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COCLIQUES OF MAXIMAL SIZE IN THE PRIME GRAPH OF A FINITE SIMPLE GROUP

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A prime graph of a finite group is defined in the following way: the set of vertices of the graph is the set of prime divisors of the order of the group, and two distinct vertices r and s are joined by an edge if there is an element of order rs in the group. We describe all cocliques of maximal size for finite simple groups.

Let G be a finite group, $\pi(G)$ the set of all prime divisors of its order, and $\omega(G)$ the spectrum of G, i.e., the set of its element orders. A graph $GK(G)$ is called the *prime graph*, or the *Gruenberg–Kegel graph*, of G if the vertex set of G equals $\pi(G)$ and two distinct vertices r and s are adjacent iff $rs \in \omega(G)$. For convenience, we say that primes $r, s \in \pi(G)$ are *adjacent* if they are adjacent as vertices of $GK(G)$. Otherwise, r and s are said to be *nonadjacent*.

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The present paper is a continuation of the investigation of the prime graphs of finite simple groups initiated in [1]. We keep the notation created and the conventions made therein. Denote by $t(G)$ the maximal number of prime divisors of G that are pairwise nonadjacent in $GK(G)$. In other words, $t(G)$ is the maximal number of vertices in cocliques of $GK(G)$. (A set of vertices of a graph is called a *coclique* or an *independent set* if its elements are pairwise nonadjacent.) In graph theory, this number is called an *independence number* of a graph. By analogy, we denote by $t(r, G)$ the maximal number of vertices in cocliques of $GK(G)$ containing a prime r and call it an r*-independence number*.

In [1], for every finite non-Abelian simple group G, we gave an arithmetic criterion for adjacency of vertices in the prime graph $GK(G)$. Using this criterion, values for $t(G)$ and $t(2, G)$, as well as a value of $t(p, G)$ for the case where G is a group of Lie type over a field of characteristic p, were determined. Denote by $\rho(G)$ and $\rho(r, G)$, respectively, a coclique of maximal size in $GK(G)$ and a coclique of maximal size in $GK(G)$ containing r. It is not hard to see that $\rho(G)$ and $\rho(r, G)$ are generally not uniquely determined. In [1], all cocliques $\rho(2, G)$, and also all cocliques $\rho(p, G)$ for groups G of Lie type over a field of characteristic p, were described. Moreover, there, at least one coclique $\rho(G)$ was determined for every simple group G, which allowed $t(G)$ to be calculated, but the problem of finding all such cocliques was left unconsidered.

The main objective of the present paper is to find all cocliques of maximal size in the prime graph of a finite simple group G. In order to achieve this goal, we introduce two sets $\Theta(G)$ and $\Theta'(G)$ consisting of some subsets of $\pi(G)$, in which case every coclique $\rho(G)$ of maximal size can appear as $\theta(G) \cup \theta'(G)$, where $\theta(G) \in \Theta(G)$ and $\theta'(G) \in \Theta'(G)$.

THEOREM. Suppose that G is a finite non-Abelian simple group. Then every coclique of maximal size in $GK(G)$ is a union of subsets $\theta(G) \in \Theta(G)$ and $\theta'(G) \in \Theta'(G)$. The sets $\Theta(G)$ and $\Theta'(G)$, together with the maximal size $t(G)$ of cocliques in $GK(G)$, are described in Proposition 1.1 (for alternating groups), in Table 1 (for sporadic groups), and in Tables 2-4 (for groups of Lie type).

Note that [1] contains several misprints and errors. Some of these were spotted by the authors, and others were pointed out by readers. We are grateful to W. Shi, H. He, A. R. Moghaddamfar, M. A. Grechkoseeva, A. A. Buturlakin, and A. Zavarnitsine for their comments. Corrections to all inaccuracies detected in [1] are contained in Sec. 4.

1. SPORADIC AND ALTERNATING GROUPS

The results presented in this section can be readily derived from well-known facts. We cite them below to make our discussion self-contained. Let G be a finite simple sporadic or alternating group. Denote by $\theta(G)$ the intersection of all cocliques of maximal size in $GK(G)$, and by $\Theta(G)$ the set $\{\theta(G)\}$. The set $\Theta'(G)$ is defined as follows. A subset $\theta'(G)$ of $\pi(G) \setminus \theta(G)$ is an element of $\Theta'(G)$ iff $\rho(G) = \theta(G) \cup \theta'(G)$ is a coclique of maximal size in $GK(G)$. Obviously, the sets $\Theta(G)$ and $\Theta'(G)$ are uniquely determined, and $\Theta'(G)$ either is empty or contains at least two elements.

We start with alternating groups. Let $G = Alt_n$ be the alternating group of degree n, with $n \geqslant 5$. Following [1], we denote by $\tau(n)$ the set of primes r with $n/2 \leqslant r \leqslant n$, and by s_n the minimal element of $\tau(n)$. Define the set $\tau'(n)$ as follows. An odd prime r lies in $\tau'(n)$ iff $r < n/2$ and $r + s_n > n$, while 2 is in $\tau'(n)$ iff $4 + s_n > n$.

PROPOSITION 1.1. Let $G = Alt_n$ be the alternating group of degree n, with $n \geqslant 5$. If $|\tau'(n)| \leqslant 1$, then $\theta(G) = \tau(n) \cup \tau'(n)$ is a unique coclique of maximal size in $GK(G)$, and $\Theta'(G) = \varnothing$. If $|\tau'(n)| \geqslant 2$, then $\theta(G) = \tau(n)$, $\Theta'(G) = \{\{r\} \mid r \in \tau'(n)\}$, and every coclique of maximal size in $GK(G)$ is of the form $\tau(n) \cup \{r\}$, where $r \in \tau'(n)$. In any case the set $\Theta(G) = \{\theta(G)\}$ is a singleton and all elements $\theta'(G)$ of $\Theta'(G)$ are singleton subsets of $\pi(G)$.

Proof. The result follows from an adjacency criterion for vertices in $GK(G)$ (see [1, Prop. 1.1]). □

PROPOSITION 1.2. Let G be a simple sporadic group. If $\Theta'(G) = \varnothing$, then $\theta(G)$ is a unique coclique of maximal size in $GK(G)$. If $\Theta'(G) \neq \varnothing$, then every coclique of maximal size is of the form $\theta(G) \cup \theta'(G)$, where $\theta'(G) \in \Theta'(G)$. If $G \neq M_{23}$, then every set $\theta'(G)$ in $\Theta'(G)$ contains precisely one element. The sets $\Theta(G)$ and $\Theta'(G)$, as well as values for $t(G)$, are presented in Table 1.

The **proof** follows readily using [2] or [3]. □

Remark. In columns 3 and 4 of Table 1, elements of $\Theta(G)$ and $\Theta'(G)$, i.e., sets $\theta(G) \in \Theta(G)$ and $\theta'(G) \in \Theta'(G)$, are listed and braces around singleton sets are omitted. In particular, $\Theta(G) = \{\theta(G)\} = \{\{5, 11\}\}$ and $\Theta'(G) = \{\{2\}, \{3\}\}$ for $G = M_{11}$, while $\Theta(G) = \{\theta(G)\} = \{\{11, 23\}\}$ and $\Theta'(G) = \{\{2, 5\}, \{3, 7\}\}$ for $G = M_{23}$.

In addition, we point out yet another essential property of the prime graphs of the groups under consideration.

PROPOSITION 1.3. Suppose that G either is an alternating group of degree n, $n \geqslant 5$, or is a sporadic group distinct from M_{23}. Then the set $\pi(G) \setminus \theta(G)$ is a clique in $GK(G)$.

The **proof** follows from [1, Prop. 1.1; 2].

2. PRELIMINARY RESULTS FOR GROUPS OF LIE TYPE

We write $[x]$ for the integer part of a rational number x. Denote by $\pi(m)$ the set of prime divisors of a natural number m, and by $(m_1, m_2, \ldots, m_s)$ the greatest common divisor of numbers $m_1, m_2, \ldots, m_s$. For a natural number r, the *r-part* of a natural number m is the greatest divisor t of m with $\pi(t) \subseteq \pi(r)$. We write m_r for the r-part of m and write $m_{r'}$ for the quotient m/m_r.

If q is a natural number, r is an odd prime, and $(q, r) = 1$, then $e(r, q)$ denotes a multiplicative order of q modulo r, i.e., a minimal natural number m with $q^m \equiv 1 \pmod r$. For an odd q, we put $e(2, q) = 1$ if $q \equiv 1 \pmod 4$ and put $e(2, q) = 2$ otherwise.

LEMMA 2.1 (corollary to Zsigmondy's theorem [4]). Let q be a natural number greater than 1. For every natural number m, there exists a prime r with $e(r, q) = m$ except for the following

TABLE 1. Cocliques of sporadic groups

G	$t(G)$	$\Theta(G)$	$\Theta'(G)$
M_{11}	3	$\{5, 11\}$	$2, 3$
M_{12}	3	$\{3, 5, 11\}$	$\varnothing$
M_{22}	4	$\{5, 7, 11\}$	$2, 3$
M_{23}	4	$\{11, 23\}$	$\{2, 5\}$, $\{3, 7\}$
M_{24}	4	$\{5, 7, 11, 23\}$	$\varnothing$
J_1	4	$\{7, 11, 19\}$	$2, 3, 5$
J_2	2	7	$2, 3, 5$
J_3	3	$\{17, 19\}$	$2, 3, 5$
J_4	7	$\{11, 23, 29, 31, 37, 43\}$	$5, 7$
Ru	4	$\{7, 13, 29\}$	$3, 5$
He	3	$\{5, 7, 17\}$	$\varnothing$
McL	3	$\{7, 11\}$	$3, 5$
HN	3	$\{11, 19\}$	$3, 5, 7$
HiS	3	$\{7, 11\}$	$2, 3, 5$
Suz	4	$\{5, 7, 11, 13\}$	$\varnothing$
Co_1	4	$\{11, 13, 23\}$	$5, 7$
Co_2	4	$\{7, 11, 23\}$	$3, 5$
Co_3	4	$\{5, 7, 11, 23\}$	$\varnothing$
Fi_{22}	4	$\{5, 7, 11, 13\}$	$\varnothing$
Fi_{23}	5	$\{11, 13, 17, 23\}$	$5, 7$
Fi'_{24}	6	$\{11, 13, 17, 23, 29\}$	$5, 7$
O'N	5	$\{7, 11, 19, 31\}$	$3, 5$
LyS	6	$\{5, 7, 11, 31, 37, 67\}$	$\varnothing$
F_1	11	$\{11, 13, 19, 23, 29, 31, 41, 47, 59, 71\}$	$7, 17$
F_2	8	$\{7, 11, 13, 17, 19, 23, 31, 47\}$	$\varnothing$
F_3	5	$\{5, 7, 13, 19, 31\}$	$\varnothing$

cases: $q = 2$ and $m = 1$; $q = 3$ and $m = 1$; $q = 2$ and $m = 6$.

Remark. In the formulation of the corollary to Zsigmondy's theorem in [1, Lemma 1.4], two exceptions were left out: $m = 1$ and $q = 2$; $m = 1$ and $q = 3$. The two exceptions, however, are not met with in the proofs and arguments of [1] using that corollary.

A prime r with $e(r, q) = m$ is called a *primitive prime divisor* of $q^m - 1$. By Lemma 2.1, such a number exists except for the cases mentioned in the lemma. Given q, we denote by $R_m(q)$ the set of all primitive prime divisors of a number $q^m - 1$, and by $r_m(q)$ any element of $R_m(q)$. For $m \neq 2$, a divisor $k_m(q)$ of $q^m - 1$ is the *greatest primitive divisor* of $q^m - 1$ if $\pi(k_m(q)) = R_m(q)$ and $k_m(q)$

is the greatest divisor with this property, i.e., $k_m(q) = (q^m - 1)_t$, where $t = \prod_{s \in R_m(q)} s$. The greatest primitive divisor $k_2(q)$ of a number $q^2 - 1$ is the greatest divisor of a number $q + 1$ for which $\pi(k_2(q)) = R_2(q)$. Defining a number $e(2, q)$ has a singularity, which gives rise to a singularity in defining the greatest primitive divisor for $m = 2$. Using the definition of $e(2, q)$, we derive the following: $k_1(q) = (q - 1)/2$ if $q \equiv -1 \pmod 4$ and $k_1(q) = q - 1$ otherwise; $k_2(q) = (q + 1)/2$ if $q \equiv 1 \pmod 4$ and $k_2(q) = q + 1$ otherwise. A formula for expressing greatest primitive divisors $k_m(q)$, $m \geqslant 3$, in terms of cyclotomic polynomials $\phi_m(x)$ is given in

LEMMA 2.2 [5]. Let q and m be natural numbers, $q > 1$, $m \geqslant 3$, and $k_m(q)$ be the greatest primitive divisor of $q^m - 1$. Then

$$k_m(q) = \frac{\phi_m(q)}{(\phi_{m_{r'}}(q), r)},$$

where r is the greatest prime divisor of m.

The number q is usually fixed (e.g., by the choice of a group of Lie type G), and we so write R_m, r_m, and k_m instead of $R_m(q)$, $r_m(q)$, and $k_m(q)$, respectively. According to our definitions, if $i \neq j$, then $\pi(R_i) \cap \pi(R_j) = \varnothing$, and hence $(k_i, k_j) = 1$.

LEMMA 2.3 [6, Lemma 6(iii)]. Let q, k, and l be natural numbers. Then:

(a) $(q^k - 1, q^l - 1) = q^{(k,l)} - 1$;

(b) $(q^k + 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if } \frac{k}{(k,l)} \text{ and } \frac{l}{(k,l)} \text{ are odd,} \\ (2, q + 1) & \text{otherwise;} \end{cases}$

(c) $(q^k - 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if } \frac{k}{(k,l)} \text{ is even and } \frac{l}{(k,l)} \text{ is odd,} \\ (2, q + 1) & \text{otherwise.} \end{cases}$

In particular, $(q^k - 1, q^k + 1) \leqslant 2$ for any $q \geqslant 2$ and any $k \geqslant 1$.

We also recall the following statements (see [1, statements (1) and (4)]). Given q, where p is a prime, and an odd prime $c \neq p$, we have

$$c \text{ divides } q^x - 1 \text{ iff } e(c, q) \text{ divides } x; \tag{1}$$

$$\text{if } c \text{ divides } q^x - \epsilon, \text{ where } \epsilon \in \{+1, -1\}, \text{ then } \eta(e(c, q)) \text{ divides } x. \tag{2}$$

The function $\eta(n)$ in (2) is defined as in Prop. 2.4.

In proving Propositions 2.4, 2.5, and 2.7, by ϵ and ϵ_i we denote elements of the set $\{+1, -1\}$. For groups of Lie type, the notation is the same as in [1]. We write $A_n^\varepsilon(q)$, $D_n^\varepsilon(q)$, and $E_6^\varepsilon(q)$, where $\varepsilon \in \{+, -\}$, and define $A_n^+(q) = A_n(q)$, $A_n^-(q) = {}^2A_n(q)$, $D_n^+(q) = D_n(q)$, $D_n^-(q) = {}^2D_n(q)$, $E_6^+(q) = E_6(q)$, and $E_6^-(q) = {}^2E_6(q)$. In studying unitary groups in [1, Prop. 2.2], we defined the function

$$\nu(m) = \begin{cases} m & \text{if } m \equiv 0 \pmod 4, \\ \frac{m}{2} & \text{if } m \equiv 2 \pmod 4, \\ 2m & \text{if } m \equiv 1 \pmod 2. \end{cases} \tag{3}$$

Clearly, $\nu(m)$ is a bijection from $\mathbb{N}$ onto $\mathbb{N}$ and $\nu^{-1}(m) = \nu(m)$. In most cases it is natural to consider linear and unitary groups together. Therefore, we define

$$\nu_\varepsilon(m) = \begin{cases} m & \text{if } \varepsilon = +, \\ \nu(m) & \text{if } \varepsilon = -. \end{cases} \tag{4}$$

PROPOSITION 2.4. Let G be one of the finite simple groups $B_n(q)$ or $C_n(q)$ of Lie type over a field of characteristic p. Define

$$\eta(m) = \begin{cases} m & \text{if } m \text{ is odd}, \\ \frac{m}{2} & \text{otherwise}. \end{cases}$$

Let r and s be odd primes and $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r,q)$ and $l = e(s,q)$, letting $1 \leqslant \eta(k) \leqslant \eta(l)$. Then r and s are nonadjacent if and only if $\eta(k) + \eta(l) > n$ and k and l satisfy the following condition:

$$\frac{l}{k} \text{ is not an odd natural number.} \tag{5}$$

Proof. Necessity. Assume $\eta(k) + \eta(l) \leqslant n$; then there exists a maximal torus T of order $\frac{1}{(2,q-1)}(q^{\eta(k)} + (-1)^k)(q^{\eta(l)} + (-1)^l)(q-1)^{n-\eta(k)-\eta(l)}$ in G (see, e.g., [1, Lemma 1.2(2)]). Both r and s divide $|T|$; hence r and s are adjacent in G. If $\frac{l}{k}$ is an odd integer, then either k and l are both odd, and by Lemma 2.3(a), $q^{\eta(k)} + (-1)^k = q^k - 1$ divides $q^{\eta(l)} + (-1)^l = q^l - 1$, or k and l are both even, and by Lemma 2.3(b), $q^{\eta(k)} + (-1)^k = q^{k/2} + 1$ divides $q^{\eta(l)} + (-1)^l = q^{l/2} + 1$. Again r and s divide $|T|$, where T is a maximal torus of order $\frac{1}{(2,q-1)}(q^{\eta(l)} + (-1)^l)(q-1)^{n-\eta(l)}$ in G (whose existence follows from [1, Lemma 1.2(2)]); so r and s are adjacent.

