma 8. *Let G be a finite group, $H \in \mathrm{Hall}_\pi(G)$, $A \trianglelefteq G$, and $G = HA$. If $(H \cap A) \operatorname{prn} A$, then $H \operatorname{prn} G$.*

Proof. By Lemma 2, $H \cap A$ is a π-Hall subgroup of A. Let $(H \cap A) \operatorname{prn} A$. Choose an arbitrary $g \in G$ and show that $H^x = H^g$ for some $x \in \langle H, H^g\rangle$.

Since $G = HA$, there exist $h \in H$ and $a \in A$ such that $g = ha$. Since $(H \cap A) \operatorname{prn} A$, there exists $y \in \langle H \cap A, H^a \cap A\rangle$ such that $H^y \cap A = H^a \cap A$. In view of

$$y \in \langle H \cap A, H^a \cap A\rangle \leq \langle H, H^a\rangle = \langle H, H^{ha}\rangle = \langle H, H^g\rangle,$$

and Lemma 4 we need to consider the case $H = H^y$. In particular,

$$H \cap A = H^a \cap A = H^g \cap A.$$

Now H, H^g, and g are included in $N_G(H \cap A)$. Since $G = HA$, we have $G = AN_G(H \cap A)$. Notice that

$$N_G(H \cap A)/N_A(H \cap A) = N_G(H \cap A)/(A \cap N_G(H \cap A)) \simeq AN_G(H \cap A)/A = G/A$$

is a π-group. Consider the normal series

$$N_G(H \cap A) \trianglerighteq N_A(H \cap A) \trianglerighteq H \cap A \trianglerighteq 1$$

of $N_G(H \cap A)$. Each factor of the series is either a π- or π'-group, and so $N_G(H \cap A)$ is π-separable. Therefore, the subgroup $\langle H, H^g\rangle$ of $N_G(H \cap A)$ is π-separable as well, and in particular $\langle H, H^g\rangle \in D_\pi$ by Lemma 3. Thus the π-Hall subgroups H and H^g are conjugate in $\langle H, H^g\rangle$. $\square$

The next lemma gives a sufficient condition for the treatment of Lemma 6 in case when H is a Hall subgroup of G.

Lemma 9. *Let $\mathfrak{X}$ be a class of finite groups close under subgroups such that $\mathfrak{X} \subseteq C_\pi$. Let G be a finite group, $H \in \mathrm{Hall}_\pi(G)$, $A \trianglelefteq G$, and let $\bar{}: G \to G/A$ be the natural homomorphism. Assume also that $A \in \mathfrak{X}$. Then $H \operatorname{prn} G$ if and only if $\overline{H} \operatorname{prn} \overline{G}$.*

Proof. The implication $\Rightarrow$ holds by Lemma 6.

We prove $\Leftarrow$. Take $g \in G$. We need to show that $H^x = H^g$ for some $x \in \langle H, H^g\rangle$. Since $\overline{H} \operatorname{prn} \overline{G}$, there exists $y \in \langle H, H^g\rangle$ such that $H^yA = H^gA$. By Lemma 4 we may replace H by H^y and so we may assume that $HA = H^gA$.

Consider $M = \langle H \cap A, H^g \cap A\rangle$. Since $M \leq A$, $A \in \mathfrak{X}$ and $\mathfrak{X}$ is closed under subgroups, we have $M \in \mathfrak{X} \subseteq C_\pi$. Further $H \cap A, H^g \cap A \in \mathrm{Hall}_\pi(A)$ by Lemma 2, and $M \leq A$. So $H \cap A, H^g \cap A \in \mathrm{Hall}_\pi(M)$. Hence $H^a \cap A = H^g \cap A$ for some $a \in M$. Since $M \leq \langle H, H^g\rangle$, by Lemma 4 we may replace H by H^a, and so we may assume that $H \cap A = H^g \cap A$. In such case $g \in N_G(H \cap A)$ and $H, H^g \leq N_G(H \cap A)$. Since $A \in C_\pi$ by the Frattini argument, $G = AN_G(H \cap A)$. Now

$$N_G(H \cap A)/N_A(H \cap A) = N_G(H \cap A)/(A \cap N_G(H \cap A)) \simeq AN_G(H \cap A)/A = G/A = \overline{G}.$$

As we noted above $\overline{H} = \overline{H}^g$, so the isomorphism implies that $HN_A(H \cap A) = H^gN_A(H \cap A)$. Denote the last subgroup by B for brevity. Then B is π-separable and $H, H^g \leq B$. Moreover, $\langle H, H^g\rangle$ is also π-separable as a subgroup of the π-separable group B. In particular, by Lemma 3

$$\langle H, H^g\rangle \in D_\pi \quad \text{and} \quad H, H^g \in \mathrm{Hall}_\pi(\langle H, H^g\rangle),$$

whence H and H^g are conjugate in $\langle H, H^g\rangle$. $\square$

Let G be a finite group and $1\pi(G) = \{p_1, \dots, p_n\}$. Following [7] we say that G has a *Sylow tower of type*[1)] $(p_1, \dots, p_n)$, if G possesses the normal series

$$G = G_0 > G_1 > \dots > G_n = 1$$

such that each section G_{i-1}/G_i is isomorphic to a Sylow p_i-subgroup of G.

[1)]The parentheses in the notation $(p_1, \dots, p_n)$ are used for an ordered set, apart from braces. For example, the symmetric group Sym_3 has a Sylow tower of type $(2, 3)$, while the alternating group Alt_4 has a Sylow tower of type $(3, 2)$.

Lemma 10. *Let G be a finite group, and let H be a Hall subgroup with a Sylow tower. Then $H \operatorname{prn} G$.*

PROOF. Take $g \in G$. We show that H and H^g are conjugate in $\langle H, H^g\rangle$. By [7, Theorem A1] every two Hall subgroups of a finite groups having Sylow tower of the same type are conjugate. Since H and H^g are two Hall subgroups of $\langle H, H^g\rangle$ having Sylow tower of the same type, H and H^g are conjugate in $\langle H, H^g\rangle$. $\square$

Lemma 11. *Let G be a finite nonabelian simple group, let H be its Hall subgroup of order not divisible either by 2 or 3. Then H has a Sylow tower.*

PROOF. In case 2 does not divide the order of H, the claim is proven in [9, Theorem B]. In case 3 does not divide the order of H, the claim follows from [10, Lemma 5.1 and Theorem 5.2]. $\square$

The symmetric group and the alternating group of degree n we denote by Sym_n and Alt_n respectively.
The finite field containing q elements is denoted by $\mathbb{F}_q$.