Sufficiency. Assume to the contrary that $\eta(k)+\eta(l) > n$ and l/k is not an odd natural number, but r and s are adjacent. Then G contains an element g of order rs. The element g is semisimple since $(rs, p) = 1$. Hence g is contained in a maximal torus T of G. From [1, Lemma 1.2(2)], it follows that $|T| = \frac{1}{(2,q-1)}(q^{n_1} - \epsilon_1)(q^{n_2} - \epsilon_2)\dots(q^{n_k} - \epsilon_k)$, where $n_1 + n_2 + \dots + n_k = n$. Up to renumbering, we may assume that r divides $(q^{n_1} - \epsilon_1)$ and s divides either $(q^{n_1} - \epsilon_1)$ or $(q^{n_2} - \epsilon_2)$. Assume first that s divides $(q^{n_2} - \epsilon_2)$. Then (2) implies that $\eta(k)$ divides n_1 and $\eta(l)$ divides n_2. Therefore, $n_1 + n_2 \geqslant \eta(k) + \eta(l) > n$, a contradiction.

Now suppose that both r and s divide $(q^{n_1} - \epsilon_1)$. Again (2) implies that $\eta(k)$ and $\eta(l)$ divide n_1. Also $\eta(k) + \eta(l) > n$ and $\eta(k) \leqslant \eta(l)$, and so $\eta(l) = n_1$. First assume that l is odd. Then $l = \eta(l) = n_1$ and s divides $q^l - 1$. Since s is odd, Lemma 2.3 implies that s does not divide $q^l + 1$, whence $q^{n_1} - \epsilon_1 = q^{n_1} - 1$. Since r divides $q^{n_1} - 1$, by using (1), we see that k divides $n_1 = l$, and hence k is odd. Therefore, $\frac{l}{k}$ is an odd integer, a contradiction with (5). Next suppose that l is even. Then $l/2 = \eta(l) = n_1$ and s divides $q^l - 1$. In view of (1), s does not divide $q^{l/2} - 1$, and hence s divides $q^{l/2} + 1$ and $q^{n_1} - \epsilon_1 = q^{n_1} + 1$. Now (2) implies that $\eta(k)$ divides n_1, and so k divides $2n_1 = l$. By Lemma 2.3(c), r does not divide $q^{l/2} - 1$, and hence k does not divide $l/2$ and $\frac{l}{k}$ is an odd integer, a contradiction with (5). □

PROPOSITION 2.5. Let $G = D_n^\varepsilon(q)$ be a finite simple group of Lie type over a field of characteristic p and $\eta(m)$ be as in Proposition 2.4. Suppose r and s are odd primes and $r, s \in \pi(D_n^\varepsilon(q)) \setminus \{p\}$. Put $k = e(r,q)$, $l = e(s,q)$, and $1 \leqslant \eta(k) \leqslant \eta(l)$. Then r and s are nonadjacent if and only if $2 \cdot \eta(k) + 2 \cdot \eta(l) > 2n - (1 - \varepsilon(-1)^{k+l})$, k and l satisfy (5), and for $\varepsilon = +$, a chain of equalities like

$$n = l = 2\eta(l) = 2\eta(k) = 2k \tag{6}$$

is not true.

Proof. We know of the following inclusions: $\widetilde{B}_{n-1}(q) \leq \widetilde{D}_n^\varepsilon(q) \leq \widetilde{B}_n(q)$ (see [7, Table 2]), where $\widetilde{B}_{n-1}(q)$, $\widetilde{D}_n^\varepsilon(q)$, and $\widetilde{B}_n(q)$ are central extensions of corresponding simple groups and $n \geqslant 4$. Since the Schur multiplier for each of the simple groups $B_{n-1}(q)$, $D_n^\varepsilon(q)$, and $B_n(q)$ has order 1, 2, or 4, it is clear that two odd prime divisors of the order of a simple group isomorphic to $B_n(q)$ or $D_n^\varepsilon(q)$ are adjacent iff the two odd prime divisors are adjacent in every central extension of the group. Hence if two odd prime divisors of $|D_n^\varepsilon(q)|$ are adjacent in $GK(B_{n-1}(q))$, then they are adjacent in $GK(D_n^\varepsilon(q))$, and if two odd prime divisors of $|D_n^\varepsilon(q)|$ are nonadjacent in $GK(B_n(q))$, then they are nonadjacent in $GK(D_n^\varepsilon(q))$. We have the following options:

(i) $\eta(k) + \eta(l) \leqslant n - 1$;

(ii) $\eta(k) + \eta(l) \geqslant n$, l/k is an odd number, and $\eta(l) \leqslant n - 1$;

(iii) $\eta(k) + \eta(l) = n$ and $\frac{l}{k}$ is not an odd natural number;

(iv) $\eta(l) = n$ and $\frac{l}{k}$ is an odd natural number;

(v) $\eta(k) + \eta(l) > n$ and $\frac{l}{k}$ is not an odd natural number.

By Lemma 2.4, if one of (i), (ii) holds, then primes r and s are adjacent in $GK(B_{n-1}(q))$; if (v) is the case, then primes r and s are nonadjacent in $GK(B_n(q))$. Therefore, we are left to consider (iii) and (iv).

(iii) Suppose first that $\eta(k) + \eta(l) = n$ and $\frac{l}{k}$ is not an odd natural number. Since $(rs, p) = 1$, r and s are adjacent in $GK(G)$ iff there exists a maximal torus T of G whose order is divisible by rs. By virtue of [1, Lemma 1.2(3)], the order of T is equal to $\frac{1}{(4, q^n - \varepsilon 1)}(q^{n_1} - \epsilon_1) \cdot \ldots \cdot (q^{n_m} - \epsilon_m)$, where $n_1 + \ldots + n_m = n$ and $\epsilon_1 \cdot \ldots \cdot \epsilon_m = \varepsilon 1$. Up to renumbering, we may assume that r divides $q^{n_1} - \epsilon_1$ and s divides either $q^{n_1} - \epsilon_1$ or $q^{n_2} - \epsilon_2$.

If s divides $q^{n_1} - \epsilon_1$, then (2) implies that numbers $\eta(k)$ and $\eta(l)$ divide n_1. As in the proof of Proposition 2.4, we conclude that r and s are adjacent iff $\frac{l}{k}$ is an odd integer.

Assume now that s divides $q^{n_2} - \epsilon_2$. Then (2) implies that $\eta(k)$ divides n_1 and $\eta(l)$ divides n_2. Thus the following inequalities hold: $n \geqslant n_1 + n_2 \geqslant \eta(k) + \eta(l) = n$. Hence $\eta(k) = n_1$, $\eta(l) = n_2$ and $q^{n_1} - \epsilon_1 = q^{\eta(k)} + (-1)^k$, $q^{n_2} - \epsilon_2 = q^{\eta(l)} + (-1)^l$. If $\varepsilon = -$, then a maximal torus T of order $\frac{1}{(4, q^n + 1)}(q^{\eta(k)} + (-1)^k)(q^{\eta(l)} + (-1)^l)$ in G exists iff k and l have different parity, i.e., if $2n - (1 - \varepsilon(-1)^{k+l}) = 2n - (1 + (-1)^{k+l}) = 2n$. In this case, therefore, r and s are nonadjacent iff $2 \cdot \eta(k) + 2 \cdot \eta(l) > 2n - (1 - \varepsilon(-1)^{k+l})$. If $\varepsilon = +$ and $n_1 \neq n_2$, then a maximal torus T of order $\frac{1}{(4, q^n - 1)}(q^{\eta(k)} + (-1)^k)(q^{\eta(l)} + (-1)^l)$ in G exists iff k and l have the same parity, i.e., if $2n-(1-\varepsilon(-1)^{k+l}) = 2n-(1-(-1)^{k+l}) = 2n$. Hence r and s are now nonadjacent iff $2\cdot\eta(k)+2\cdot\eta(l) > 2n-(1-\varepsilon(-1)^{k+l})$. If $n_1=n_2=n/2$ and $\frac{l}{k}$ is an odd integer, then r and s are adjacent. Assume that $n_1=n_2=n/2$ and $\frac{l}{k}$ is not an odd integer. The last condition entails $l\neq k$, and so the chain of equalities in (6) is true. In this event G contains a maximal torus T of order $\frac{1}{(4,q^n-1)}(q^n-1) = \frac{1}{(4,q^n-1)}(q^{n/2}-1)(q^{n/2}+1)$, hence condition (6) is not satisfied, and so r and s are adjacent.

(iv) Suppose that $\eta(l)=n$ and $\frac{l}{k}$ is an odd natural number. Then there exists a maximal torus T of order $\frac{1}{(4,q^n-\varepsilon 1)}(q^n+(-1)^l)$ in G. (If such a torus does not exist then s does not divide $|G|$.) The fact that $\frac{l}{k}$ is an odd prime implies that r divides $|T|$, and so r and s are adjacent. □

Now we pass to simple exceptional groups of Lie type. Note that the orders of maximal tori of simple exceptional groups were listed in [1, Lemma 1.3]. There, however, a list of orders of tori is incorrect for groups $E_7(q)$ and $E_8(q)$ and for Ree groups ${}^2F_4(2^{2n+1})$ (see, respectively, items (4), (5), and (9) in Lemma 1.3). The list in [1] is remedied in

LEMMA 2.6 [8]. Let $\overline{G}$ be a connected simple exceptional algebraic group of adjoint type and $G = O^{p'}(\overline{G}_\sigma)$ a finite simple exceptional group of Lie type.

(1) For every maximal torus T of $G=E_7(q)$, the number $m=(2,q-1)|T|$ is equal to one of the following: $(q+1)^{n_1}(q-1)^{n_2}$, $n_1+n_2=7$; $(q^2+1)^{n_1}(q+1)^{n_2}(q-1)^{n_3}$, $1\leqslant n_1\leqslant 2$, $2n_1+n_2+n_3=7$, and $m\neq(q^2+1)(q\pm1)^5$; $(q^3+1)^{n_1}(q^3-1)^{n_2}(q^2+1)^{n_3}(q+1)^{n_4}(q-1)^{n_5}$, $1\leqslant n_1+n_2\leqslant 2$, $3n_1+3n_2+2n_3+n_4+n_5=7$, and $m\neq(q^3+\epsilon)(q-\epsilon)^4$, $m\neq(q^3\pm1)(q^2+1)^2$, $m\neq(q^3+\epsilon)(q^2+1)(q+\epsilon)^2$; $(q^4+1)(q^2\pm1)(q\pm1)$; $(q^5\pm1)(q^2-1)$; $(q^5+\epsilon)(q+\epsilon)^2$; $q^7\pm1$; $(q-\epsilon)\cdot(q^2+\epsilon q+1)^3$; $(q^5-\epsilon)\cdot(q^2+\epsilon q+1)$; $(q^3\pm1)\cdot(q^4-q^2+1)$; $(q-\epsilon)\cdot(q^6+\epsilon q^3+1)$; $(q^3-\epsilon)\cdot(q^2-\epsilon q+1)^2$, where $\epsilon\in\{+1,-1\}$. Moreover, for every number m given above, there exists a torus T for which $(2,q-1)|T|=m$.

(2) Every maximal torus T of $G=E_8(q)$ has one of the following orders: $(q+1)^{n_1}(q-1)^{n_2}$, $n_1+n_2=8$; $(q^2+1)^{n_1}(q+1)^{n_2}(q-1)^{n_3}$, $1\leqslant n_1\leqslant 4$, $2n_1+n_2+n_3=8$, and $|T|\neq(q^2+1)^3(q\pm1)^2$, $|T|\neq(q^2+1)(q\pm1)^6$; $(q^3+1)^{n_1}(q^3-1)^{n_2}(q^2+1)^{n_3}(q+1)^{n_4}(q-1)^{n_5}$, $1\leqslant n_1+n_2\leqslant 2$, $3n_1+3n_2+2n_3+n_4+n_5=8$ and $|T|\neq(q^3\pm1)^2(q^2+1)$, $|T|\neq(q^3+\epsilon)(q-\epsilon)^5$, $|T|\neq(q^3+\epsilon)(q^2+1)(q+\epsilon)^3$, $|T|\neq(q^3+\epsilon)(q^2+1)^2(q-\epsilon)$; q^8-1; $(q^4+1)^2$; $(q^4+1)(q^2\pm1)(q\pm1)^2$; $(q^4+1)(q^2-1)^2$; $(q^4+1)(q^3+\epsilon)(q-\epsilon)$; $(q^5+\epsilon)(q+\epsilon)^3$; $(q^5\pm1)(q+\epsilon)^2(q-\epsilon)$; $(q^5+\epsilon)(q^2+1)(q-\epsilon)$; $(q^5+\epsilon)(q^3+\epsilon)$; $(q^6+1)(q^2\pm1)$; $(q^7\pm1)(q\pm1)$; $(q-\epsilon)\cdot(q^2+\epsilon q+1)^3\cdot(q\pm1)$; $(q^5-\epsilon)\cdot(q^2+\epsilon q+1)\cdot(q+\epsilon)$; $(q^3\pm1)\cdot(q^4-q^2+1)\cdot(q\pm1)$; $(q-\epsilon)\cdot(q^6+\epsilon q^3+1)\cdot(q\pm1)$; $(q^3-\epsilon)\cdot(q^2-\epsilon q+1)^2\cdot(q\pm1)$; q^8-q^4+1; $q^8+q^7-q^5-q^4-q^3+q+1$; $q^8-q^6+q^4-q^2+1$; $(q^4-q^2+1)^2$; $(q^6+\epsilon q^3+1)(q^2+\epsilon q+1)$; $q^8-q^7+q^5-q^4+q^3-q+1$; $(q^4+\epsilon q^3+q^2+\epsilon q+1)^2$; $(q^4-q^2+1)(q^2\pm q+1)^2$; $(q^2-q+1)^2\cdot(q^2+q+1)^2$; $(q^2\pm q+1)^4$, where $\epsilon\in\{+1,-1\}$. Moreover, for every number given above, there exists a torus of corresponding order.

(3) Every maximal torus T of $G={}^2F_4(2^{2n+1})$ with $n\geqslant 1$ has one of the following orders: $q^2+\epsilon q\sqrt{2q}+q+\epsilon\sqrt{2q}+1$; $q^2-\epsilon q\sqrt{2q}+\epsilon\sqrt{2q}-1$; q^2-q+1; $(q\pm\sqrt{2q}+1)^2$; $(q-1)(q\pm\sqrt{2q}+1)$; $(q\pm1)^2$; $q^2\pm1$, where $q=2^{2n+1}$ and $\epsilon\in\{+1,-1\}$. Moreover, for every number given above, there

exists a torus of corresponding order.

PROPOSITION 2.7. Let G be a finite simple exceptional group of Lie type over a field of order q and characteristic p, suppose that r and s are odd primes, and assume that $r, s \in \pi(G)\setminus\{p\}$, $k = e(r,q)$, $l = e(s,q)$, and $1 \leqslant k \leqslant l$. Then r and s are nonadjacent if and only if $k \neq l$ and one of the following holds:

(1) $G = G_2(q)$ and either $r \neq 3$ and $l \in \{3,6\}$ or $r = 3$ and $l = 9 - 3k$.

(2) $G = F_4(q)$ and either $l \in \{8,12\}$, or $l = 6$ and $k \in \{3,4\}$, or $l = 4$ and $k = 3$.

(3) $G = E_6(q)$ and either $l = 4$ and $k = 3$, or $l = 5$ and $k \geqslant 3$, or $l = 6$ and $k = 5$, or $l = 8$, $k \geqslant 3$, or $l = 8$, $r = 3$, and $(q-1)_3 = 3$, or $l = 9$, or $l = 12$ and $k \neq 3$.

(4) $G = {}^2E_6(q)$ and either $l = 6$ and $k = 4$, or $l = 8$ and $k \geqslant 3$, or $l = 8$, $r = 3$, and $(q+1)_3 = 3$, or $l = 10$ and $k \geqslant 3$, or $l = 12$ and $k \neq 6$, or $l = 18$.

(5) $G = E_7(q)$ and either $l = 5$ and $k = 4$, or $l = 6$ and $k = 5$, or $l \in \{14,18\}$ and $k \neq 2$, or $l \in \{7,9\}$ and $k \geqslant 2$, or $l = 8$ and $k \geqslant 3$, $k \neq 4$, or $l = 10$ and $k \geqslant 3$, $k \neq 6$, or $l = 12$ and $k \geqslant 4$, $k \neq 6$.

(6) $G = E_8(q)$ and either $l = 6$ and $k = 5$, or $l \in \{7,14\}$ and $k \geqslant 3$, or $l = 9$ and $k \geqslant 4$, or $l \in \{8,12\}$ and $k \geqslant 5$, $k \neq 6$, or $l = 10$ and $k \geqslant 3$, $k \neq 4, 6$, or $l = 18$ and $k \neq 1,2,6$, or $l = 20$ and $r \cdot k \neq 20$, or $l \in \{15,24,30\}$.

(7) $G = {}^3D_4(q)$ and either $l = 6$ and $k = 3$ or $l = 12$.

Proof. Recall that k_m is the greatest primitive divisor of $q^m - 1$ and R_m is the set of all prime primitive divisors of $q^m - 1$. The orders of maximal tori in exceptional groups are given, for instance, in [1, Lemma 1.3] and in Lemma 2.6 above.