Given an odd number q, define $\varepsilon(q) = (-1)^{(q-1)/2}$, i.e., $\varepsilon(q) = 1$, if $q - 1$ is divisible by 4, and $\varepsilon(q) = -1$ otherwise. Without additional explanation we use the symbols ε and η to denote either an element from $\{+1, -1\}$ or the sign of the element.

Given a group of Lie type the order of the base field is always denoted by q (see [1], for example), while its characteristic is denoted by p. Given a matrix group G the reduction modulo scalars is denoted by $\mathrm{P}G$.

Our notation for classical groups agrees with that of [11]. We recall the special notation that we often use:
$\mathrm{GL}^+_n(q) = \mathrm{GL}_n(q)$ is the general linear group of degree n over $\mathbb{F}_q$;
$\mathrm{SL}^+_n(q) = \mathrm{SL}_n(q)$ is the special linear group of degree n over $\mathbb{F}_q$;
$\mathrm{PGL}^+_n(q) = \mathrm{PGL}_n(q)$ is the projective general linear group of degree n over $\mathbb{F}_q$;
$\mathrm{PSL}^+_n(q) = \mathrm{PSL}_n(q)$ is the projective special linear group of degree n over $\mathbb{F}_q$;
$\mathrm{GL}^-_n(q) = \mathrm{GU}_n(q)$ is the general unitary group of degree n over $\mathbb{F}_{q^2}$;
$\mathrm{SL}^-_n(q) = \mathrm{SU}_n(q)$ is the special unitary group of degree n over $\mathbb{F}_{q^2}$;
$\mathrm{PSL}^-_n(q) = \mathrm{PSU}_n(q)$ is the projective special unitary group of degree n over $\mathbb{F}_{q^2}$;
$\mathrm{PGL}^-_n(q) = \mathrm{PGU}_n(q)$ is the projective general unitary group of degree n over $\mathbb{F}_{q^2}$;
$\mathrm{Sp}_n(q)$ is the symplectic group of degree n over $\mathbb{F}_q$;
$\mathrm{PSp}_n(q)$ is the projective symplectic group of degree n over $\mathbb{F}_q$.

The necessary facts about properties and structure of finite groups of Lie type can be found in [12–15], the properties and structure of linear algebraic groups can be found in [12], and the results on connection between the groups of Lie type and the linear algebraic groups can be found in [13–14]. The definitions of Borel and Cartan subgroups, a parabolic subgroup, and a maximal torus in a finite group of Lie type can be also found in [13–14].

We denote groups $E_6(q)$ and ${}^2E_6(q)$ by $E_6^+(q)$ and $E_6^-(q)$ respectively.

A *Frobenius map* of an algebraic group $\overline{G}$ is a surjective endomorphism $\sigma : \overline{G} \to \overline{G}$ such that the set of its stable points $\overline{G}_\sigma$ is finite. Each simple group of Lie type of a finite field F of characteristic p is known to coincide with $O^{p'}(\overline{G}_\sigma)$ for an appropriate linear algebraic group $\overline{G}$ over the algebraic closure of F and a Frobenius map σ, where $O^{p'}(\overline{G}_\sigma)$ is a subgroup of $\overline{G}_\sigma$ generated by all p-elements.

Let $\overline{R}$ be a closed σ-stable subgroup of an algebraic group $\overline{G}$ for a Frobenius map σ of $\overline{G}$. Consider the subgroups $R = G \cap \overline{R}$ and $N(G, R) = G \cap N_{\overline{G}}(\overline{R})$, where $G = O^{p'}(\overline{G}_\sigma)$. Notice that $N(G, R) \leq N_G(R)$ and $N(G, R) \neq N_G(R)$ in general.

Lemma 12 [16, the Corollary of Theorems 1–3]. *Let G be a finite nonabelian simple group and $S \in \mathrm{Syl}_2(G)$. Then $N_G(S) = S$, except the following cases:*
(1) *$G \simeq J_2$, J_3, Suz or HN and $|N_G(S) : S| = 3$;*
(2) *$G \simeq {}^2G_2(q)$ or J_1 and $N_G(S) \simeq 2^3.7.3 < \mathrm{Hol}(2^3)$;*
(3) *G is a group of Lie type over a field of characteristic 2 and $N_G(S)$ is a Borel subgroup of G;*

(4) $G \simeq \mathrm{PSL}_2(q)$, *where* $3 < q \equiv \pm 3 \pmod 8$ *and* $N_G(S) \simeq \mathrm{Alt}_4$;

(5) $G \simeq E_6^\eta(q)$, $\eta = \pm$, q *is odd and* $N_G(S) = S \times C$, *where* C *is a nontrivial cyclic group of order* $(q-\eta)_{2'}/(q-\eta,3)_{2'}$;

(6) $G \simeq \mathrm{PSp}_{2m}(q)$, $m \geq 2$, $q \equiv \pm 3 \pmod 8$, *the factor group* $N_G(S)/S$ *is isomorphic to an elementary abelian 3-group of order* 3^t *and* t *can be found from the 2-adic decomposition*

$$m = 2^{s_1} + \cdots + 2^{s_t},$$

where $s_1 > \cdots > s_t \geq 0$;

(7) $G \simeq \mathrm{PSL}_n^\eta(q)$, $n \geq 3$, $\eta = \pm$, q *is odd,*

$$N_G(S) \simeq S \times C_1 \times \cdots \times C_{t-1},$$

t *can be found from a 2-adic decomposition*

$$n = 2^{s_1} + \cdots + 2^{s_t},$$

where $s_1 > \cdots > s_t \geq 0$, *and* $C_1, \ldots, C_{t-2}, C_{t-1}$ *are cyclic groups of orders* $(q-\eta)_{2'}, \ldots, (q-\eta)_{2'}$ *and* $(q-\eta)_{2'}/(q-\eta,n)_{2'}$ *respectively.*

Lemma 13 [7, Theorem A4; 17]. *Let* $2,3 \in \pi$. *Then the list of all cases, when* Sym_n *possesses a proper* π*-Hall subgroup is given in Table 1. In particular, each proper* π*-Hall subgroup of* Sym_n *is maximal in* Sym_n.

Table 1. π-Hall subgroups
in symmetric groups

n	$\pi \cap \pi(\mathrm{Sym}_n)$	$H \in \mathrm{Hall}_\pi(\mathrm{Sym}_n)$
simple	$\pi((n-1)!)$	Sym_{n-1}
7	$\{2,3\}$	$\mathrm{Sym}_3 \times \mathrm{Sym}_4$
8	$\{2,3\}$	$\mathrm{Sym}_4 \wr \mathrm{Sym}_2$

Lemma 14 [18, Theorem 4.1]. *Let* G *be either one of* 26 *sporadic groups or the Tits group. Assume that* π *contains both 2 and 3. Then* G *possesses a proper* π*-Hall subgroup* H *if and only if one of the conditions on* G *and* $\pi \cap \pi(G)$ *from Table 2 holds. In the table the structure of* H *is also given.*

Table 2. π-Hall subgroups
in sporadic groups, case $2,3 \in \pi$

G	$\pi \cap \pi(G)$	Structure H
M_{11}	$\{2,3\}$	$3^2 : Q_8.2$
	$\{2,3,5\}$	$\mathrm{Alt}_6\,.2$
M_{22}	$\{2,3,5\}$	$2^4 : \mathrm{Alt}_6$
M_{23}	$\{2,3\}$	$2^4 : (3 \times \mathrm{Alt}_4) : 2$
	$\{2,3,5\}$	$2^4 : \mathrm{Alt}_6$
	$\{2,3,5\}$	$2^4 : (3 \times \mathrm{Alt}_5) : 2$
	$\{2,3,5,7\}$	$\mathrm{L}_3(4) : 2_2$
	$\{2,3,5,7\}$	$2^4 : \mathrm{Alt}_7$
	$\{2,3,5,7,11\}$	M_{22}
M_{24}	$\{2,3,5\}$	$2^6 : 3{}^{\cdot}\mathrm{Sym}_6$
J_1	$\{2,3\}$	$2 \times \mathrm{Alt}_4$
	$\{2,3,5\}$	$2 \times \mathrm{Alt}_5$
	$\{2,3,7\}$	$2^3 : 7 : 3$
J_4	$\{2,3,5\}$	$2^{11} : (2^6 : 3{}^{\cdot}\mathrm{Sym}_6)$

Lemma 15 [19, Theorem 3.3]. *Let G be a finite group of Lie type over a field of characteristic $p \in \pi$. If H is a π-Hall subgroup of G, then either H is included in a Borel subgroup or H is a parabolic subgroup of G.*

Lemma 16 [20, Lemma 3.1]. *Let $G \simeq \mathrm{P\,SL}_2(q) \simeq \mathrm{P\,SL}_2^\eta(q) \simeq \mathrm{P\,Sp}_2(q)$, where q is a power of an odd prime p, and set $\varepsilon = \varepsilon(q)$. Assume that $2, 3 \in \pi$ and $p \notin \pi$. Then $G \in E_\pi$ if and only if one of the cases from Table 3 holds.*

Table 3. π-Hall subgroups H of $\mathrm{PSL}_2(q)$, $2, 3 \in \pi$, $p \notin \pi$

$\pi \cap \pi(G)$	H	Conditions
$\subseteq \pi(q-\varepsilon)$	$D_{q-\varepsilon}$	—
$\{2,3\}$	Alt_4	$(q^2-1)_{\{2,3\}} = 24$
$\{2,3\}$	Sym_4	$(q^2-1)_{\{2,3\}} = 48$
$\{2,3,5\}$	Alt_5	$(q^2-1)_{\{2,3,5\}} = 120$

Lemma 17 [20, Lemma 3.2]. *Assume that $G = \mathrm{GL}_2^\eta(q)$, where q is a power of a prime p, $\mathrm{P} : G \to G/Z(G) = \mathrm{PGL}_2^\eta(q)$ is the natural homomorphism, and let $\varepsilon = \varepsilon(q)$. Assume also that $2, 3 \in \pi$ and $p \notin \pi$. A subgroup H of G is a π-Hall subgroup if and only if one of the following holds:*

(1) *$\pi \cap \pi(G) \subseteq \pi(q-\varepsilon)$, $\mathrm{P}H$ is a π-Hall subgroup of the dihedral group $D_{2(q-\varepsilon)}$ of order $2(q-\varepsilon)$ of $\mathrm{P}G$;*

(2) *$\pi \cap \pi(G) = \{2,3\}$, $(q^2-1)_{\{2,3\}} = 24$, $\mathrm{P}H \simeq \mathrm{Sym}_4$.*

Moreover every two π-Hall subgroups of G, satisfying the same statement (1) or (2), are conjugate.

Lemma 18 [20, Lemma 4.3]. *Let $G^* = \mathrm{SL}_n^\eta(q)$ be a special linear or unitary group with the base field $\mathbb{F}_q$ of characteristic p, and let $n \geq 2$. Assume that $2, 3 \in \pi$ and $p \notin \pi$. Suppose that $G^* \in E_\pi$ and H^* is a π-Hall subgroup of G^*. Then for G^*, H^* and π one of the following holds:*

(1) *$n = 2$ and for groups $G = G^*/Z(G^*)$ and $H = H^*Z(G^*)/Z(G^*)$ the conditions from Table 3 hold.*

(2) *Either $q \equiv \eta \pmod{12}$, or $n = 3$ and $q \equiv \eta \pmod 4$; Sym_n satisfies E_π, $\pi \cap \pi(G^*) \subseteq \pi(q-\eta) \cup \pi(n!)$ and if $r \in (\pi \cap \pi(n!)) \setminus \pi(q-\eta)$, then $|G^*|_r = |\mathrm{Sym}_n|_r$; H^* is included in*

$$M = L \cap G^* \simeq Z^{n-1} \,.\, \mathrm{Sym}_n,$$

where $L = Z \wr \mathrm{Sym}_n \leq \mathrm{GL}_n^\eta(q)$ and $Z = \mathrm{GL}_1^\eta(q)$ is a cyclic group of order $q - \eta$.