(1) Since $|G_2(q)| = q^6(q^2-1)(q^6-1)$, the numbers k and l are in the set $\{1,2,3,6\}$. If $\{k,l\} \subseteq \{1,2\}$, then the existence of a maximal torus of order $q^2 - 1 = (2,q-1)\cdot k_1 \cdot k_2$ implies the existence of an element of order rs; i.e., r and s are adjacent in $GK(G)$. If $l = 3$ (resp. $l = 6$), then an element of order s in G is contained in a unique (up to conjugation) maximal torus of order $q^2+q+1 = (3,q-1)\cdot k_3$ (resp. $q^2-q+1 = (3,q+1)\cdot k_6$). In this case r and s are nonadjacent iff r does not divide $|T|$.

(2) Since $|F_4(q)| = q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)$, the numbers k and l are in the set $\{1,2,3,4,6,8,12\}$. If $l \leqslant 3$, then the existence of a maximal torus of order $(q^3-1)(q+1) = (2,q-1)\cdot(3,q-1)\cdot k_1\cdot k_2\cdot k_3$ implies that primes r and s are adjacent for any $k \leqslant 3$.

If $l = 4$, then an element of order s in G lies in a maximal torus T of order $(q-\epsilon)^2(q^2+1)$ or $(q^2-\epsilon)(q^2+1)$. In particular, the inclusion $\pi(T) \subseteq R_1 \cup R_2 \cup R_4$ holds for T. Moreover, G contains a maximal torus of order $q^4 - 1 = (2,q-1)^2\cdot k_1\cdot k_2\cdot k_4$. In this case, therefore, r and s are nonadjacent iff r does not divide $k_1\cdot k_2\cdot k_4$, i.e., if $k = 3$.

For $l = 6$, each element of order s in G lies in a maximal torus T of order $(q^3+1)(q-\epsilon) = (3,q+1)\cdot k_6\cdot(q+1)\cdot(q-\epsilon)$ or $(q^2-q+1)^2 = (3,q+1)^2\cdot k_6^2$. In particular, the inclusion $\pi(T) \subseteq R_1 \cup R_2 \cup R_6$ holds for T. Moreover, G contains a maximal torus of order $(q^3+1)(q-1) = (2,q-1)(3,q+1)\cdot k_1\cdot k_2\cdot k_6$. Thus r and s are nonadjacent iff $k \in \{3,4\}$.

Finally, for $l = 8$ (resp. $l = 12$), every element of order s in G lies in a maximal torus of order $(2, q-1) \cdot k_8$ (resp. k_{12}). Thus r and s are nonadjacent iff $k \neq 8$ (resp. $k \neq 12$).

(3) Since $|E_6(q)| = \frac{1}{(3,q-1)} q^{36}(q^2-1)(q^5-1)(q^6-1)(q^8-1)(q^9-1)(q^{12}-1)$, the numbers k and l are in the set $\{1, 2, 3, 4, 5, 6, 8, 9, 12\}$. If $l \leqslant 3$, then the existence in G of a maximal torus T of order $\frac{1}{(3,q-1)}(q^3-1)(q^2-1)(q-1) = (2, q-1) \cdot k_3 \cdot k_2 \cdot k_1 \cdot (q-1)^2$ implies that r and s are adjacent.

For $l = 4$, each element of order s in G lies in a maximal torus of order equal to $\frac{1}{(3,q-1)}(q^4 - 1)(q-\epsilon_1)(q-\epsilon_2) = \frac{1}{(3,q-1)} \cdot (2, q-1)^2 \cdot k_1 \cdot k_2 \cdot k_4 \cdot (q-\epsilon_1) \cdot (q-\epsilon_2)$, $\frac{1}{(3,q-1)}(q^3+1)(q^2+1)(q-1) = \frac{1}{(3,q-1)} \cdot (2, q-1)^2 \cdot (3, q+1) \cdot k_6 \cdot k_4 \cdot k_2 \cdot k_1$, or $\frac{1}{(3,q-1)}(q^2+1)^2(q-1)^2 = \frac{1}{(3,q-1)} \cdot (2, q-1)^2 \cdot k_4^2 \cdot (q-1)^2$. Thus r and s are nonadjacent iff $k = 3$.

For $l = 5$, each element of order s in G lies in a maximal torus of order $\frac{1}{(3,q-1)}(q^5-1)(q-\epsilon) = \frac{1}{(3,q-1)}(5, q-1)k_5(q-1)(q-\epsilon)$. Thus r and s are nonadjacent iff $k \in \{3, 4\}$.

For $l = 6$, every element of order s in G lies in a maximal torus of order equal to $\frac{1}{(3,q-1)}(q^3 + 1)(q^2+q+1)(q-\epsilon) = (3, q+1) \cdot k_6 \cdot k_3 \cdot (q+1) \cdot (q-\epsilon)$, $\frac{1}{(3,q-1)}(q^3+1)(q^2+1)(q-1) = \frac{1}{(3,q-1)} \cdot (3, q+1) \cdot k_6 \cdot (2, q-1)^2 \cdot k_1 \cdot k_2 \cdot k_4$, $\frac{1}{(3,q-1)}(q^3+1)(q^2-1)(q-1) = \frac{1}{(3,q-1)} \cdot (3, q+1) \cdot (2, q-1)^2 \cdot k_6 \cdot k_1^2 \cdot k_2^2$, or $\frac{1}{(3,q-1)}(q^2+q+1)(q^2-q+1)^2 = (3, q+1)^2 \cdot k_6^2 \cdot k_3$. Thus r and s are nonadjacent iff $k = 5$.

For $l = 8$, each element of order s in G lies in a maximal torus of order $\frac{1}{(3,q-1)}(q^4+1)(q^2-1) = \frac{1}{(3,q-1)} \cdot (2, q-1)^2 \cdot k_8 \cdot k_2 \cdot k_1$. Hence r and s are nonadjacent iff either $k \geqslant 3$ and $k \neq 8$ or $r = 3$ and $(q-1)_3 = 3$.

For $l = 9$, each element of order s in G lies in a maximal torus of order $\frac{1}{(3,q-1)}(q^6+q^3+1) = k_9$. Hence r and s are nonadjacent iff $k \neq 9$.

Finally, if $l = 12$, then each element of order s in G lies in a maximal torus of order $\frac{1}{(3,q-1)}(q^4 - q^2+1)(q^2+q+1) = k_{12} \cdot k_3$. Hence r and s are nonadjacent iff $k \neq 3, 12$.

(4) Since $|{}^2E_6(q)| = \frac{1}{(3,q+1)} q^{36}(q^2-1)(q^5+1)(q^6-1)(q^8-1)(q^9+1)(q^{12}-1)$, the numbers k and l are in the set $\{1, 2, 3, 4, 6, 8, 10, 12, 18\}$.

For $l \leqslant 4$, the existence in G of maximal tori of orders $\frac{1}{(3,q+1)}(q^3-1)(q^2+1)(q+1) = \frac{1}{(3,q+1)} \cdot (2, q-1)^2 \cdot (3, q-1) \cdot k_1 \cdot k_2 \cdot k_3 \cdot k_4$, $\frac{1}{(3,q+1)}(q^2+1)^2(q+1)^2 = \frac{1}{(3,q+1)} \cdot (2, q-1)^2 \cdot k_4^2 \cdot (q+1)^2$, and $\frac{1}{(3,q+1)}(q^3-1)(q^2-1)(q+1) = \frac{1}{(3,q+1)} \cdot (3, q-1) \cdot (2, q-1)^2 k_3 \cdot k_1^2 \cdot k_2^2$ implies that r and s are adjacent.

For $l = 6$, every element of G of order s is contained in a maximal torus of G of order equal to $\frac{1}{(3,q+1)}(q^3+1)^2 = (3, q+1) \cdot (q+1)^2 \cdot k_6^2$, $\frac{1}{(3,q+1)}(q^3+1)(q+1)(q-\epsilon_1)(q-\epsilon_2) = k_6(q+1)^2(q-\epsilon_1)(q-\epsilon_2)$, $\frac{1}{(3,q+1)}(q^2-q+1)(q^3-\epsilon)(q-1) = k_6 \cdot (q^3-\epsilon) \cdot (q-1)$, $\frac{1}{(3,q+1)}(q^2-q+1)(q^2+q+1)^2 = k_6 \cdot (3, q-1)^2 \cdot k_3^2$, or $\frac{1}{(3,q+1)}(q^4-q^2+1)(q^2-q+1) = k_{12} \cdot k_6$. Thus r and s are nonadjacent iff $k = 4$.

For $l = 8$, every element of G of order s is contained in a maximal torus of G of order $\frac{1}{(3,q+1)}(q^4+1)(q^2-1) = \frac{1}{(3,q+1)} \cdot (2, q-1)^2 \cdot k_8 \cdot k_2 \cdot k_1$. Therefore, r and s are nonadjacent iff either $k \geqslant 3$ and $k \neq 8$ or $r = 3$ and $(q+1)_3 = 3$.

For $l = 10$, every element of G of order s is contained in a maximal torus of G of order $\frac{1}{(3,q+1)}(q^5+1)(q-\epsilon) = \frac{1}{(3,q+1)} \cdot (5, q+1) \cdot k_{10} \cdot (q+1) \cdot (q-\epsilon)$. Hence r and s are nonadjacent iff

$k \geqslant 3$ and $k \neq 10$.

For $l = 12$, every element of G of order s is contained in a maximal torus of G of order $\frac{1}{(3,q+1)}(q^4 - q^2 + 1)(q^2 - q + 1) = k_{12} \cdot k_6$. Consequently, r and s are nonadjacent iff $k \neq 6, 12$.

Lastly, for $l = 18$, every element of G of order s is contained in a maximal torus of G of order $\frac{1}{(3,q+1)}(q^6 - q^3 + 1) = k_{18}$. Hence r and s are nonadjacent iff $k \neq 18$.

(5) Since $|E_7(q)| = \frac{1}{(2,q-1)}q^{63}(q^2 - 1)(q^6 - 1)(q^8 - 1)(q^{10} - 1)(q^{12} - 1)(q^{14} - 1)(q^{18} - 1)$, the numbers k and l are in $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 18\}$. There exist maximal tori of G of orders equal to $\frac{1}{(2,q-1)}(q^5-1)(q^2+q+1) = \frac{1}{(2,q-1)}\cdot(3, q-1)\cdot(5, q-1)\cdot(q-1)\cdot k_5 \cdot k_3$, $\frac{1}{(2,q-1)}(q^4-1)(q^3-1) = (2, q-1)\cdot k_4 \cdot k_1 \cdot k_2 \cdot k_3 \cdot (3, q-1)\cdot(q-1)$, and $\frac{1}{(2,q-1)}(q^5-1)(q^2-1) = k_5 \cdot k_2 \cdot k_1 \cdot (5, q-1)\cdot(q-1)$. Hence r and s are adjacent for $l \leqslant 5$ and $(k, l) \neq (4, 5)$.

If $l = 5$, then every element of order s in G is contained in a maximal torus of order $\frac{1}{(2,q-1)}(q^5 - 1)(q-1)(q-\epsilon)$ or $\frac{1}{(2,q-1)}(q^5-1)(q^2+q+1) = \frac{1}{(2,q-1)}\cdot(3, q-1)\cdot(5, q-1)\cdot(q-1)\cdot k_5 \cdot k_3$. Therefore, r and s are nonadjacent if $(k, l) \neq (4, 5)$.

If $l = 6$, then the existence in G of maximal tori of orders $\frac{1}{(2,q-1)}(q^3 + 1)(q^4 - 1) = (2, q-1)\cdot(3, q+1)\cdot k_1 \cdot k_2 \cdot k_4 \cdot k_6 \cdot (q+1)$ and $\frac{1}{(2,q-1)}(q^6-1)(q-1) = (3, q^2-1)\cdot k_6 \cdot k_3 \cdot k_2 \cdot k_1 \cdot (q-1)$ implies that r and s are adjacent if $k \leqslant 4$ and $k = 6$. Every element of order s in G is contained in a maximal torus of order equal to $\frac{1}{(2,q-1)}(q^3+1)(q^2+1)(q-\epsilon_1)(q-\epsilon_2) = (3, q+1)\cdot(q+1)\cdot k_6 \cdot k_4 \cdot (q-\epsilon_1)\cdot(q-\epsilon_2)$ (with $(\epsilon_1, \epsilon_2) \neq (-1, -1)$), $\frac{1}{(2,q-1)}(q^3 + 1)(q - \epsilon_1)(q - \epsilon_2)(q - \epsilon_3)(q - \epsilon_4) = \frac{1}{(2,q-1)} \cdot (3, q + 1) \cdot k_6 \cdot (q + 1)(q - \epsilon_1)(q - \epsilon_2)(q - \epsilon_3)(q - \epsilon_4)$, $\frac{1}{(2,q-1)}(q^3 + 1)(q^3 - \epsilon_1)(q - \epsilon_2)$, $\frac{1}{(2,q-1)}(q^2 - q + 1)^3(q + 1) = \frac{1}{(2,q-1)}\cdot(3, q+1)^3 \cdot k_6^3 \cdot (q+1)$, $\frac{1}{(2,q-1)}(q^5+1)(q^2-q+1) = \frac{1}{(2,q-1)}\cdot(3, q+1)\cdot(5, q+1)\cdot k_6 \cdot k_{10} \cdot (q+1)$, or $\frac{1}{(2,q-1)}(q^3 - 1)(q^2 - q + 1)^2 = \frac{1}{(2,q-1)} \cdot (q - 1) \cdot (3, q - 1) \cdot k_3 \cdot (3, q + 1)^2 \cdot k_6^2$. For $k = 5$, r does not divide these numbers, and so r and s are nonadjacent iff $k = 5$.

If $l = 7$, then each element of order s in G is contained in a maximal torus of order $\frac{1}{(2,q-1)}(q^7 - 1) = \frac{1}{(2,q-1)} \cdot (7, q - 1) \cdot k_7 \cdot (q - 1)$. Hence r and s are nonadjacent iff $k \neq 1, 7$.

If $l = 8$, then each element of order s in G is contained in a maximal torus of order $\frac{1}{(2,q-1)}(q^4 + 1)(q^2 - \epsilon_1)(q - \epsilon_2) = k_8 \cdot (q^2 - \epsilon_1)(q - \epsilon_2)$. Consequently, r and s are nonadjacent iff $k \geqslant 3$ and $k \neq 4$.

If $l = 9$, then an element of order s in G is contained in a maximal torus of order $\frac{1}{(2,q-1)}(q - 1)(q^6 + q^3 + 1) = \frac{1}{(2,q-1)} \cdot (q - 1) \cdot (3, q - 1) \cdot k_9$. Therefore, r and s are nonadjacent iff $k \neq 1, 9$.

If $l = 10$, then an element of order s in G is contained in a maximal torus of order $\frac{1}{(2,q-1)}(q^5 + 1)(q - 1)(q - \epsilon) = (5, q + 1) \cdot k_{10} \cdot k_2 \cdot k_1 \cdot (q - \epsilon)$ or $\frac{1}{(2,q-1)}(q^5 + 1)(q^2 - q + 1) = \frac{1}{(2,q-1)} \cdot (5, q + 1) \cdot (q + 1) \cdot k_{10} \cdot (3, q + 1) \cdot k_6$. Consequently, r and s are nonadjacent iff $k \geqslant 3$ and $k \neq 6$.

If $l = 12$, then each element of order s in G is contained in a maximal torus of order $\frac{1}{(2,q-1)}(q^3 - \epsilon)(q^4 - q^2 + 1) = \frac{1}{(2,q-1)} \cdot (q^3 - \epsilon) \cdot k_{12}$. Thus r and s are nonadjacent iff $k \geqslant 4$ and $k \neq 6, 12$.

If $l = 14$, then each element of order s in G is contained in a maximal torus of order $\frac{1}{(2,q-1)}(q^7 + 1) = \frac{1}{(2,q-1)} \cdot (7, q + 1) \cdot k_{14} \cdot (q + 1)$. Hence r and s are nonadjacent iff $k \neq 2, 14$.

Finally, if $l = 18$, then an element of order s in G is contained in a maximal torus of order $\frac{1}{(2,q-1)}(q + 1)(q^6 - q^3 + 1) = \frac{1}{(2,q-1)} \cdot (3, q + 1) \cdot (q + 1) \cdot k_{18}$. Therefore, r and s are nonadjacent iff

$k \neq 2, 18$.

(6) Since $|E_8(q)| = q^{120}(q^2-1)(q^8-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)(q^{20}-1)(q^{24}-1)(q^{30}-1)$, the numbers k and l are in the set $\{1,2,3,4,5,6,7,8,9,10,12,14,15,18,20,24,30\}$. By virtue of the fact that G contains maximal tori of orders $(q^3-\epsilon_1)(q^4-1)(q-\epsilon_2)$, $(q^5-1)(q^2+1)(q+1) = (5,q-1)\cdot k_5\cdot(2,q-1)^2\cdot k_4\cdot k_2\cdot k_1$, and $(q^5-1)(q^3-1) = (3,q-1)\cdot(5,q-1)\cdot k_5\cdot k_3\cdot(q-1)^2$, primes r and s are adjacent for $l \leqslant 6$ if $(k,l) \neq (5,6)$.

If $k=5$, then every element of order r in G is contained in a maximal torus of order equal to $(q^5-1)(q^3-1) = (3,q-1)\cdot(5,q-1)\cdot k_5\cdot k_3\cdot(q-1)^2$, $(q^5-1)(q^2+1)(q+1) = (5,q-1)\cdot k_5\cdot(2,q-1)^2\cdot k_4\cdot k_2\cdot k_1$, $(q^5-1)(q^2-1)(q-\epsilon) = (5,q-1)\cdot k_5\cdot(q^2-1)\cdot k_2\cdot k_1\cdot(q-1)\cdot(q-\epsilon)$, $(q^5-1)(q-1)^3$, or $(q^4+q^3+q^2+q+1)^2$, and these orders are all not divisible by s for $l=6$. Consequently, if $(k,l)=(5,6)$, then r and s are nonadjacent.

If $l=7$, then every element of order s in G is contained in a maximal torus of order $(q^7-1)(q-\epsilon) = (7,q-1)\cdot k_7\cdot(q-1)(q-\epsilon)$. Hence r and s are nonadjacent iff $k \geqslant 3$ and $k \neq 7$.

If $l=8$, then an element of order s in G is contained in a maximal torus of order equal to $(q^4+1)(q^4-\epsilon) = (2,q-1)\cdot k_8\cdot(q^4-\epsilon)$, $(q^4+1)(q^3-\epsilon_1)(q-\epsilon_2) = (2,q-1)\cdot k_8\cdot(q^3-\epsilon_1)(q-\epsilon_2)$ (with $(\epsilon_1,\epsilon_2)\neq(-1,-1)$), $(q^4+1)(q^2-1)^2 = (2,q-1)\cdot k_8\cdot(q^2-1)^2$, or $(q^4+1)(q^2-\epsilon_1)(q-\epsilon_2)^2 = (2,q-1)\cdot k_8\cdot(q^2-\epsilon_1)\cdot(q-\epsilon_2)^2$. Therefore, r and s are nonadjacent iff $k=5,7$.

If $l=9$, then an element of order s in G is contained in a maximal torus of order $(q^6+q^3+1)(q-1)(q-\epsilon) = (3,q-1)\cdot k_9\cdot(q-1)\cdot(q-\epsilon)$ or $(q^6+q^3+1)(q^2+q+1) = (3,q-1)^2\cdot k_9\cdot k_3$. Hence r and s are nonadjacent iff $k \geqslant 4$ and $k \neq 9$.

If $l=10$, then every element of order s in G is contained in a maximal torus of order equal to $(q^5+1)(q^2-\epsilon_1)(q-\epsilon_2) = (5,q+1)\cdot k_{10}\cdot(q+1)(q^2-\epsilon_1)(q-\epsilon_2)$ (with $(\epsilon_1,\epsilon_2)\neq(-1,-1)$), $(q^5+1)(q^3+1) = (5,q+1)\cdot k_{10}\cdot(q+1)^2\cdot(3,q+1)\cdot k_6$, $(q^5+1)(q^2-q+1)(q-1) = (5,q+1)\cdot k_{10}\cdot(3,q+1)\cdot k_6\cdot(2,q-1)\cdot k_1\cdot k_2$, $(q^5+1)(q+1)^3 = (5,q+1)\cdot k_{10}\cdot(q+1)^4$, or $(q^4-q^3+q^2-q+1)^2 = ((5,q+1)\cdot k_{10})^2$. Consequently, r and s are nonadjacent iff $k \geqslant 3$ and $k \neq 4,6,10$.

If $l=12$, then every element of order s in G is contained in a maximal torus of order equal to $(q^4-q^2+1)(q^2+1)(q^2-\epsilon) = (2,q-1)\cdot k_{12}\cdot k_4\cdot(q^2-\epsilon)$, $(q^4-q^2+1)(q^2+1)(q-\epsilon)^2 = (2,q-1)\cdot k_{12}\cdot k_4\cdot(q-\epsilon)^2$, $(q^4-q^2+1)(q^3-\epsilon_1)(q-\epsilon_2) = k_{12}\cdot(q^3-\epsilon_1)\cdot(q-\epsilon_2)$, $(q^4-q^2+1)(q^2+q+1)^2 = (3,q-1)^2\cdot k_{12}\cdot k_3^2$, or $(q^4-q^2+1)(q^2-q+1)^2 = (3,q+1)^2\cdot k_{12}\cdot k_6^2$. Therefore, r and s are nonadjacent iff $k \geqslant 5$ and $k \neq 6,12$.

If $l=14$, then an element of order s in G is contained in a maximal torus of order $(q^7+1)(q-\epsilon) = (7,q+1)\cdot k_{14}\cdot(q+1)\cdot(q-\epsilon)$. Hence r and s are nonadjacent iff $k \geqslant 3$ and $k \neq 14$.

If $l=15,24,30$, then an element of order s in G is contained in a maximal torus of order k_l. Consequently, r and s are nonadjacent iff $k \neq l$.

If $l=18$, then an element of order s in G is contained in a maximal torus of order $(q^6-q^3+1)(q+1)(q-\epsilon) = (3,q+1)\cdot k_{18}\cdot(q+1)\cdot(q-\epsilon)$ or $(q^6-q^3+1)(q^2-q+1) = (3,q+1)^2\cdot k_{18}\cdot k_6$. Hence r and s are nonadjacent iff $k \geqslant 3$ and $k \neq 6,18$.

If $l=20$, then every element of order s in G is contained in a maximal torus of order q^8-q^6+

$q^4-q^2+1=(5,q^2+1)\cdot k_{20}$. Therefore, r and s are nonadjacent iff $r\cdot k\neq 20$ (i.e., $r\neq 5$ or $k\neq 4$) and $k\neq 20$.

(7) Since $|{}^3D_4(q)|=q^{12}(q^2-1)(q^6-1)(q^8+q^4+1)$, the numbers k and l are in the set $\{1,2,3,6,12\}$. By virtue of the fact that G contains maximal tori of orders $(q^3-\epsilon_1)(q-\epsilon_2)$, primes r and s are adjacent for $l\leqslant 3$.

If $l=6$, then every element of order s in G is contained in a maximal torus of order $(q^3+1)(q-\epsilon)=(3,q+1)\cdot k_6\cdot(q+1)\cdot(q-\epsilon)$. Hence r and s are nonadjacent iff $k=3$.

If $l=12$, then every element of order s in G is contained in a unique (up to conjugation) maximal torus of order $q^4-q^2+1=k_{12}$, and r and s are nonadjacent iff $k\neq 12$. □

Now we turn to Suzuki and Ree groups.

LEMMA 2.8. Suppose n is a natural number.

(1) Let $m_1(B,n)=2^{2n+1}-1$, $m_2(B,n)=2^{2n+1}-2^{n+1}+1$, and $m_3(B,n)=2^{2n+1}+2^{n+1}+1$. Then $(m_i(B,n),m_j(B,n))=1$ for $i\neq j$.

(2) Let $m_1(G,n)=3^{2n+1}-1$, $m_2(G,n)=3^{2n+1}+1$, $m_3(G,n)=3^{2n+1}-3^{n+1}+1$, and $m_4(G,n)=3^{2n+1}+3^{n+1}+1$. Then $(m_1(G,n),m_2(G,n))=2$ and $(m_i(G,n),m_j(G,n))=1$ otherwise.

(3) Let $m_1(F,n)=2^{2n+1}-1$, $m_2(F,n)=2^{2n+1}+1$, $m_3(F,n)=2^{4n+2}+1$, $m_4(F,n)=2^{4n+2}-2^{2n+1}+1$, $m_5(F,n)=2^{4n+2}-2^{3n+2}+2^{2n+1}-2^{n+1}+1$, and $m_6(F,n)=2^{4n+2}+2^{3n+2}+2^{2n+1}+2^{n+1}+1$. Then $(m_2(F,n),m_4(F,n))=3$ and $(m_i(F,n),m_j(F,n))=1$ otherwise.

Proof. Items (1) and (2) are the same as in [1, Lemma 1.5]. Item (3) is amended in response to Lemma 2.6. □

If G is a Suzuki group or a Ree group over a field of order q, then we denote by $S_i(G)$ the set $\pi(m_i(B,n))$ for $G={}^2B_2(2^{2n+1})$, the set $\pi(m_i(G,n))\setminus\{2\}$ for $G={}^2G_2(3^{2n+1})$, and the set $\pi(m_i(F,n))\setminus\{3\}$ for $G={}^2F_4(2^{2n+1})$. If G is fixed, then we put $S_i=S_i(G)$ and denote by s_i any prime in S_i.

PROPOSITION 2.9. Let G be a finite simple Suzuki or Ree group over a field of characteristic p and r and s be odd primes with $r,s\in\pi(G)\setminus\{p\}$. Then r and s are nonadjacent if and only if one of the following holds:

(1) $G={}^2B_2(2^{2n+1})$, $r\in S_k(G)$, $s\in S_l(G)$, and $k\neq l$.

(2) $G={}^2G_2(3^{2n+1})$, $r\in S_k(G)$, $s\in S_l(G)$, and $k\neq l$.

(3) $G={}^2F_4(2^{2n+1})$ and either $r\in S_k(G)$ and $s\in S_l(G)$, where $k\neq l$ and $\{k,l\}\neq\{1,2\},\{1,3\}$, or $r=3$ and $s\in S_l(G)$, where $l\in\{3,5,6\}$.

The **proof** follows from [1, Lemma 1.3] and Lemmas 2.6 and 2.8. □

3. COCLIQUES FOR GROUPS OF LIE TYPE

Let G be a finite simple group of Lie type with the base field of order q and characteristic p. Every $r\in\pi(G)\setminus\{p\}$ is known to be a primitive prime divisor of q^i-1, where i is bounded by some function depending on the Lie rank of G. For G, we define a set $I(G)$ as follows. If G is neither a Suzuki group nor a Ree group, then $i \in I(G)$ iff $\pi(G) \cap R_i(q) \neq \varnothing$. If G is a Suzuki group or a Ree group, then $i \in I(G)$ iff $\pi(G) \cap S_i(G) \neq \varnothing$. Notice that if $\pi(G) \cap R_i(q) \neq \varnothing$ (resp. $\pi(G) \cap S_i(G) \neq \varnothing$), then $R_i(q) \subseteq \pi(G)$ (resp. $S_i(G) \subseteq \pi(G)$). Thus $\pi(G)$ can be partitioned as follows:

$$\pi(G) = \{p\} \sqcup \bigsqcup_{i \in I(G)} R_i,$$

or

$$\pi(G) = \{2\} \sqcup \bigsqcup_{i \in I(G)} S_i \text{ for Suzuki groups,}$$

or

$$\pi(G) = \{2\} \sqcup \{3\} \cup \bigsqcup_{i \in I(G)} S_i \text{ for Ree groups.}$$

The adjacency criterion implies that two distinct primes in the same class of the partition are always adjacent. Moreover, in most cases an answer to the question whether two primes in distinct classes R_i and R_j (or S_i and S_j) of the partition are adjacent depends only on the choice of indices i and j. We formalize this inference by means of the following definitions.

Definition 3.1. Suppose G is a finite simple group of Lie type with the base field of order q and characteristic p, and G is not isomorphic to ${}^2B_2(2^{2m+1})$, ${}^2G_2(3^{2m+1})$, ${}^2F_4(2^{2m+1})$, and $A_2^{\varepsilon}(q)$. Define the set $M(G)$ to be a subset of $I(G)$ such that $i \in M(G)$ iff the intersection of R_i and every coclique of maximal size in $GK(G)$ is nonempty.

Definition 3.2. Put
$M(G) = I(G)$ if G is equal to ${}^2B_2(2^{2m+1})$ or G is equal to ${}^2G_2(3^{2m+1})$, $m \geqslant 1$;
$M(G) = \{2, 3, 4, 5, 6\}$ if $G = {}^2F_4(2^{2m+1})$, $m \geqslant 2$;
$M(G) = \{5, 6\}$ if $G = {}^2F_4(8)$.

Definition 3.3. Suppose G is a finite simple group of Lie type with the base field of order q and characteristic p, and G is not isomorphic to ${}^2B_2(2^{2m+1})$, ${}^2G_2(3^{2m+1})$, ${}^2F_4(2^{2m+1})$, and $A_2^{\varepsilon}(q)$. A set $\Theta(G)$ consists of all subsets $\theta(G)$ of $\pi(G)$ satisfying the following conditions:
(a) p lies in $\theta(G)$ iff p lies in every coclique of maximal size in $GK(G)$;
(b) for every $i \in M(G)$, exactly one prime in R_i lies in $\theta(G)$.

Definition 3.4. Let $G = {}^2B_2(2^{2m+1})$. A set $\Theta(G)$ consists of all subsets $\theta(G)$ of $\pi(G)$ satisfying the following conditions:
(a) $p = 2$ lies in $\theta(G)$;
(b) for every $i \in M(G)$, exactly one prime in S_i lies in $\theta(G)$.

Definition 3.5. Let $G = {}^2G_2(3^{2m+1})$. A set $\Theta(G)$ consists of all subsets $\theta(G)$ of $\pi(G)$ satisfying the following conditions:
(a) $p = 3$ lies in $\theta(G)$;
(b) for every $i \in M(G)$, exactly one prime in S_i lies in $\theta(G)$.

Definition 3.6. Let $G = {}^2F_4(2^{2m+1})$, with $m \geqslant 1$. A set $\Theta(G)$ consists of all subsets $\theta(G)$ of $\pi(G)$, and for every $i \in M(G)$, exactly one prime in S_i lies in $\theta(G)$.

Definition 3.7. Let $G = A_2^\varepsilon(q)$ and $(q, \varepsilon) \neq (2, -)$. Put $M(G) = \{\nu_\varepsilon(2), \nu_\varepsilon(3)\}$, if $q+\varepsilon 1 \neq 2^k$, and $M(G) = \{\nu_\varepsilon(3)\}$ if $q+\varepsilon 1 = 2^k$. A set $\Theta(G)$ consists of all subsets $\theta(G)$ of $\pi(G)$ satisfying the following conditions:

(a) p lies in $\theta(G)$ iff $q + \varepsilon 1 \neq 2^k$;

(b) if $(q - \varepsilon 1)_3 = 3$, then $3 \in \theta(G)$;

(c) for every $i \in M(G)$, exactly one prime in $R_{\nu_\varepsilon(i)}$ lies in $\theta(G)$ except for the following case: if $2 \in R_{\nu_\varepsilon(2)}$, then 2 does not lie in $\theta(G)$.

Note that a function ν_ε is defined via formula (4).

Definition 3.8. Let G be a finite simple group of Lie type. A subset $\theta'(G)$ of $\pi(G)$ is an element of $\Theta'(G)$ if the union $\rho(G) = \theta(G) \cup \theta'(G)$ is a coclique of maximal size in $GK(G)$ for any $\theta(G) \in \Theta(G)$.

Now we describe cocliques of maximal size for groups of Lie type. First we consider classical groups, reserving groups $A_1(q)$ and $A_2^\varepsilon(q)$ for analysis at the end of the section.

PROPOSITION 3.9. If G is one of the finite simple groups $A_{n-1}(q)$ or ${}^2A_{n-1}(q)$ with the base field of characteristic p and order q, $n \geqslant 4$, then $t(G)$ and the sets $\Theta(G)$ and $\Theta'(G)$ are listed in Table 2.

Proof. It is obvious that the function ν_ε defined in (4) is a bijection on $\mathbb{N}$; hence ν_ε^{-1} is well defined. Moreover, since ν_ε^2 is the identity map, we have $\nu_\varepsilon^{-1} = \nu_\varepsilon$.

Using Lemma 2.1 and information on orders of groups $A_{n-1}^\varepsilon(q)$, we see that a number i is in $I(G)$ if the following conditions hold:

(a) $\nu_\varepsilon(i) \leqslant n$;

(b) $i \neq 1$, for $q = 2, 3$, and $i \neq 6$ for $q = 2$.

By virtue of [1, Props. 2.1, 2.2, 4.1, 4.2], two distinct primes in R_i are adjacent for any $i \in I(G)$.

Denote by $N(G)$ the set $\{i \in I(G) \mid n/2 < \nu_\varepsilon(i) \leqslant n\}$, and by χ any set of the form $\{r_i \mid i \in N(G)\}$ such that $|\chi \cap R_i| = 1$ for all $i \in N(G)$. Note that 1 and 2 cannot be in $N(G)$, since $n \geqslant 4$. In particular, 2 does not lie in any set χ. Let $i \neq j$, $n/2 < \nu_\varepsilon(i), \nu_\varepsilon(j) \leqslant n$. Then $\nu_\varepsilon(i) + \nu_\varepsilon(j) > n$ and $\nu_\varepsilon(i)$ does not divide $\nu_\varepsilon(j)$. In view of [1, Props. 2.1, 2.2], primes r_i and r_j are not adjacent. Thus every set χ forms a coclique in $GK(G)$.

Denote by ξ the set

$$\{p\} \cup \bigcup_{i \in I(G) \setminus N(G)} R_i.$$

By virtue of [1, Props. 2.1, 2.2, 3.1, 4.1, 4.2], every two distinct primes in ξ are adjacent in $GK(G)$. Thus every coclique in $GK(G)$ contains at most one prime in ξ.

Case 1. Let $n \geqslant 7$.

If $q = 2$ and $G = A_{n-1}(q)$, then first we assume that $n \geqslant 13$ to avoid exceptions arising by reason of the fact that $R_6 = \varnothing$ for $q = 2$.

The condition on n implies $|N(G)| \geqslant 4$. By [1, Prop. 3.1], we have $t(p, G) \leqslant 3$, and so p cannot lie in any coclique of maximal size. In virtue of [1, Props. 4.1, 4.2], the same conclusion is true for any primitive prime divisor r_i, where $\nu_\varepsilon(i) = 1$. Therefore, in solving the problem whether a prime r lies in a coclique of maximal size in $GK(G)$, we may assume that r is not a characteristic, nor a divisor of $q - \varepsilon 1$. Hence the main technical tools are Propositions 2.1 and 2.2 in [1].