(3) *$n = 2m + k$, where $k \in \{0, 1\}$, $m \geq 1$, $q \equiv -\eta \pmod 3$, $\pi \cap \pi(G^*) \subseteq \pi(q^2-1)$, both Sym_m and $\mathrm{GL}_2^\eta(q)$ satisfy E_π*[2]*; H^* is included in*

$$M = L \cap G^* \simeq \big(\underbrace{\mathrm{GL}_2^\eta(q) \circ \cdots \circ \mathrm{GL}_2^\eta(q)}_{m \text{ times}}\big) \,.\, \mathrm{Sym}_m \circ Z,$$

where $L = \mathrm{GL}_2^\eta(q) \wr \mathrm{Sym}_m \times Z \leq \mathrm{GL}_n(q)$ and Z is a cyclic group of order $q-\eta$ if $k = 1$, and $Z = 1$ if $k = 0$. A subgroup H^*, acting by conjugation on the set of factors of type $\mathrm{GL}_2^\eta(q)$ in the central product

$$\underbrace{\mathrm{GL}_2^\eta(q) \circ \cdots \circ \mathrm{GL}_2^\eta(q)}_{m \text{ times}}, \qquad (1)$$

has at most two orbits. The intersection of H^* with each factor $\mathrm{GL}_2^\eta(q)$ in (1) is a π-Hall subgroup of $\mathrm{GL}_2^\eta(q)$. All intersections of H^* with factors from the same orbit satisfy to the same statement (1) or (2) in Lemma 17.

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[2] By Lemma 16 conditions $\mathrm{GL}_2^\eta(q) \in E_\pi$ and $q \equiv -\eta \pmod 3$ mean that $q \equiv -\eta \pmod r$ for all odd primes $r \in \pi(q^2-1) \cap \pi$.

(4) $n=4$, $\pi\cap\pi(G^*)=\{2,3,5\}$, $q\equiv 5\eta \pmod 8$, $(q+\eta)_3=3$, $(q^2+1)_5=5$ *and* $H^*\simeq 4\,.\,2^4\,.\,\mathrm{Alt}_6$.

(5) $n=11$, $\pi\cap\pi(G^*)=\{2,3\}$, $(q^2-1)_{\{2,3\}}=24$, $q\equiv-\eta \pmod 3$, $q\equiv\eta \pmod 4$, H^* *is included in* $M=L\cap G^*$, *where* L *is a subgroup of* G^* *of type* $\big(\big(\mathrm{GL}_2^\eta(q)\wr\mathrm{Sym}_4\big)\perp\big(\mathrm{GL}_1^\eta(q)\wr\mathrm{Sym}_3\big)\big)$ *and*

$$H^*=(((Z\circ 2\,.\,\mathrm{Sym}_4)\wr\mathrm{Sym}_4)\times(Z\wr\mathrm{Sym}_3))\cap G,$$

where Z *is a Sylow 2-subgroup of a cyclic group of order* $q-\eta$.

Lemma 19 [20, Lemma 4.4]. *Let* $G^*=\mathrm{Sp}_{2n}(q)$ *be a symplectic group over a field* $\mathbb{F}_q$ *of characteristic* p. *Assume that* $2,3\in\pi$ *and* $p\notin\pi$. *Suppose that* $G^*\in E_\pi$ *and* $H^*\in\mathrm{Hall}_\pi(G)$. *Then both* Sym_n *and* $\mathrm{SL}_2(q)$ *satisfy* E_π *and* $\pi\cap\pi(G^*)\subseteq\pi(q^2-1)$. *Moreover* H^* *is a* π*-Hall subgroup of*

$$M=\mathrm{Sp}_2(q)\wr\mathrm{Sym}_n\simeq(\underbrace{\mathrm{SL}_2(q)\times\cdots\times\mathrm{SL}_2(q)}_{n\text{ times}}):\mathrm{Sym}_n\leq G^*.$$

Lemma 20 [20, Lemma 7.3]. *Let* $G=E_6^\eta(q)$, *where* q *is a power of a prime* p, *and* $\varepsilon=\varepsilon(q)$. *Assume that* $2,3\in\pi$ *and* $p\notin\pi$. *Suppose that* G *possesses a* π*-Hall subgroup* H. *Then* $\pi\cap\pi(G)\subseteq\pi(q-\varepsilon)$ *and one of the following holds:*

(1) $\eta=\varepsilon$, $5\in\pi$ *and* H *is a* π*-Hall subgroup of* $M=((q-\eta)^6.W(E_6))/(3,q-\eta)$;

(2) $\eta=-\varepsilon$ *and* H *is a* π*-Hall subgroup of* $M=(q^2-1)^2(q+\eta)^2.W(F_4)$.

2. Proof of Theorem 1

Let G be a finite simple group and $H\in\mathrm{Hall}_\pi(G)$. We show that $H\,\mathrm{prn}\,G$, and so we prove Theorem 1. By Lemmas 10 and 11 we may assume that $2,3\in\pi$. Let $S\in\mathrm{Syl}_2(H)\subseteq\mathrm{Syl}_2(G)$ and $g\in N_G(S)$ be arbitrary. By Lemma 5 it is enough to prove that H and H^g are conjugate in $\langle H,H^g\rangle$. If $N_G(S)=S$, then this statement is true: $g\in N_G(S)=S\leq H$, and so $H^g=H$. Therefore we may assume that one of the exceptional cases (1)–(7) from Lemma 12 holds, and H is a proper π-Hall subgroup of G.

We consider cases (1)–(7) from Lemma 12, proving a series of auxiliary lemmas. In order to unify the notation in the lemmas with the already introduced notation we say that $(\star)$ *holds*, if

(a) G is a finite simple group;

(b) $2,3\in\pi$;

(c) $H\in\mathrm{Hall}_\pi(G)$ and $H<G$;

(d) $S\in\mathrm{Syl}_2(H)\subseteq\mathrm{Syl}_2(G)$;

(e) $g\in N_G(S)$.

The following lemma is immediate from Lemma 14.

Lemma 21. *Assume that* $(\star)$ *holds. If* $G\simeq J_2,J_3,Suz$ *or* HN, *then* G *does not possesses proper* π*-Hall subgroups.*

Thus if case (1) of Lemma 12 holds, then by Lemma 5 $H\,\mathrm{prn}\,G$.

Lemma 22. *Assume that* $(\star)$ *holds. Then the following hold:*

(1) *If* $G\simeq{}^2G_2(q)$, *then* G *possesses no proper* π*-Hall subgroup.*

(2) *If* $G\simeq J_1$, *then one of the following holds:*

  (a) $H\simeq 2\times\mathrm{Alt}_4$ *and* H *possesses a Sylow tower;*

  (b) $H\simeq 2^3:7:3$ *and* H *possesses a Sylow tower;*

  (c) $H\simeq 2\times\mathrm{Alt}_5$ *and* H *is maximal in* G.

(3) *If* $G\simeq J_1$, *then* H *is conjugate with* H^g *by an element from* $\langle H,H^g\rangle$.

Proof. Statement (1) follows from [19, Theorem 1.2], since $3\in\pi$ and 3 is the characteristic of the base field for ${}^2G_2(q)$. Lemma 14 implies the structure of H in (2); moreover, it is clear that in cases (a) and (b) the subgroup has a Sylow tower. In case (c) H is maximal in view of [6]. Statement (3) follows from (2), Lemma 10, and pronormality of maximal subgroups. □

Thus Lemmas 5 and 22 imply that $H\,\mathrm{prn}\,G$, if (2) of Lemma 12 holds.

Lemma 23. *Assume that* ($\star$) *holds, and* G *is a group of Lie type over a field of characteristic* 2*. Then* S *is a maximal unipotent subgroup,* $N_G(S)$ *is a Borel subgroup of* G*, and one of the following holds:*

(1) *H lies in a Borel subgroup and has a Sylow tower;*

(2) *H is parabolic and includes $N_G(S)$.*

In both cases H is conjugate with H^g by an element from $\langle H, H^g\rangle$.

Proof. In view of Lemma 15, the structure of Borel subgroups and the fact that every parabolic subgroup includes a Borel subgroup we see that either (1) or (2) holds. By Lemma 10 we conclude that H is conjugate with H^g by an element from $\langle H, H^g\rangle$, if (1) holds. If (2) holds, then the final claim is evident since $g \in H$. $\square$

Thus if (3) of Lemma 12 holds, then $H \operatorname{prn} G$.

Lemma 24. *Assume that* ($\star$) *holds,* $G = \mathrm{PSL}_2(q)$*,* $q \equiv \pm 3 \pmod 8$*, and* $q > 3$*. Then one of the following holds:*

(1) *H is a π-Hall subgroup in a dihedral group of order $q - \varepsilon$, where $\varepsilon = \varepsilon(q) = (-1)^{(q-1)/2}$, and it has a Sylow tower;*

(2) *$H \simeq \mathrm{Alt}_4$ and H has a Sylow tower;*

(3) *$H \simeq \mathrm{Alt}_5$ and H includes $N_G(S) \simeq \mathrm{Alt}_4$. In particular, $H^g = H$.*

In any case H is conjugate with H^g by an element from $\langle H, H^g\rangle$.

Proof. Conditions $q \equiv \pm 3 \pmod 8$ and $q > 3$, and Lemma 16 imply the structure of H. Moreover, if either H is included in a dihedral subgroup, or $H \simeq \mathrm{Alt}_4$, then it clearly has a Sylow tower. Assume that $H \simeq \mathrm{Alt}_5$. Then $\mathrm{Alt}_4 = N_H(S) \leq N_G(S) \simeq \mathrm{Alt}_4$, and so $N_H(S) = N_G(S)$. Using (1)–(3) and Lemma 10, we obtain the final conclusion. $\square$

Thus we have shown that if (4) of Lemma 12 holds, then $H \operatorname{prn} G$.

Lemma 24 implies also the following statement that is extensively used for consideration of items (6) and (7) in Lemma 12.

Lemma 25. *Let* $2, 3 \in \pi$*,* q *be a power of an odd prime* $p \notin \pi$*,* $G^* \in \{\mathrm{PSL}_2(q),\ \mathrm{PGL}_2^{\eta}(q),\ \mathrm{SL}_2(q),\ \mathrm{GL}_2^{\eta}(q)\}$*, and* $H^* \in \mathrm{Hall}_\pi(G^*)$*. Then* $H^* \operatorname{prn} G^*$*.*

Proof. If $G^* = \mathrm{PSL}_2(q)$ and $S^* \in \mathrm{Syl}_2(H^*) \subseteq \mathrm{Syl}_2(G^*)$, then by Lemma 12 either $N_{G^*}(S^*) = S^*$ or G^* satisfies the conditions of Lemma 24. In both cases H^* is pronormal.

Now let $G^* = \mathrm{PGL}_2^{\eta}(q)$ and let $A^* = \mathrm{PSL}_2^{\eta}(q) \simeq \mathrm{PSL}_2(q)$ be a normal subgroup of index 2 in G^*. As we have already shown, $H^* \cap A^* \operatorname{prn} A^*$ and $G^* = A^*H^*$. Using Lemma 8 we conclude that $H^* \operatorname{prn} G^*$.

Assume finally that G^* is isomorphic to either $\mathrm{SL}_2(q)$ or $\mathrm{GL}_2^{\eta}(q)$. Choose in Lemma 9 the class of all 2-groups as $\mathfrak{X}$. Then this lemma and the above arguments imply $H^* \operatorname{prn} G^*$. $\square$

Consider (5) of Lemma 12.