Suppose $n = 2t + 1$ is odd. If $\nu_\varepsilon(i) \leqslant n/2$, then $N(G)$ contains at least two distinct numbers j and k such that r_i is adjacent to r_j and r_k. Indeed, if $\nu_\varepsilon(i) < t$, then we choose j and k so that $\nu_\varepsilon(j) = t + 1$ and $\nu_\varepsilon(k) = t + 2$, and if $\nu_\varepsilon(i) = t$, then we take j and k such that $\nu_\varepsilon(j) = t + 1$ and $\nu_\varepsilon(k) = 2t$. Thus $M(G) = N(G)$, every set $\theta(G) \in \Theta(G)$ is of the form $\{r_i \mid n/2 < \nu_\varepsilon(i) \leqslant n\}$, $\Theta'(G) = \varnothing$, and $t(G) = t + 1 = [(n + 1)/2]$.

Assume $n = 2t$ is even. If $\nu_\varepsilon(i) < n/2$, then $N(G)$ contains at least two distinct numbers j and k such that r_i is adjacent to r_j and r_k. It suffices to take j and k such that $\nu_\varepsilon(j) = t + 1$ and $\nu_\varepsilon(k) = t + 2$, if $\nu_\varepsilon(i) < t - 1$, and $\nu_\varepsilon(k) = 2t - 2$ if $\nu_\varepsilon(i) = t - 1$. On the other hand, if $\nu_\varepsilon(i) = t = n/2$, then r_i is adjacent to r_j, where $\nu_\varepsilon(j) = 2t = n$, and is nonadjacent to every r_k, where $k \in N(G)$ and $k \neq j$. Thus $M(G) = N(G) \setminus \{\nu_\varepsilon(n)\}$, each set $\theta(G) \in \Theta(G)$ has the form $\{r_i \mid n/2 < \nu_\varepsilon(i) < n\}$, and $\Theta'(G)$ consists of singletons of type $\{r_{\nu_\varepsilon(n/2)}\}$ or $\{r_{\nu_\varepsilon(n)}\}$. Hence $t(G) = t = [(n + 1)/2]$.

It remains to consider the following cases: $q = 2$, $G = A_{n-1}(q)$, and $7 \leqslant n \leqslant 12$. All results (see Table 2) are derived by following essentially the same line of argument as was used in the general case (with due regard for the fact that $R_6 = \varnothing$) and can be easily verified by appealing to [1, Props. 2.1, 2.2, 3.1, 4.1, 4.2]. The most interesting case obtains for $n = 8$. In this event $\Theta(G)$ consists of singleton sets $\theta(G)$ of type $\{r_7\}$, while $\Theta'(G)$ consists of two-elements sets $\theta'(G)$ of type $\{p, r_8\}$, $\{r_4, r_5\}$, $\{r_3, r_8\}$, or $\{r_5, r_8\}$.

Case 2. Let $n = 6$.

First we assume that $q \neq 2$. Then $N(G) = \{\nu_\varepsilon(4), \nu_\varepsilon(5), \nu_\varepsilon(6)\}$ and $|N(G)| = 3$. Therefore, a set of the form $\{r_{\nu_\varepsilon(4)}, r_{\nu_\varepsilon(5)}, r_{\nu_\varepsilon(6)}\}$ forms a coclique in $GK(G)$, and $t(G) \geqslant 3$. Arguing as in the previous case, we conclude that any prime $r_{\nu_\varepsilon(3)}$ is adjacent to $r_{\nu_\varepsilon(6)}$ and a set of the form $\{r_{\nu_\varepsilon(3)}, r_{\nu_\varepsilon(4)}, r_{\nu_\varepsilon(5)}\}$ is a coclique. By virtue of [1, Prop. 3.1], a set of the form $\{p, r_{\nu_\varepsilon(5)}, r_{\nu_\varepsilon(6)}\}$ is a coclique and p is adjacent to any prime $r_{\nu_\varepsilon(4)}$. If $\nu_\varepsilon(i) = 2$, then r_i is adjacent to p, and it is not adjacent to r_j iff $\nu_\varepsilon(j) = 5$. Hence $t(r_i, G) = 2$. Let r be a divisor of $q - \varepsilon 1$. If $r \neq 3$ or $(q - \varepsilon 1)_3 \neq 3$, then it follows from [1, Props. 4.1, 4.2] that $t(r, G) = 2$, and if $r = 3$ and $(q - \varepsilon 1)_3 = 3$, then $t(3, G) = 3$ and a set of the form $\{3, r_{\nu_\varepsilon(5)}, r_{\nu_\varepsilon(6)}\}$ is a coclique in $GK(G)$. Therefore, if $q \neq 2$, then $M(G) = \{\nu_\varepsilon(5)\}$ and every set $\theta(G) \in \Theta(G)$ has the form $\{r_{\nu_\varepsilon(5)}\}$. Each set $\theta'(G) \in \Theta'(G)$ is a two-element set of the form $\{p, r_{\nu_\varepsilon(6)}\}$, $\{r_{\nu_\varepsilon(3)}, r_{\nu_\varepsilon(4)}\}$, or $\{r_{\nu_\varepsilon(4)}, r_{\nu_\varepsilon(6)}\}$, and also of the form $\{3, r_{\nu_\varepsilon(6)}\}$ if $(q - \varepsilon 1)_3 = 3$.

Let $G = A_5(2)$. Since $R_6 = R_1 = \varnothing$, each set $\theta(G) \in \Theta(G)$ is represented as $\{r_3, r_4, r_5\}$ and $\Theta'(G) = \varnothing$.

Let $G = {}^2A_5(2)$. Since $R_6 = R_{\nu(3)} = \varnothing$ and $(q + 1)_3 = 3$, every $\theta(G) \in \Theta(G)$ has the form

$\{r_3, r_{10}\}$ and every $\theta'(G) \in \Theta'(G)$ is a singleton set of type $\{p\}$, $\{r_4\}$, or $\{3\}$.

Thus $t(G) = 3$ holds for any q.

Case 3. Let $n = 5$.

We have $N(G) = \{\nu_\varepsilon(4), \nu_\varepsilon(5)\}$ and $|N(G)| = 2$, whence $t(G) \leqslant 3$. Assume now that $G \neq {}^2A_4(2)$. Then $R_{\nu_\varepsilon(3)}$ is always nonempty, and a set of the form $\{r_{\nu_\varepsilon(3)}, r_{\nu_\varepsilon(4)}, r_{\nu_\varepsilon(5)}\}$ is a coclique in $GK(G)$. By virtue of [1, Prop. 3.1], a set of the form $\{p, r_{\nu_\varepsilon(4)}, r_{\nu_\varepsilon(5)}\}$, too, is a coclique. A prime $r_{\nu_\varepsilon(2)}$ is adjacent to p, and it is not adjacent to r_j iff $\nu_\varepsilon(j) = 5$. Let r be a divisor of $q - \varepsilon 1$. If $r \neq 5$ or $(q - \varepsilon 1)_5 \neq 5$, then it follows from [1, Props. 4.1, 4.2] that $t(r, G) = 2$, and if $r = 5$ and $(q - \varepsilon 1)_5 = 5$, then $t(5, G) = 3$ and a set of the form $\{5, r_{\nu_\varepsilon(4)}, r_{\nu_\varepsilon(5)}\}$ is a coclique in $GK(G)$. Therefore, if $G \neq {}^2A_4(2)$, then $M(G) = N(G)$ and any set $\theta(G) \in \Theta(G)$ is of the form $\{r_{\nu_\varepsilon(4)}, r_{\nu_\varepsilon(5)}\}$. Every $\theta'(G) \in \Theta'(G)$ is a singleton set of type $\{p\}$ or $\{r_{\nu_\varepsilon(3)}\}$, and also of type $\{5\}$ if $(q - \varepsilon 1)_5 = 5$.

Let $G = {}^2A_4(2)$. Since $R_6 = R_{\nu(3)} = \varnothing$ and $(q+1)_5 = 1$, each set $\theta(G) \in \Theta(G)$ is represented as $\{p, r_4, r_{10}\}$ and $\Theta'(G) = \varnothing$.

Thus $t(G) = 3$ holds for any q.

Case 4. Let $n = 4$.

First we assume that $G \neq {}^2A_3(2)$. Then $N(G) = \{\nu_\varepsilon(3), \nu_\varepsilon(4)\}$ and $|N(G)| = 2$, whence $t(G) \leqslant 3$. In view of [1, Prop. 3.1], a set of the form $\{p, r_{\nu_\varepsilon(3)}, r_{\nu_\varepsilon(4)}\}$ is a coclique in $GK(G)$. A prime $r_{\nu_\varepsilon(2)}$ is adjacent to p and to any prime $r_{\nu_\varepsilon(4)}$. If r is an odd prime divisor of $q - \varepsilon 1$, then it follows from [1, Props. 4.1, 4.2] that $t(r, G) = 2$. The same conclusion is true for $r = 2$ iff $(q - \varepsilon 1)_2 \neq 4$, but if $(q - \varepsilon 1)_2 = 4$, then $\{2, r_{\nu_\varepsilon(3)}, r_{\nu_\varepsilon(4)}\}$ is a coclique. Therefore, if $(q - \varepsilon 1)_2 \neq 4$, then every set $\theta(G) \in \Theta(G)$ is represented as $\{p, r_{\nu_\varepsilon(3)}, r_{\nu_\varepsilon(4)}\}$ and $\Theta'(G) = \varnothing$. But if $(q - \varepsilon 1)_2 = 4$, then $M(G) = N(G)$, every set $\theta(G) \in \Theta(G)$ is represented as $\{r_{\nu_\varepsilon(3)}, r_{\nu_\varepsilon(4)}\}$, and $\Theta'(G) = \{\{2\}, \{p\}\}$. Thus $t(G) = 3$ in any case.

Let $G = {}^2A_3(2)$. Since $R_6 = R_{\nu(3)} = \varnothing$ and $(q+1)_4 = 1$, we conclude that $\Theta(G) = \{\{r_4\}\}$ and $\Theta'(G) = \{\{p\}, \{r_2\}\}$. In this event $t(G) = 2$. □

PROPOSITION 3.10. If G is one of the finite simple groups $B_n(q)$, $C_n(q)$, $D_n(q)$, or ${}^2D_n(q)$ with the base field of characteristic p and order q, then $t(G)$ and the sets $\Theta(G)$ and $\Theta'(G)$ are listed in Table 3.

Proof. Using Lemma 2.1 and information on the orders of the groups under consideration, we see that a number i is in $I(G)$ if the following conditions hold:

(a) $\eta(i) \leqslant n$;

(b) $i \neq 1$, for $q = 2, 3$, and $i \neq 6$ for $q = 2$;

(c) $i \neq 2n$ for $G = D_n(q)$;

(d) $i \neq n$ for $G = {}^2D_n(q)$ and n odd.

From [1, Props. 4.3, 4.4] and Propositions 2.4 and 2.5, it follows that two distinct primes in R_i are adjacent for any $i \in I(G)$.

Denote by $N(G)$ the set $\{i \in I(G) \mid n/2 < \eta(i) \leqslant n\}$, and by χ any set of the form $\{r_i \mid i \in$

$N(G)\}$ such that $|\chi \cap R_i| = 1$ for all $i \in N(G)$. Let $i \neq j$ and $n/2 < \eta(i), \eta(j) \leqslant n$. We have $\eta(i) + \eta(j) > n$. Suppose i/j is an odd natural number. Then i and j have the same parity, and so $\eta(i)/\eta(j)$, too, is an odd natural number. Since $i \neq j$, we have $\eta(j) > 2\eta(i) > n$, contrary to the choice of j. Thus i/j is not an odd number. By virtue of Propositions 2.4 and 2.5, primes r_i and r_j are nonadjacent. Thus every set χ forms a coclique in $GK(G)$.

Denote by ξ the set

$$\{p\} \cup \bigcup_{i \in I(G) \setminus N(G)} R_i.$$

In view of Propositions 2.4 and 2.5, every two distinct primes in ξ are adjacent in $GK(G)$ (see also [1, Props. 3.1, 4.3, 4.4]). Thus every coclique in $GK(G)$ contains at most one prime in ξ.

Our next goal is to determine cocliques of maximal size, treating groups of various types separately. In view of [1, Thm. 7.5], $GK(B_n(q)) = GK(C_n(q))$, and so groups of types B_n and C_n can be analyzed in tandem.

Case 1. Let G be one of the simple groups $B_n(q)$ or $C_n(q)$.

Suppose $n = 2$. If $q = 2$, then the group G is not simple, and so we may assume that $q \geqslant 3$. If $q = 3$, then $I(G) = \{2, 4\}$, and if $q > 3$, then $I(G) = \{1, 2, 4\}$. In either case $N(G) = \{4\}$. Since r_4 is not adjacent to any $r \in \xi$, we have $M(G) = N(G) = \{4\}$ and every $\theta(G) \in \Theta(G)$ is a singleton containing exactly one element r_4 of R_4. Every $\theta'(G) \in \Theta'(G)$ is a singleton containing exactly one element of ξ. Thus $t(G) = 2$.

Assume $n = 3$. If $q \neq 2$ then $N(G) = \{3, 6\}$ and $N(G) = \{3\}$ otherwise. The set $\{1, 2, 3, 4, 6\}$ includes $I(G)$, and hence $\xi = \{p\} \cup R_1 \cup R_2 \cup R_4$, where $\{p\}$, R_2, and R_4 are always nonempty. A prime p and any prime r_4 are adjacent one to another and are nonadjacent to every r_i with $i \in N(G)$. On the other hand, primes r_i and r_j are adjacent for $i \in \{1, 2\}$ and $j \in \{3, 6\}$. Therefore, $M(G) = N(G)$, the set $\theta(G)$ is of the form $\{r_3\}$ for $q = 2$, and of the form $\{r_3, r_6\}$ otherwise. The set $\Theta'(G)$ consists of singleton sets of types $\{p\}$, $\{r_2\}$, and $\{r_4\}$ if $q = 2$, and of types $\{p\}$ and $\{r_4\}$ otherwise. Thus $t(G) = 2$ for $q = 2$ and $t(G) = 3$ otherwise.

Let $n \geqslant 4$. Consider four different cases depending on the residue of n modulo 4. We write $n = 4t + k$, where $k = 0, 1, 2, 3$ and $t \geqslant 1$. If $q = 2$, then we assume that $t > 1$ in order to avoid exceptional instances necessitated by the condition that $R_6 = \varnothing$ for $q = 2$.

Suppose $n = 4t$. Then

$$N(G) = \{2t + 1, 2t + 3, \ldots, 4t - 1, 4t + 2, 4t + 4, \ldots, 8t\},$$

and hence $|N(G)| = 3t$. In view of the adjacency criterion, r_{4t} is nonadjacent to every r_i, where $i \in N(G)$. Therefore, $t(G) \geqslant 3t + 1 \geqslant 4$. By [1, Props. 3.1, 4.3], we have $t(2, G) \leqslant t(p, G) < 4$, and so p and 2 cannot lie in any coclique of maximal size. Furthermore, if $\eta(i) < n/2 = 2t$, then any odd r_i is adjacent to r_{4t}, r_{2t+1}, and r_{4t+2}. Hence $M(G) = N(G) \cup \{n\}$, every set $\theta(G) \in \Theta(G)$ is of the form $\{r_i \mid n/2 \leqslant \eta(i) \leqslant n\}$, $\Theta'(G) = \varnothing$, and $t(G) = 3t + 1 = [(3n + 5)/4]$.

Assume $n = 4t + 1$. Then

$$N(G) = \{2t + 1, 2t + 3, \dots, 4t + 1, 4t + 2, 4t + 4, \dots, 8t + 2\};$$

hence $|N(G)| = 3t + 2$ and $t(G) \geqslant 5$. By virtue of [1, Props. 3.1, 4.3], $t(2, G) \leqslant t(p, G) < 4$. Therefore, p and 2 cannot lie in any coclique of maximal size. If $\eta(i) < n/2$, then any odd r_i is adjacent to r_{2t+1} and r_{4t+2}, and so it cannot lie in any coclique of maximal size. Thus $M(G) = N(G)$, every set $\theta(G) \in \Theta(G)$ is of the form $\{r_i \mid n/2 < \eta(i) \leqslant n\} = \{r_i \mid n/2 \leqslant \eta(i) \leqslant n\}$, $\Theta'(G) = \varnothing$, and $t(G) = 3t + 2 = [(3n + 5)/4]$.

Suppose $n = 4t + 2$. Then

$$N(G) = \{2t + 3, 2t + 5, \dots, 4t + 1, 4t + 4, 4t + 6, \dots, 8t + 4\};$$

hence $|N(G)| = 3t + 1$ and $t(G) \geqslant 4$. Since $t(2, G) \leqslant t(p, G) < 4$, primes p and 2 cannot lie in any coclique of maximal size. All primes r_{2t+1} and r_{4t+2} are adjacent one to another and are nonadjacent to every r_i with $i \in N(G)$. If $\eta(i) < n/2$, then r_i is adjacent to r_{4t+4}, r_{4t+2}, and r_{2t+1}. Therefore, $N(G) = M(G)$, every $\theta(G) \in \Theta(G)$ is of the form $\{r_i \mid n/2 < \eta(i) \leqslant n\}$, and $\Theta'(G)$ consists of singleton sets of type $\{r_{2t+1}\}$ or $\{r_{4t+2}\}$. Thus $t(G) = 3t + 2 = [(3n + 5)/4]$.

Assume $n = 4t + 3$. Then

$$N(G) = \{2t + 3, 2t + 5, \dots, 4t + 3, 4t + 4, 4t + 6, \dots, 8t + 6\};$$

hence $|N(G)| = 3t + 3$ and $t(G) \geqslant 6$. Since $t(2, G) \leqslant t(p, G) < 4$, primes p and 2 cannot lie in any coclique of maximal size. If $\eta(i) < 2t + 1$, then r_i is adjacent to r_{4t+4}, r_{4t+6}, and r_{2t+3}. Let $\eta(i) = 2t + 1$. If r_i is adjacent to r_j with $j \in N(G)$, then $j = 4t + 4$. Since there are two distinct numbers $2t+1$ and $4t+2$ the value of the function η on which is equal to $2t+1$, we conclude that $\Theta'(G)$ consists of singleton sets of one of the following three types: $\{r_{4t+4}\}$, $\{r_{2t+1}\}$, or $\{r_{4t+2}\}$. Thus $M(G) = N(G) \setminus \{4t + 4\}$, every $\theta(G) \in \Theta(G)$ is of the form $\{r_i \mid (n + 1)/2 < \eta(i) \leqslant n\}$, and $t(G) = 3t + 3 = [(3n + 5)/4]$.

It remains to handle the cases where $q = 2$ and $n = 4 + k$, with $k = 0, 1, 2, 3$. All results (see Table 3) are derived by following essentially the same arguments as were used in the general case (with the fact that $R_{4t+2} = R_6 = \varnothing$ in mind) and can be easily verified by appealing to Proposition 2.4 and [1, Props. 3.1, 4.3].

Case 2. Let $G = D_n(q)$.

Suppose $n = 4$. If $q \neq 2$ then $N(G) = \{3, 6\}$; otherwise, $N(G) = \{3\}$. The set $\{1, 2, 3, 4, 6\}$ includes $I(G)$, and so $\xi = \{p\} \cup R_1 \cup R_2 \cup R_4$, where $\{p\}$, R_2, and R_4 are always nonempty. A prime p and any prime r_4 are adjacent one to another and are nonadjacent to every r_i with $i \in N(G)$. On the other hand, primes r_i and r_j are adjacent for $i \in \{1, 2\}$ and $j \in \{3, 6\}$. Therefore, $M(G) = N(G)$ and $\theta(G)$ is of the form $\{r_3\}$ for $q = 2$, and of the form $\{r_3, r_6\}$ otherwise. The set $\Theta'(G)$ consists of singleton sets of types $\{p\}$, $\{r_2\}$, and $\{r_4\}$ if $q = 2$, and of types $\{p\}$ and $\{r_4\}$ otherwise. Thus $t(G) = 2$ for $q = 2$ and $t(G) = 3$ otherwise.

Let $n > 4$. Consider four different cases depending on the residue of n modulo 4. Put $n = 4t+k$, where $k = 0, 1, 2, 3$ and $t \geqslant 1$. If $q = 2$ then we assume that $t > 1$ to avoid exceptional instances arising by virtue of the fact that $R_6 = \varnothing$ for $q = 2$.

Suppose $n = 4t > 4$. Then

$$N(G) = \{2t+1, 2t+3, \ldots, 4t-1, 4t+2, 4t+4, \ldots, 8t-2\},$$

and hence $|N(G)| = 3t - 1 > 4$. It follows from [1, Props. 3.1, 4.4] that $t(2, G) \leqslant t(p, G) < 4$, and so p and 2 cannot lie in any coclique of maximal size. In view of the adjacency criterion, r_{4t} is nonadjacent to every r_i, where $i \in N(G)$. On the other hand, any prime r_{4t-2} is adjacent to r_{4t} and r_{4t+2}, and any prime r_{2t-1} is adjacent to r_{4t} and r_{2t+1}. Moreover, if $\eta(i) < 2t - 1$, then r_i is adjacent to at least three primes in any set χ. Therefore, $M(G) = N(G) \cup \{n\}$, every set $\theta(G) \in \Theta(G)$ has the form $\{r_i \mid n/2 \leqslant \eta(i) \leqslant n,\ i \neq 2n\}$, $\Theta'(G) = \varnothing$, and $t(G) = 3t = [(3n+1)/4]$.

Assume $n = 4t + 1$. Then

$$N(G) = \{2t+1, 2t+3, \ldots, 4t+1, 4t+2, 4t+4, \ldots, 8t\},$$

whence $|N(G)| = 3t + 1 \geqslant 4$. In view of [1, Props. 3.1, 4.4], we have $t(2, G) \leqslant t(p, G) < 4$. Hence p and 2 cannot lie in any coclique of maximal size. If $\eta(i) < 2t$, then any prime r_i is adjacent to r_{4t+2} and r_{2t+1}. If $i = 4t$, then r_i is adjacent to r_j, where $j \in N(G)$, iff $j = 4t + 2$. Thus $M(G) = N(G) \setminus \{n+1\}$, every set $\theta(G) \in \Theta(G)$ has the form $\{r_i \mid n/2 < \eta(i) \leqslant n,\ i \neq n+1, 2n\}$, and $\Theta'(G)$ consists of singleton sets of type $\{r_{4t}\}$ or $\{r_{4t+2}\}$. Therefore, $t(G) = 3t+1 = [(3n+1)/4]$.

Suppose $n = 4t + 2$. Then

$$N(G) = \{2t+3, 2t+5, \ldots, 4t+1, 4t+4, 4t+6, \ldots, 8t+2\},$$

whence $|N(G)| = 3t \geqslant 3$. All primes r_{2t+1} and r_{4t+2} are adjacent one to another and are nonadjacent to every r_i for which $i \in N(G)$. Hence $t(G) \geqslant 4$. Since $t(2, G) \leqslant t(p, G) < 4$, primes p and 2 cannot lie in any coclique of maximal size. If $\eta(i) < n/2$, then r_i is adjacent to r_{2t+1}, r_{4t+2}, and r_{4t+4}. Therefore, $N(G) = M(G)$, every set $\theta(G) \in \Theta(G)$ is of the form $\{r_i \mid n/2 < \eta(i) \leqslant n,\ i \neq 2n\}$, and $\Theta'(G)$ consists of singleton sets of type $\{r_{2t+1}\}$ or $\{r_{4t+2}\}$. Thus $t(G) = 3t + 1 = [(3n+1)/4]$.

Assume $n = 4t + 3$. Then

$$N(G) = \{2t+3, 2t+5, \ldots, 4t+3, 4t+4, 4t+6, \ldots, 8t+4\},$$

and hence $|N(G)| = 3t + 2 \geqslant 5$. Since $t(2, G) \leqslant t(p, G) < 4$, primes p and 2 cannot lie in a coclique of maximal size. By the adjacency criterion, r_{2t+1} is nonadjacent to every r_i, where $i \in N(G)$. On the other hand, if $\eta(i) < 2t + 1$ or $i = 4t + 2$, then r_i is adjacent to r_{4t+4} and r_{2t+1}. Therefore, $M(G) = N(G) \cup \{(n-1)/2\}$, every set $\theta(G) \in \Theta(G)$ is of the form $\{r_i \mid (n-1)/2 \leqslant \eta(i) \leqslant n,\ i \neq 2n, n-1\}$, $\Theta'(G) = \varnothing$, and $t(G) = 3t + 3 = (3n+3)/4$. For the case where $n = 4t+3$, every coclique of maximal size does not contain any primes r_{4t+2}, and we are so done away with the group $D_7(2)$ too.

It remains to consider the cases where $q = 2$ and $n = 4+k$, with $k = 1, 2$. Both results (see Table 3) are derived by following essentially the same arguments as were used in the general case (with due regard for the fact that $R_6 = \varnothing$) and can be easily verified by appealing to Proposition 2.5 and [1, Props. 3.1, 4.4].

Case 3. Let $G = {}^2D_n(q)$.

Suppose $n = 4$. If $q \neq 2$ then $N(G) = \{3, 6, 8\}$; otherwise, $N(G) = \{3, 8\}$. The set $\{1, 2, 3, 4, 6, 8\}$ includes $I(G)$, and so $\xi = \{p\} \cup R_1 \cup R_2 \cup R_4$, where $\{p\}$, R_2, and R_4 are always nonempty. A prime p and any prime r_4 are adjacent one to another and are nonadjacent to every r_i with $i \in N(G)$. On the other hand, primes r_i and r_j are adjacent for $i \in \{1, 2\}$ and $j \in \{3, 6\}$. Therefore, $M(G) = N(G)$ and $\theta(G)$ is of the form $\{r_3, r_8\}$ for $q = 2$, and of the form $\{r_3, r_6, r_8\}$ otherwise. The set $\Theta'(G)$ consists of singleton sets of types $\{p\}$ and $\{r_4\}$. Thus $t(G) = 3$ for $q = 2$ and $t(G) = 4$ otherwise.

Let $n > 4$. Consider four different cases depending on the residue of n modulo 4. Put $n = 4t+k$, where $k = 0, 1, 2, 3$ and $t \geqslant 1$. If $q = 2$, then we assume that $t > 1$ to avoid exceptional instances necessitated by the condition that $R_6 = \varnothing$ for $q = 2$.

Assume $n = 4t > 4$. Then

$$N(G) = \{2t+1, 2t+3, \ldots, 4t-1, 4t+2, 4t+4, \ldots, 8t\},$$

and hence $|N(G)| = 3t > 4$. By [1, Props. 3.1, 4.4], $t(2, G) \leqslant t(p, G) \leqslant 4$, and so p and 2 cannot lie in any coclique of maximal size. By virtue of the adjacency criterion, r_{4t} is nonadjacent to every r_i, where $i \in N(G)$. On the other hand, any prime r_{2t-1} is adjacent to r_{4t} and r_{4t+2}, and any prime r_{4t-2} is adjacent to r_{4t} and r_{2t+1}. Moreover, if $\eta(i) < 2t-1$, then r_i is adjacent to at least three primes in χ. Therefore, $M(G) = N(G) \cup \{n\}$, every set $\theta(G) \in \Theta(G)$ has the form $\{r_i \mid n/2 \leqslant \eta(i) \leqslant n\}$, $\Theta'(G) = \varnothing$, and $t(G) = 3t+1 = [(3n+4)/4]$.

Suppose $n = 4t+1$. Then

$$N(G) = \{2t+1, 2t+3, \ldots, 4t-1, 4t+2, 4t+4, \ldots, 8t+2\},$$

and so $|N(G)| = 3t+1 \geqslant 4$. In view of [1, Props. 3.1, 4.4], $t(2, G) \leqslant t(p, G) < 4$. Therefore, p and 2 cannot lie in any coclique of maximal size. If $\eta(i) < 2t$, then any prime r_i is adjacent to r_{4t+2} and r_{2t+1}. If $i = 4t$, then r_i is adjacent to r_j, where $j \in N(G)$, iff $j = 2t+1$. Thus $M(G) = N(G) \setminus \{(n+1)/2\}$, every set $\theta(G) \in \Theta(G)$ has the form $\{r_i \mid n/2 < \eta(i) \leqslant n,\ i \neq (n+1)/2, n\}$, and $\Theta'(G)$ consists of singleton sets of type $\{r_{4t}\}$ or $\{r_{2t+1}\}$. Hence $t(G) = 3t+1 = [(3n+4)/4]$.

Assume $n = 4t+2$. Then

$$N(G) = \{2t+3, 2t+5, \ldots, 4t+1, 4t+4, 4t+6, \ldots, 8t+4\},$$

and so $|N(G)| = 3t+1 \geqslant 4$. All primes r_{2t+1}, r_{4t}, and r_{4t+2} are adjacent one to another and are nonadjacent to every r_i with $i \in N(G)$. Hence $t(G) > 4$. Since $t(2, G) \leqslant t(p, G) \leqslant 4$, primes p and

2 cannot lie in any coclique of maximal size. If $\eta(i) < 2t$, then r_i is adjacent to r_{2t+1}, r_{4t+2}, r_{4t}, and r_{4t+4}. Therefore, $N(G) = M(G)$, every set $\theta(G) \in \Theta(G)$ has the form $\{r_i \mid n/2 < \eta(i) \leqslant n\}$, and $\Theta'(G)$ consists of singleton sets of type $\{r_{2t+1}\}$, $\{r_{4t}\}$, or $\{r_{4t+2}\}$. Thus $t(G) = 3t+2 = [(3n+4)/4]$.

Suppose $n = 4t+3$. Then

$$N(G) = \{2t+3, 2t+5, \ldots, 4t+1, 4t+4, 4t+6, \ldots, 8t+6\},$$

and so $|N(G)| = 3t+2 \geqslant 5$. Since $t(2,G) \leqslant t(p,G) < 4$, primes p and 2 cannot lie in a coclique of maximal size. In view of the adjacency criterion, r_{4t+2} is nonadjacent to every r_i, where $i \in N(G)$. On the other hand, if $\eta(i) < 2t+1$ or $i = 2t+1$, then r_i is adjacent to r_{4t+4} and r_{4t+2}. Therefore, $M(G) = N(G) \cup \{n-1\}$, every set $\theta(G) \in \Theta(G)$ is of the form $\{r_i \mid (n-1)/2 \leqslant \eta(i) \leqslant n, i \neq n, (n-1)/2\}$, $\Theta'(G) = \varnothing$, and $t(G) = 3t+3 = [(3n+4)/4]$.

It remains to handle the cases where $q = 2$ and $n = 4+k$, with $k = 1, 2, 3$. All results (see Table 3) are obtained by following essentially the same arguments as were used in the general case (with the fact that $R_{4t+2} = R_6 = \varnothing$ in mind) and can be readily verified by appealing to Proposition 2.5 and [1, Props. 3.1, 4.4]. □

PROPOSITION 3.11. If G is a finite simple exceptional group of Lie type over a field of characteristic p, then $t(G)$ and the sets $\Theta(G)$ and $\Theta'(G)$ are listed in Table 4.

Proof. All kinds of exceptional groups of Lie type will be treated separately. Following [9], we represent the prime graph $GK(G)$ in a compact form. By a *compact form* we mean a graph whose vertices are marked with labels R_i. A vertex labeled R_i is a clique in $GK(G)$ each vertex of which is a prime in R_i. An edge joining R_i and R_j is a set of edges of $GK(G)$ connecting each vertex in R_i to each vertex in R_j. If an edge occurs under some condition, we draw such an edge as a dotted line and write an appropriate occurrence condition. Technical tools for determining the compact form of $GK(G)$ for an exceptional group of Lie type G are Propositions 2.7 and 2.9, and also [1, Props. 3.2, 3.3, 4.5]. Notice that the compact form for $GK(G)$ can be conceived of as a graphical representation for the adjacency criterion in $GK(G)$.

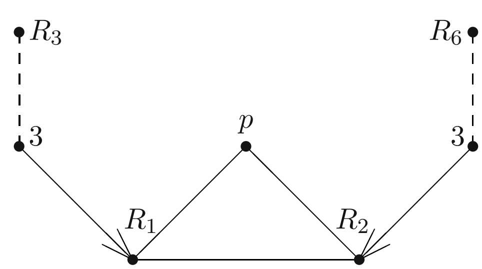

Fig. 1. Diagram of a compact form for $GK(G_2(q))$.

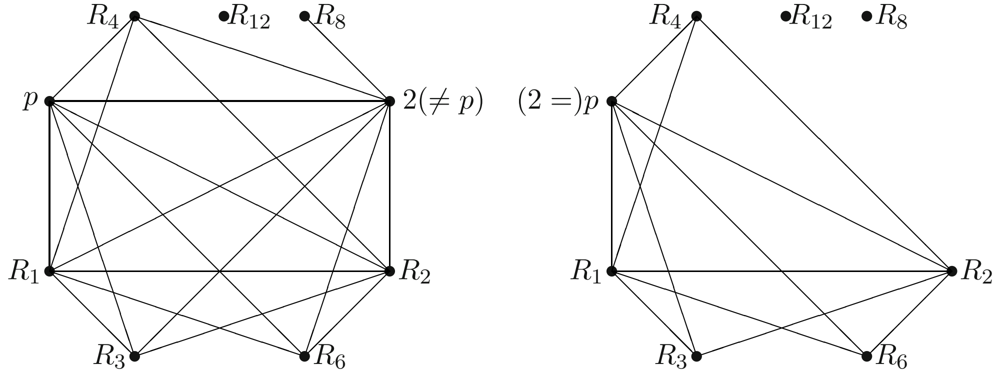

Fig. 2. Diagram of a compact form for $GK(F_4(q))$.

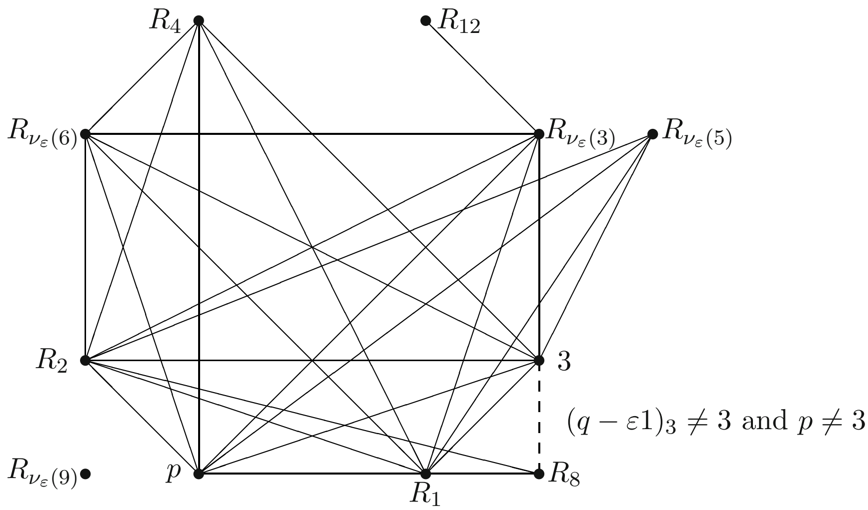

Fig. 3. Diagram of a compact form for $GK(E_6^\varepsilon(q))$.