Lemma 26. *Assume that* ($\star$) *holds and* $G = E_6^{\eta}(q)$*, where* q *is a prime of* $p \notin \pi$*. Denote* $\varepsilon(q)$ *by* ε*. Then*

(1) *G includes an S-invariant maximal torus T such that*

$$|T| = \begin{cases} (q-\varepsilon)^6/(3, q-\varepsilon), & \text{if } \eta = \varepsilon, \\ (q-\varepsilon)^4(q+\varepsilon)^2, & \text{if } \eta = -\varepsilon, \end{cases}$$

moreover $H \leq N_G(T)$ and $N_G(T)$ is an extension of T by a π-group;

(2) *$N_G(T)$ includes $N_G(S)$;*

(3) *H and H^g are conjugate subgroups in $\langle H, H^g\rangle$.*

Proof. (1) The existence of an S-invariant torus T follows from [12, Theorem 4.10.2]. In view of [10, Lemma 3.10] such a torus is unique up to conjugation and $N_G(T) = N(G, T)$. Moreover by [10, Lemma 3.11] the order of T equals $(q-\varepsilon)^6/(3, q-\varepsilon)$, if $\varepsilon = \eta$, and it equals $(q-\varepsilon)^4(q+\varepsilon)^2$, if $\varepsilon = -\eta$.

Since $G \in E_\pi$ and $2,3 \in \pi$, while $p \notin \pi$, by Lemma 20 we see that H lies in $N_G(T)$ for some such a torus T and $N_G(T)/T$ is a π-group.

(2) Since $H \leq N_G(T)$ and $3 \in \pi$, $N_G(T)$ includes a Sylow 3-subgroup of G. So it follows by [10, Lemma 3.13] that $N_G(S) \leq N_G(T)$.

(3) In view of (2) of the lemma we remain to prove that $H \operatorname{prn} N_G(T)$. By (1), $N_G(T)$ is an extension of an abelian group T by a π-group, and in particular $N_G(T) = HT$. Now, by Lemma 8, $H \operatorname{prn} N_G(T)$. $\square$

Therefore, if (5) of Lemma 12 holds, then $H \operatorname{prn} G$.

In the next lemma we consider (6) and, partially, (7) of Lemma 12. We need to recall the notion of fundamental subgroup which was introduced in [21]. We use the notion in simple linear, unitary, and symplectic groups in odd characteristic only, and their central extensions. Recall that if G is one of these groups, X^+ is a long root subgroup of G, and X^- is the opposite root subgroup, then each G-conjugate of $\langle X^+, X^- \rangle \simeq \mathrm{SL}_2(q)$ is called a *fundamental subgroup*. If $S \in \mathrm{Syl}_2(G)$, then by $\mathrm{Fun}_G(S)$ the set of all fundamental subgroups K of G such that $K \cap S \in \mathrm{Syl}_2(K)$ is denoted. $\mathrm{Fun}_G(S)$ is known to be an inclusion-maximal S-invariant set of pairwise commuting fundamental subgroups of G (see [21]).

Lemma 27. *Assume that* $(\star)$ *holds and* G *is isomorphic to either* $\mathrm{PSL}_n^\eta(q)$ *or* $\mathrm{PSp}_n(q)$, *where* $n > 2$. *Let* $\Delta = \mathrm{Fun}_G(S)$ *and suppose that* Δ *is* H*-invariant (i.e.* $H \leq N_G(\Delta)$ *in the notation of* [21]*). Then* H *and* H^g *are conjugate in* $\langle H, H^g \rangle$.

Proof. Let $m = [n/2]$. Then $|\Delta| = m$.

In view of [11, Propositions 4.1.4, 4.2.9, and 4.2.10] the stabilizer $N_G(\Delta)$ in G of Δ coincides with the image in G of a subgroup M of either $\mathrm{SL}_n^\eta(q)$ or $\mathrm{Sp}_n(q)$, where M is defined in the following way. If $G = \mathrm{PSL}_n^\eta(q)$, then

$$M = L \cap \mathrm{SL}_n^\eta(q) \simeq \big(\underbrace{\mathrm{GL}_2^\eta(q) \circ \cdots \circ \mathrm{GL}_2^\eta(q)}_{m \text{ times}}\big) . \mathrm{Sym}_m \circ Z.$$

Moreover, $L = \mathrm{GL}_2^\eta(q) \wr \mathrm{Sym}_m \times Z \leq \mathrm{GL}_n^\eta(q)$, and Z is a cyclic group of order $q - \eta$ if n is odd, and $Z = 1$ if n is even. If $G = \mathrm{PSp}_n(q)$, then

$$M = \mathrm{Sp}_2(q) \wr \mathrm{Sym}_m \simeq (\underbrace{\mathrm{SL}_2(q) \times \cdots \times \mathrm{SL}_2(q)}_{m \text{ times}}) : \mathrm{Sym}_m \leq \mathrm{Sp}_n(q).$$

Suppose that the action of $N_G(\Delta)$ on Δ is denoted by

$$\rho : N_G(\Delta) \to \mathrm{Sym}(\Delta) \simeq \mathrm{Sym}_m .$$

By [21, Theorem 2], $N_G(\Delta)^\rho = \mathrm{Sym}(\Delta)$. By Lemma 13 it follows that a π-Hall subgroup H^ρ is either maximal in $\mathrm{Sym}(\Delta)$ or equal to $\mathrm{Sym}(\Delta)$. In particular,

$$H^\rho \operatorname{prn} \mathrm{Sym}(\Delta) \quad \text{and} \quad N_{\mathrm{Sym}(\Delta)}(H^\rho) = H^\rho.$$