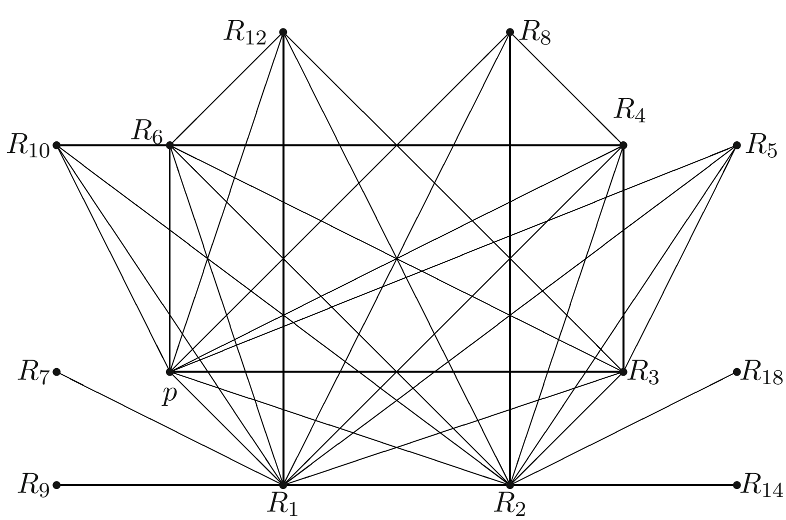

Fig. 4. Diagram of a compact form for $GK(E_7(q))$.

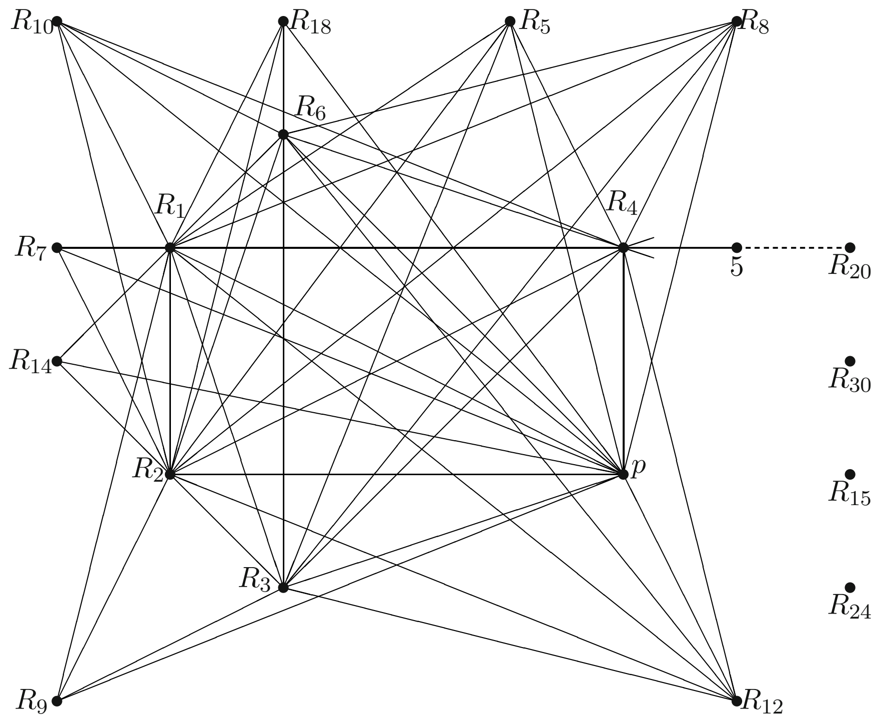

Fig. 5. Diagram of a compact form for $GK(E_8(q))$.

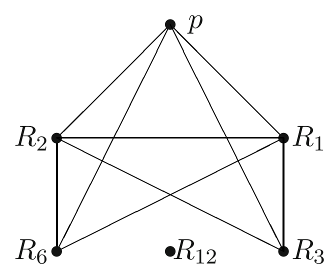

Fig. 6. Diagram of a compact form for $GK(^3D_4(q))$.

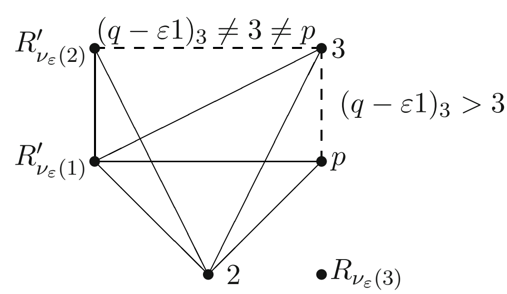

Fig. 7. Diagram of a compact form for $GK(A_2^\varepsilon(q))$.

Let $G = G_2(q)$. In the compact form for $GK(G_2(q))$, the vector from 3 to R_1 (resp. R_2) and the dotted edge $(3, R_3)$ (resp. $(3, R_6)$) signify that R_1 (resp. R_2) and R_3 (resp. R_6) are not connected, but if $3 \in R_1$, i.e., $q \equiv 1 \pmod 3$ (resp. $3 \in R_2$, i.e., $q \equiv -1 \pmod 3$), then there exists an edge between 3 and R_3 (resp. R_6). If $R_1 = \varnothing$, then we need to remove vertex R_1 together with all incident edges. From the compact form of $GK(G)$, it is evident that $\Theta(G) = \{\{r_3, r_6\} \mid r_i \in R_i\}$ while $\Theta'(G) = \{\{p\}, \{r_1\}, \{r_2\} \mid r_i \in R_i \setminus \{3\}\}$.

Let $G = F_4(q)$. The compact form for $GK(F_4(q))$ indicates that $\{2, p, R_1, R_2, R_3\}$ is a clique, and the remaining vertices are pairwise nonadjacent. Since R_3 in nonadjacent to R_4, R_6, R_8, and R_{12}, while the remaining vertices in $\{2, p, R_1, R_2\}$ are adjacent to at least two vertices in $\{R_4, R_6, R_8, R_{12}\}$, we see that $\Theta(G) = \{\{r_3, r_4, r_6, r_8, r_{12}\} \mid r_i \in R_i\}$, if $R_6 \neq \varnothing$, and $\Theta(G) = \{\{r_3, r_4, r_8, r_{12}\} \mid r_i \in R_i\}$ if $R_6 = \varnothing$ (i.e., if $q = 2$). In either case $\Theta'(G) = \varnothing$.

Let $G = E_6^\varepsilon(q)$. In the compact form for $GK(E_6^\varepsilon(q))$, the set $\{3, p, R_1, R_2, R_{\nu_\varepsilon(3)}, R_{\nu_\varepsilon(6)}\}$ forms a clique, and the remaining vertices are pairwise nonadjacent. Moreover, $R_{\nu_\varepsilon(3)}$ and $R_{\nu_\varepsilon(6)}$ are the only vertices in $\{3, p, R_1, R_2, R_{\nu_\varepsilon(3)}, R_{\nu_\varepsilon(6)}\}$ that are adjacent to precisely one of the remaining vertices (viz., $R_{\nu_\varepsilon(3)}$ is adjacent to R_{12} and $R_{\nu_\varepsilon(6)}$ is adjacent to R_4). Thus $\Theta(G) = \{\{r_{\nu_\varepsilon(5)}, r_8, r_{\nu_\varepsilon(9)}\} \mid r_i \in R_i\}$ and $\Theta'(G) = \{\{r_4, r_{\nu_\varepsilon(3)}\}, \{r_{\nu_\varepsilon(6)}, r_{12}\}, \{r_4, r_{12}\} \mid r_i \in R_i\}$. Since $R_6 = \varnothing$ for $q = 2$, we are faced up to the exceptions mentioned in Table 4.

Let $G = E_7(q)$. In the compact form for $GK(E_7(q))$, the set $\{p, R_1, R_2, R_3, R_4, R_6\}$ forms a clique, and the remaining vertices are pairwise nonadjacent. Moreover, R_4 is the only vertex in $\{p, R_1, R_2, R_3, R_4, R_6\}$ that is adjacent to precisely one of the remaining vertices (viz., R_4 is adjacent to R_8). Thus $\Theta(G) = \{\{r_5, r_7, r_9, r_{10}, r_{12}, r_{14}, r_{18}\} \mid r_i \in R_i\}$ and $\Theta'(G) = \{\{r_4\}, \{r_8\} \mid r_i \in R_i\}$.

Let $G = E_8(q)$. In the compact form for $GK(E_8(q))$, the vector from 5 to R_4 and the dotted edge $(5, R_{20})$ indicate that R_4 and R_{20} are not connected, but if $5 \in R_4$ (i.e., $q^2 \equiv -1 \pmod 5$), then there exists an edge joining 5 and R_{20}. Now $\{p, R_1, R_2, R_3, R_4, R_6\}$ forms a clique, and the remaining vertices are pairwise nonadjacent. Notice that each vertex in the clique $\{p, R_1, R_2, R_3, R_4, R_6\}$ is adjacent to at least two of the remaining vertices. Hence

$$\Theta(G) = \{\{r_5, r_7, r_8, r_9, r_{10}, r_{12}, r_{14}, r_{15}, r_{18}, r_{20}, r_{24}, r_{30}\} \mid r_i \in R_i\}$$

and $\Theta'(G) = \varnothing$.

Let $G = {}^3D_4(q)$. The compact form for $GK({}^3D_4(q))$ shows that $\Theta(G) = \{\{r_3, r_6, r_{12}\} \mid r_i \in R_i\}$ and $\Theta'(G) = \varnothing$ if $q \neq 2$. For $q = 2$, the result follows from the compact form for the prime graph $GK({}^3D_4(q))$ and the fact that $R_6 = \varnothing$.

Let $G = {}^2B_2(q)$. In this case primes $s_i \in S_i$ and $s_j \in S_j$ are adjacent iff $i = j$, while $p = 2$ is nonadjacent to all vertices, yielding the result required.

Let $G = {}^2G_2(q)$. In this event odd primes $s_i \in S_i$ and $s_j \in S_j$ are adjacent iff $i = j$, while $p = 3$ is nonadjacent to all odd primes. The required result now follows once we note that 2 is adjacent to s_1, s_2, and p.

Let $G = {}^2F_4(q)$. If $q > 8$, then any set $\{s_2, s_3, s_4, s_5, s_6\}$ forms a coclique in $GK(G)$, as follows by Prop. 2.9. The same proposition, together with [1, Prop. 3.3], implies that the set $\{2\} \cup S_1 \cup S_2$ forms a clique in $GK(G)$, any prime s_3 is adjacent to s_1 and 2, and 3 is adjacent to s_2 and s_4, yielding the result required. If $G = {}^2F_4(8)$, then $S_2 = \pi(9) \setminus \{3\} = \varnothing$, in which case every $\theta(G) \in \Theta(G)$ is of the form $\{s_5, s_6\}$ and every $\theta'(G) \in \Theta'(G)$ is a two-element set of type $\{s_1, s_4\}$, $\{3, s_3\}$, $\{2, s_4\}$, or $\{s_3, s_4\}$. A group $G = {}^2F_4(2)$ is not simple, and its derived subgroup $T = {}^2F_4(2)'$ is a simple Tits group. Using [2], we see that the prime graph of the Tits group T contains a unique coclique $\rho(T) = \{3, 5, 13\}$ of maximal size. $\square$

PROPOSITION 3.12. If $G = A^{\varepsilon}_{n-1}(q)$ is a finite simple group of Lie type over a field of characteristic p, and $n \in \{2, 3\}$, then $t(G)$ and the sets $\Theta(G)$ and $\Theta'(G)$ are listed in Table 2.

Proof. Let $G = A_1(q)$. Then the compact form for $GK(A_1(q))$ is a coclique with vertex set $\{R_1, R_2, p\}$. Thus $\Theta(G) = \{\{r_1, r_2, p\} \mid r_i \in R_i\}$ and $\Theta'(G) = \varnothing$.

Let $G = A^{\varepsilon}_2(q)$. Put $R'_{\nu_\varepsilon(1)} = R_{\nu_\varepsilon(1)} \setminus \{2, 3\}$ and $R'_{\nu_\varepsilon(2)} = R_{\nu_\varepsilon(2)} \setminus \{2, 3\}$. First assume that $(q - \varepsilon 1)_3 > 3$. Then the set $\{2, 3, p, R'_{\nu_\varepsilon(1)}\}$ is a clique in a compact form for $GK(A^{\varepsilon}_2(q))$, while $R'_{\nu_\varepsilon(2)}$ and $R_{\nu_\varepsilon(3)}$ are nonadjacent. If $R_{\nu_\varepsilon(2)} \neq \{2\}$ (i.e., $q + \varepsilon 1 \neq 2^k$ and $R'_{\nu_\varepsilon(2)} \neq \varnothing$), then p is the only vertex in the clique $\{2, 3, p, R'_{\nu_\varepsilon(1)}\}$ that is nonadjacent to both $R'_{\nu_\varepsilon(2)}$ and $R_{\nu_\varepsilon(3)}$. Hence $\Theta(G) = \{\{p, r_{\nu_\varepsilon(2)} \neq 2, r_{\nu_\varepsilon(3)}\} \mid r_i \in R_i\}$ and $\Theta'(G) = \varnothing$. If $R_{\nu_\varepsilon(2)} = \{2\}$ (i.e., $q + \varepsilon 1 = 2^k$ and $R'_{\nu_\varepsilon(2)} = \varnothing$), then $\Theta(G) = \{\{r_{\nu_\varepsilon(3)}\} \mid r_{\nu_\varepsilon(3)} \in R_{\nu_\varepsilon(3)}\}$ and $\Theta'(G) = \{\{2\}, \{p\}, \{r_{\nu_\varepsilon(1)}\} \mid r_{\nu_\varepsilon(1)} \in R_{\nu_\varepsilon(1)}\}$.

Next suppose that $(q - \varepsilon 1)_3 = 3$. Then the set $\{2, p, R'_{\nu_\varepsilon(1)}\}$ is a clique in a compact form for $GK(A^{\varepsilon}_2(q))$, while 3, $R'_{\nu_\varepsilon(2)}$, and $R_{\nu_\varepsilon(3)}$ are pairwise nonadjacent. Since p is the only vertex in the clique $\{2, p, R'_{\nu_\varepsilon(1)}\}$ that is nonadjacent to 3, $R'_{\nu_\varepsilon(2)}$, and $R_{\nu_\varepsilon(3)}$, we conclude that $\Theta(G) = \{\{3, p, r_{\nu_\varepsilon(2)} \neq 2, r_{\nu_\varepsilon(3)}\} \mid r_i \in R_i\}$, if $R_{\nu_\varepsilon(2)} \neq \{2\}$, and $\Theta(G) = \{\{3, p, r_{\nu_\varepsilon(3)}\} \mid r_{\nu_\varepsilon(3)} \in R_{\nu_\varepsilon(3)}\}$ if $R_{\nu_\varepsilon(2)} = \{2\}$. In both cases $\Theta'(G) = \varnothing$.

Lastly, let $(q - \varepsilon 1)_3 = 1$, i.e., either $(q + \varepsilon 1)_3 > 1$ and $3 \in R_{\nu_\varepsilon(2)} \neq \{2\}$, or $p = 3$. As above, we see that the set $\{2, p, R'_{\nu_\varepsilon(1)}\}$ is a clique in a compact form for $GK(A^{\varepsilon}_2(q))$, while $R'_{\nu_\varepsilon(2)}$ and $R_{\nu_\varepsilon(3)}$ are pairwise nonadjacent. Since p is the only vertex in the clique $\{2, p, R'_{\nu_\varepsilon(1)}\}$ that is nonadjacent to $R'_{\nu_\varepsilon(2)}$ and $R_{\nu_\varepsilon(3)}$, and either $3 \in R_{\nu_\varepsilon(2)}$ or $p = 3$, we see that $\Theta(G) = \{\{p, r_{\nu_\varepsilon(2)} \neq 2, r_{\nu_\varepsilon(3)}\} \mid r_i \in R_i \setminus \{2\}\}$ and $\Theta'(G) = \varnothing$, if $R_{\nu_\varepsilon(2)} \neq \{2\}$, and $\Theta(G) = \{\{r_{\nu_\varepsilon(3)}\} \mid r_{\nu_\varepsilon(3)} \in R_{\nu_\varepsilon(3)}\}$ and $\Theta'(G) = \{\{p\}, \{r_{\nu_\varepsilon(1)}\}, \{2 = r_{\nu_\varepsilon(2)}\} \mid r_{\nu_\varepsilon(1)} \in R_{\nu_\varepsilon(1)}\}$ if $R_{\nu_\varepsilon(2)} = \{2\}$. $\square$

Tables 2-4 are organized in the following way. Column 1 represents a group of Lie type G with the base field of order q and characteristic p, column 2 presents conditions on G, and column 3 contains values for $t(G)$. In columns 4 and 5, we list elements of $\Theta(G)$ and $\Theta'(G)$, which are sets $\theta(G) \in \Theta(G)$ and $\theta'(G) \in \Theta'(G)$, and in so doing omit braces around singletons. In particular, the expression $\{p, 3, r_2 \neq 2, r_3\}$ in column 4 signifies that $\Theta(G) = \{\{p, 3, r_2, r_3\} \mid r_2 \in R_2 \setminus \{2\},\ r_3 \in R_3\}$, and by writing p, r_4 in column 5 we mean that $\Theta'(G) = \{\{p\}, \{r_4\} \mid r_4 \in R_4\}$.