Since $N_G(S) \leq N_G(\Delta)$ and $g \in N_G(S)$ there exists an element $y \in \langle H, H^g \rangle$ such that $(H^g)^\rho = (H^y)^\rho$. So

$$(gy^{-1})^\rho \in N_{\mathrm{Sym}(\Delta)}(H^\rho) = H^\rho.$$

Denote by A the kernel of ρ. The structure of $N_G(\Delta)$ implies that if $\overline{} : A \to A/O_\infty(A)$ is a natural homomorphism, then $\overline{A}$ possesses a normal subgroup isomorphic to

$$\underbrace{\mathrm{PSL}_2(q) \times \cdots \times \mathrm{PSL}_2(q)}_{m \text{ times}},$$

and index of the subgroup in $\overline{A}$ is a 2-power. By Lemmas 7–9 (we take the class of solvable groups as $\mathfrak{X}$) and Lemma 25 we conclude that π-Hall subgroups of A are pronormal. Now the π-Hall subgroups of HA are pronormal by Lemma 8. Moreover, $gy^{-1} \in HA$ since $(gy^{-1})^\rho \in H^\rho$. Therefore $H^z = H^{gy^{-1}}$ for some $z \in \langle H, H^{gy^{-1}} \rangle \leq \langle H, H^g \rangle$. Let $x = zy$. Then $H^x = H^g$ and $x \in \langle H, H^g \rangle$. $\square$

Thus, if either (6) of Lemma 12 holds or (7) of the same lemma holds and for the preimage $H^* \leq G^* = \mathrm{SL}^\eta(q)$ of H statement (3) of Lemma 18 holds, then $H \operatorname{prn} G$. Notice also that if (7) of Lemma 12 and (1) of Lemma 18 hold, then $H \operatorname{prn} G$ by Lemma 25.

The next lemma allows us to exclude also the case, when (7) of Lemma 12 and (4) of Lemma 18 hold.

Lemma 28. *Let $G = \mathrm{PSL}_4^\eta(q)$, where q is odd. Then $N_G(S) = S$.*

Proof. The claim follows by Lemma 12 since the 2-adic expansion of 4 has only one unit. $\square$

In case, when (7) of Lemma 12 and (2) of Lemma 18 hold, H normalizes a maximal torus of order $(q-\eta)^{n-1}/(n, q-\eta)$ of $G = \mathrm{PSL}_n^\eta(q)$. We consider this case as (5) of Lemma 12 in the next lemma.

Lemma 29. *Assume that $(\star)$ holds and $G = \mathrm{PSL}_n^\eta(q)$, where q is a power of a prime $p \notin \pi$. Suppose also that $q \equiv \eta \pmod 4$ and there exists a maximal H-invariant torus T of order $(q-\eta)^{n-1}/(n, q-\eta)$. Then*

(1) $N_G(T) = N(G, T)$;
(2) $N_G(T)/T \simeq \mathrm{Sym}_n$;
(3) $N_G(T)$ includes $N_G(S)$;
(4) H and H^g are conjugate in $\langle H, H^g \rangle$.

Proof. (1) follows from [10, Lemma 3.10], since T is invariant under a given Sylow 2-subgroup S of G.

(2) Since $N_G(T) = N(G, T)$, the factor group $N_G(T)/T = N(G, T)/T$ is isomorphic to Sym_n (this factor group lies in the Weyl group of G, which is isomorphic to Sym_n; on the other hand, a subgroup of type $T.\mathrm{Sym}_n$ lies in G and so in $N_G(T)$).

(3) Since $H \leq N_G(T)$ and $3 \in \pi$, $N_G(T)$ includes a Sylow 3-subgroup of G. So, by [10, Lemma 3.13], $N_G(S) \leq N_G(T)$.

(4) By (3) of the lemma we have to prove that $H \operatorname{prn} N_G(T)$. By (2) of the lemma, $N_G(T)$ is an extension of an abelian group T by Sym_n. Consider the natural epimorphism $\overline{} : N_G(T) \to N_G(T)/T \simeq \mathrm{Sym}_n$. By [20, Lemma 2.1(a)], $\overline{H}$ is a π-Hall subgroup of Sym_n. Since by $2, 3 \in \pi$ and Lemma 13 each π-Hall subgroup is either maximal in Sym_n or equal to Sym_n, we have $\overline{H} \operatorname{prn} \overline{N_G(T)}$. Taking the class of all abelian groups as $\mathfrak{X}$ in Lemma 9 we see that $H \operatorname{prn} N_G(T)$. $\square$

Thus we have considered all possible cases, except the case, when (7) of Lemma 12 holds, $G = \mathrm{PSL}_{11}^\eta(q)$, and for the preimage $H^* \leq \mathrm{SL}_{11}^\eta(q)$ of H statement (5) of Lemma 18 holds. In particular the following is true.

Lemma 30. *Let $2, 3 \in \pi$ and let q be a power of a prime $p \notin \pi$. Then π-Hall subgroups in $\mathrm{PSL}_n^\eta(q)$, $\mathrm{PGL}_n^\eta(q)$, $\mathrm{SL}_n^\eta(q)$, and $\mathrm{GL}_n^\eta(q)$ for $n \leq 4$ and $n = 8$ are pronormal.*

Proof. For $\mathrm{PSL}_n^\eta(q)$ the lemma is immediate from Lemma 18. For $\mathrm{PGL}_n^\eta(q)$ the claim follows from Lemma 8 since

$$\left| \mathrm{PGL}_n^\eta(q) : \mathrm{PSL}_n^\eta(q) \right| = (n, q-\eta)$$

divides n and so it is a π-number. Finally, $\mathrm{SL}_n^\eta(q)$ and $\mathrm{GL}_n^\eta(q)$ are extensions of abelian groups by $\mathrm{PSL}_n^\eta(q)$ and $\mathrm{PGL}_n^\eta(q)$. The claim of the lemma follows from above arguments and Lemma 9. $\square$

Consider the remaining case. We need

Lemma 31. *Let $G^* = \mathrm{SL}_{11}^\eta(q)$, let q be odd, and $S^* \in \mathrm{Syl}_2(G^*)$. Put $\Delta = \mathrm{Fun}_G(S^*)$. Then*

(1) $|\Delta| = 5$ and S^* acting on Δ has exactly two orbits: Γ of order 4 and Γ_0 of order 1;
(2) Γ are Γ_0 $N_{G^*}(S^*)$-invariant;
(3) if Γ' is an S^*-invariant set of pairwise commuting fundamental subgroups of G^* such that $|\Gamma'| = 4$, then $\Gamma' = \Gamma$.

Proof. Denote by ρ the action of $N_{G^*}(\Delta)$ on Δ. According to [21, Theorem 2]

$$N_{G^*}(\Delta)^\rho = \mathrm{Sym}(\Delta) \simeq \mathrm{Sym}_5\,.$$

S^ρ is a Sylow 2-subgroup of Sym_5 and so it has two orbits on Δ: one orbit of length 4 and the other of length 1. This implies (1). Statement (2) follows from the fact that S^* and Δ are $N_{G^*}(S^*)$-invariant. Finally, Γ' is included in Δ, since Δ is a unique maximal S^*-invariant set of pairwise commuting fundamental subgroup. So Γ' is a union of some orbits of S^* on Δ and by (1) equals Γ. $\square$

Lemma 32. *Let $G^* = \mathrm{SL}^{\eta}_{11}(q)$ be a special linear or unitary group and let V be its natural module equipped with a trivial or unitary form respectively. Assume that $H^* \in \mathrm{Hall}_{\pi}(G^*)$, where $\pi \cap \pi(G^*) = \{2,3\}$, and suppose that H^* lies in a subgroup of type*

$$L = \big(\big(\,\mathrm{GL}^{\eta}_2(q) \wr \mathrm{Sym}_4\,\big) \times \big(\,\mathrm{GL}^{\eta}_1(q) \wr \mathrm{Sym}_3\,\big)\big) \cap G^*.$$

Let $S^* \in \mathrm{Syl}_2(H^*) \subseteq \mathrm{Syl}_2(G^*)$ and $g^* \in N_{G^*}(S^*)$. Then

(1) *H^* leaves invariant the set $\Gamma' = \{K_1, K_2, K_3, K_4\}$ consisting from pairwise commuting fundamental subgroups;*

(2) *Γ' is $N_{G^*}(S^*)$-invariant;*

(3) *if $V_i = [K_i, V]$ and $U = \sum V_i$, then U is invariant under both H^* and $N_{G^*}(S^*)$;*

(4) *the stabilizer M in G^* of U is a subgroup with pronormal π-Hall subgroups;*

(5) *$H^* \,\mathrm{prn}\, G^*$.*

Proof. Consider the subgroup $\big(\,\mathrm{GL}^{\eta}_2(q) \wr \mathrm{Sym}_4\,\big) \cap G^*$ of

$$L = \big(\big(\,\mathrm{GL}^{\eta}_2(q) \wr \mathrm{Sym}_4\,\big) \times \big(\,\mathrm{GL}^{\eta}_1(q) \wr \mathrm{Sym}_3\,\big)\big) \cap G^*,$$

and in the base of the wreath product consider the distinct normal subgroups K_1, K_2, K_3, K_4 isomorphic to $\mathrm{SL}_2(q)$. Clearly, $K_i \notin G^*$ and $K_i \notin L$ for all $i = 1,2,3,4$. Moreover, the set $\Gamma' = \{K_1, K_2, K_3, K_4\}$ is L-invariant and so is H^*-invariant. Statement (1) is proven. (2) follows from Lemma 31. Notice that V_i can be considered as the natural module for K_i; therefore, $\dim(V_i) = 2$ and $V_i \cap V_j = 0$ for $i \neq j$. In particular, $\dim(U) = 8$. Since Γ' is invariant under both H^* and $N_{G^*}(S^*)$, the set $\{V_1, V_2, V_3, V_4\}$ and so the subspace U are also invariant under H^* and $N_{G^*}(S^*)$. Thus (3) is proven. If $\eta = +$, then the stabilizer M of U is an extension of a p-group by a central product $\mathrm{GL}_8(q) \circ \mathrm{GL}_3(3)$ (see [11, Proposition 4.1.17]), and by Lemmas 30, 7, and 9 we conclude that π-Hall subgroups of M are pronormal. If $\eta = -$, then M is isomorphic to a central product $\mathrm{GU}_8(q) \circ \mathrm{GU}_3(q)$ (see [11, Proposition 4.1.4]), and applying again Lemmas 30 and 7, we obtain (4). In view of (3), H^* and every $g^* \in N_{G^*}(S^*)$ are included in M. Now from (4) and Lemma 5 we conclude that $H^* \,\mathrm{prn}\, G^*$. □

We continue the proof of the theorem and consider the remaining case. Assume that (7) of Lemma 12 holds, $G = \mathrm{PSL}^{\eta}_{11}(q)$, and for the preimage $H^* \leq \mathrm{SL}^{\eta}_{11}(q)$ of H statement (5) of Lemma 18 holds. By Lemma 32 we have $H^* \,\mathrm{prn}\, \mathrm{SL}^{\eta}_{11}(q)$. Applying Lemma 6 we conclude the proof of Theorem 1. □

Conclusion

In connection with the proof of Theorem 1 we make a short remark. The proof is naturally divided into the two cases. The first case is that when a Hall subgroup H of a finite simple group G has odd order (equivalently, even index). The proof in this case is reduced to application of the Hall theorem [7, Theorem A1] (Lemma 10) and the Gross theorem [9, Theorem B] (Lemma 11). In the second case, when a Hall subgroup H has even order (equivalently, odd index), the technique is absolutely different. We use the fact that H includes a Sylow 2-subgroup S of G, and so, by Lemma 5, we need to check that H and H^g are conjugate in $\langle H, H^g \rangle$ only for those g, that normalize S. Then we apply the structure of normalizers of Sylow 2-subgroups in finite simple groups obtained by A. S. Kondrat′ev (Lemma 12). This technique could be probably applied in a more general situation, For example, the following conjecture is of interest.

Conjecture 1. *The subgroups of odd index are pronormal in finite simple groups.*

By Lemma 5, Conjecture 1 holds for all finite simple groups possessing a self-normalizing Sylow 2-subgroup (for example, according to the Kondrat′ev theorem (Lemma 12) in the alternating groups of degree greater than 5, in orthogonal groups, and in the most classes of sporadic and exceptional groups).

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