TABLE 2. Cocliques for finite simple linear and unitary groups

G	Conditions	$t(G)$	$\Theta(G)$	$\Theta'(G)$
$A_1(q)$	$q > 3$	3	$\{p, r_1, r_2\}$	$\varnothing$
$A_2(q)$	$(q-1)_3 = 3,\ q+1 \neq 2^k$ $(q-1)_3 = 3,\ q+1 = 2^k$ $(q-1)_3 \neq 3,\ q+1 \neq 2^k$ $(q-1)_3 \neq 3,\ q+1 = 2^k$	4 3 3 2	$\{p, 3, r_2 \neq 2, r_3\}$ $\{3, p, r_3\}$ $\{p, r_2 \neq 2, r_3\}$ r_3	$\varnothing$ $\varnothing$ $\varnothing$ $p, r_1, 2 = r_2$
$A_3(q)$	$(q-1)_2 \neq 4$ $(q-1)_2 = 4$	3 3	$\{p, r_3, r_4\}$ $\{r_3, r_4\}$	$\varnothing$ $p, 2$
$A_4(q)$	$(q-1)_5 \neq 5$ $(q-1)_5 = 5$	3 3	$\{r_4, r_5\}$ $\{r_4, r_5\}$	p, r_3 $5, p, r_3$
$A_5(q)$	$q = 2$ $q > 2$ and $(q-1)_3 \neq 3$ $(q-1)_3 = 3$	3 3 3	$\{r_3, r_4, r_5\}$ r_5 r_5	$\varnothing$ $\{p, r_6\}$,$\{r_3, r_4\}$, $\{r_4, r_6\}$ $\{p, r_6\}$,$\{r_3, r_4\}$, $\{r_4, r_6\}$, $\{3, r_6\}$
$A_{n-1}(q)$, $n \geqslant 7$	n is odd and $q \neq 2$ for $7 \leqslant n \leqslant 11$ n is even and $q \neq 2$ for $8 \leqslant n \leqslant 12$ $n = 7,\ q = 2$ $n = 8,\ q = 2$ $n = 9,\ q = 2$ $n = 10,\ q = 2$ $n = 11,\ q = 2$ $n = 12,\ q = 2$	$[\frac{n+1}{2}]$ $[\frac{n+1}{2}]$ 3 3 4 4 5 6	$\{r_i \mid \frac{n}{2} < i \leqslant n\}$ $\{r_i \mid \frac{n}{2} < i < n\}$ $\{r_5, r_7\}$ r_7 $\{r_5, r_7, r_8, r_9\}$ $\{r_7, r_9\}$ $\{r_7, r_8, r_9, r_{11}\}$ $\{r_7, r_8, r_9, r_{10}, r_{11}, r_{12}\}$	$\varnothing$ $r_{n/2}$,r_n r_3, r_4 $\{p, r_8\}$,$\{r_5, r_8\}$, $\{r_3, r_8\}$,$\{r_4, r_5\}$ $\varnothing$ $\{r_4, r_{10}\}$,$\{r_8, r_{10}\}$ $\{r_5, r_8\}$ r_5, r_{10} $\varnothing$
${}^2A_2(q)$, $q > 2$	$(q+1)_3 = 3,\ q-1 \neq 2^k$ $(q+1)_3 = 3,\ q-1 = 2^k$ $(q+1)_3 \neq 3,\ q-1 \neq 2^k$ $(q+1)_3 \neq 3$, $q-1 = 2^k > 2$ $q = 3$	4 3 3 2 2	$\{p, 3, r_1 \neq 2, r_6\}$ $\{3, p, r_6\}$ $\{p, r_1 \neq 2, r_6\}$ r_6 r_6	$\varnothing$ $\varnothing$ $\varnothing$ $p, r_2, 2 = r_1$ $p, r_2 = 2$
${}^2A_3(q)$	$(q+1)_2 \neq 4$ and $q \neq 2$ $(q+1)_2 = 4$ $q = 2$	3 3 2	$\{p, r_6, r_4\}$ $\{r_6, r_4\}$ r_4	$\varnothing$ $p, 2$ p, r_2
${}^2A_4(q)$	$q = 2$ $q > 2$ and $(q+1)_5 \neq 5$ $(q+1)_5 = 5$	3 3 3	$\{p, r_4, r_{10}\}$ $\{r_4, r_{10}\}$ $\{r_4, r_{10}\}$	$\varnothing$ p, r_6 $5, p, r_6$
${}^2A_5(q)$	$q = 2$ $(q+1)_3 \neq 3$ $q > 2$ and $(q+1)_3 = 3$	3 3 3	$\{r_{10}, r_3\}$ r_{10} r_{10}	$3, p, r_4$ $\{p, r_3\}$,$\{r_6, r_4\}$, $\{r_4, r_3\}$ $\{p, r_3\}$,$\{r_6, r_4\}$, $\{r_4, r_3\}$, $\{3, r_3\}$
${}^2A_{n-1}(q)$, $n \geqslant 7$	n is odd n is even	$[\frac{n+1}{2}]$ $[\frac{n+1}{2}]$	$\{r_i \mid \frac{n}{2} < \nu(i) \leqslant n\}$ $\{r_i \mid \frac{n}{2} < \nu(i) < n\}$	$\varnothing$ $r_{\nu(n/2)}, r_{\nu(n)}$

TABLE 3. Cocliques for finite simple symplectic and orthogonal groups

G	Conditions	$t(G)$	$\Theta(G)$	$\Theta'(G)$
$B_n(q)$ or $C_n(q)$	$n = 2$ and $q = 3$	2	r_4	p, r_2
	$n = 2$ and $q > 3$	2	r_4	p, r_1, r_2
	$n = 3$ and $q = 2$	2	r_3	p, r_2, r_4
	$n = 3$ and $q > 2$	3	$\{r_3, r_6\}$	p, r_4
	$n = 4$ and $q = 2$	3	$\{r_3, r_4, r_8\}$	$\varnothing$
	$n = 5$ and $q = 2$	4	$\{r_5, r_8, r_{10}\}$	r_3, r_4
	$n = 6$ and $q = 2$	5	$\{r_3, r_5, r_8, r_{10}, r_{12}\}$	$\varnothing$
	$n = 7$ and $q = 2$	6	$\{r_5, r_7, r_{10}, r_{12}, r_{14}\}$	r_3, r_8
	$n > 3$, $n \equiv 0, 1 \pmod 4$, and $(n, q) \neq (4, 2), (5, 2)$	$\left[\frac{3n+5}{4}\right]$	$\{r_i \mid \frac{n}{2} \leqslant \eta(i) \leqslant n\}$	$\varnothing$
	$n > 3$, $n \equiv 2 \pmod 4$, and $(n, q) \neq (6, 2)$	$\left[\frac{3n+5}{4}\right]$	$\{r_i \mid \frac{n}{2} < \eta(i) \leqslant n\}$	$r_{n/2}, r_n$
	$n > 3$, $n \equiv 3 \pmod 4$, and $(n, q) \neq (7, 2)$	$\left[\frac{3n+5}{4}\right]$	$\{r_i \mid \frac{n+1}{2} < \eta(i) \leqslant n\}$	$r_{(n-1)/2}, r_{n-1}, r_{n+1}$
$D_n(q)$	$n = 4$ and $q = 2$	2	r_3	p, r_2, r_4
	$n = 4$ and $q > 2$	3	$\{r_3, r_6\}$	p, r_4
	$n = 5$ and $q = 2$	4	$\{r_3, r_4, r_5, r_8\}$	$\varnothing$
	$n = 6$ and $q = 2$	4	$\{r_3, r_5, r_8, r_{10}\}$	$\varnothing$
	$n > 4$ and $n \equiv 0 \pmod 4$	$\left[\frac{3n+1}{4}\right]$	$\{r_i \mid \frac{n}{2} \leqslant \eta(i) \leqslant n, i \neq 2n\}$	$\varnothing$
	$n > 4$, $n \equiv 1 \pmod 4$, and $(n, q) \neq (5, 2)$	$\left[\frac{3n+1}{4}\right]$	$\{r_i \mid \frac{n}{2} < \eta(i) \leqslant n, i \neq 2n, n+1\}$	r_{n-1}, r_{n+1}
	$n > 4$, $n \equiv 2 \pmod 4$, and $(n, q) \neq (6, 2)$	$\left[\frac{3n+1}{4}\right]$	$\{r_i \mid \frac{n}{2} < \eta(i) \leqslant n, i \neq 2n\}$	$r_{n/2}, r_n$
	$n > 4$ and $n \equiv 3 \pmod 4$	$\frac{3n+3}{4}$	$\{r_i \mid \frac{n-1}{2} \leqslant \eta(i) \leqslant n, i \neq 2n, n-1\}$	$\varnothing$
${}^2D_n(q)$	$n = 4$ and $q = 2$	3	$\{r_3, r_8\}$	p, r_4
	$n = 4$ and $q > 2$	4	$\{r_3, r_6, r_8\}$	p, r_4
	$n = 5$ and $q = 2$	3	$\{r_8, r_{10}\}$	p, r_3, r_4
	$n = 6$ and $q = 2$	5	$\{r_5, r_8, r_{10}, r_{12}\}$	r_3, r_4
	$n = 7$ and $q = 2$	5	$\{r_5, r_{10}, r_{12}, r_{14}\}$	r_3, r_8
	$n > 4$, $n \equiv 0 \pmod 4$ and	$\left[\frac{3n+4}{4}\right]$	$\{r_i \mid \frac{n}{2} \leqslant \eta(i) \leqslant n\}$	$\varnothing$
	$n > 4$, $n \equiv 1 \pmod 4$ and $(n, q) \neq (5, 2)$	$\left[\frac{3n+4}{4}\right]$	$\{r_i \mid \frac{n}{2} < \eta(i) \leqslant n, i \neq n, \frac{n+1}{2}\}$	$r_{(n+1)/2}, r_{n-1}$
	$n > 4$, $n \equiv 2 \pmod 4$, and $(n, q) \neq (6, 2)$	$\left[\frac{3n+4}{4}\right]$	$\{r_i \mid \frac{n}{2} < \eta(i) \leqslant n\}$	$r_{n/2}, r_{n-2}, r_n$
	$n > 4$, $n \equiv 3 \pmod 4$, and $(n, q) \neq (7, 2)$	$\left[\frac{3n+4}{4}\right]$	$\{r_i \mid \frac{n-1}{2} \leqslant \eta(i) \leqslant n, i \neq n, \frac{n-1}{2}\}$	$\varnothing$

TABLE 4. Cocliques for finite simple exceptional groups

G	Conditions	$t(G)$	$\Theta(G)$	$\Theta'(G)$
$G_2(q)$	$q = 3, 4$	3	$\{r_3, r_6\}$	p, r_2
	$q = 8$	3	$\{r_3, r_6\}$	p, r_1
	$q = 3^m > 3$	3	$\{r_3, r_6\}$	p, r_1, r_2
	$q \equiv 1 \pmod 3$ and $q \neq 4$	3	$\{r_3, r_6\}$	$p, r_2, r_1 \neq 3$
	$q \equiv 2 \pmod 3$ and $q \neq 8$	3	$\{r_3, r_6\}$	$p, r_1, r_2 \neq 3$
$F_4(q)$	$q = 2$	4	$\{r_3, r_4, r_8, r_{12}\}$	$\varnothing$
	$q > 2$	5	$\{r_3, r_4, r_6, r_8, r_{12}\}$	$\varnothing$
$E_6(q)$	$q = 2$	5	$\{r_4, r_5, r_8, r_9\}$	r_3, r_{12}
	$q > 2$	5	$\{r_5, r_8, r_9\}$	$\{r_3, r_4\}$, $\{r_4, r_{12}\}$, $\{r_6, r_{12}\}$
$^2E_6(q)$	$q = 2$	5	$\{r_8, r_{10}, r_{12}, r_{18}\}$	r_3, r_4
	$q > 2$	5	$\{r_8, r_{10}, r_{18}\}$	$\{r_3, r_{12}\}$, $\{r_4, r_6\}$, $\{r_4, r_{12}\}$
$E_7(q)$		8	$\{r_5, r_7, r_9, r_{10}, r_{12}, r_{14}, r_{18}\}$	r_4, r_8
$E_8(q)$		12	$\{r_5, r_7, r_8, r_9, r_{10}, r_{12}, r_{14}, r_{15}, r_{18}, r_{20}, r_{24}, r_{30}\}$	$\varnothing$
$^3D_4(q)$	$q = 2$	2	r_{12}	p, r_2, r_3
	$q > 2$	3	$\{r_3, r_6, r_{12}\}$	$\varnothing$
$^2B_2(2^{2n+1})$	$n \geqslant 1$	4	$\{2, s_1, s_2, s_3\}$	$\varnothing$
$^2G_2(3^{2n+1})$	$n \geqslant 1$	5	$\{3, s_1, s_2, s_3, s_4\}$	$\varnothing$
$^2F_4(2^{2n+1})$	$n \geqslant 2$,	5	$\{s_2, s_3, s_4, s_5, s_6\}$	$\varnothing$
$^2F_4(8)$		4	$\{s_5, s_6\}$	$\{3, s_3\}$, $\{s_1, s_4\}$, $\{2, s_4\}$, $\{s_3, s_4\}$
$^2F_4(2)'$		3	$\{3, 5, 13\}$	$\varnothing$

4. APPENDIX

We give a list of corrections for [1] obtained in the present paper.

Items (4), (5), and (9) of Lemma 1.3 in [1] should be replaced by, respectively, items (1), (2), and (3) of Lemma 2.6 in this paper. Lemma 1.4 [1] should be replaced by Lemma 2.1; Lemma 1.5 [1], by Lemma 2.8; Proposition 2.3 [1], by Proposition 2.4; Proposition 2.4 [1], by Proposition 2.5; Proposition 2.5 [1], by Proposition 2.7.

In [1, Tables 4, 8], the following corrections are necessary:

the lines

$A_{n-1}(q)$	$n = 3$, $(q-1)_3 = 3$, and $q + 1 \neq 2^k$	4	$\{p, 3, r_2, r_3\}$
	$n = 3$, $(q-1)_3 \neq 3$, and $q + 1 \neq 2^k$	3	$\{p, r_2, r_3\}$

should be replaced by

$A_{n-1}(q)$	$n=3$, $(q-1)_3=3$, and $q+1\neq 2^k$	4	$\{p,3,r_2\neq 2,r_3\}$
	$n=3$, $(q-1)_3\neq 3$, and $q+1\neq 2^k$	3	$\{p,r_2\neq 2,r_3\}$

,

and the lines

${}^2A_{n-1}(q)$	$n=3$, $(q+1)_3=3$, and $q-1\neq 2^k$	4	$\{p,3,r_1,r_6\}$
	$n=3$, $(q+1)_3\neq 3$, and $q-1\neq 2^k$	3	$\{p,r_1,r_6\}$

by

${}^2A_{n-1}(q)$	$n=3$, $(q+1)_3=3$, and $q-1\neq 2^k$	4	$\{p,3,r_1\neq 2,r_6\}$
	$n=3$, $(q+1)_3\neq 3$, and $q-1\neq 2^k$	3	$\{p,r_1\neq 2,r_6\}$

.

In the last but one line corresponding to $D_n(q)$ in [1, Table 4], "$n\equiv 1 \pmod 2$, $n>4$" must be substituted for "$n\equiv 1 \pmod 1$, $n>4$."

In [1, Table 8], the following corrections are necessary:

the line

$D_n(q)$	$n\geqslant 4$, $(n,q)\neq(4,2),(5,2),(6,2)$	$\left[\frac{3n+1}{4}\right]$	$\{r_{2i}\mid\left[\frac{n+1}{2}\right]\leqslant i<n\}\cup$ $\cup\{r_i\mid\left[\frac{n}{2}\right]<i\leqslant n,$ $i\equiv 1 \pmod 2\}$

should be replaced by

$D_n(q)$	$n\geqslant 4$, $n\not\equiv 3 \pmod 4$, $(n,q)\neq(4,2),(5,2),(6,2)$	$\left[\frac{3n+1}{4}\right]$	$\{r_{2i}\mid\left[\frac{n+1}{2}\right]\leqslant i<n\}\cup$ $\cup\{r_i\mid\left[\frac{n}{2}\right]<i\leqslant n,$ $i\equiv 1 \pmod 2\}$
	$n\equiv 3 \pmod 4$	$\frac{3n+3}{4}$	$\{r_{2i}\mid\left[\frac{n+1}{2}\right]\leqslant i<n\}\cup$ $\cup\{r_i\mid\left[\frac{n}{2}\right]\leqslant i\leqslant n,$ $i\equiv 1 \pmod 2\}$

,

and the line

${}^2D_n(q)$	$n\geqslant 4$, $n\not\equiv 1 \pmod 4$, $(n,q)\neq(4,2),(6,2),(7,2)$	$\left[\frac{3n+4}{4}\right]$	$\{r_{2i}\mid\left[\frac{n}{2}\right]\leqslant i\leqslant n\}\cup$ $\cup\{r_i\mid\left[\frac{n}{2}\right]<i\leqslant n,$ $i\equiv 1 \pmod 2\}$

by

${}^2D_n(q)$	$n \geqslant 4$, $n \not\equiv 1 \pmod 4$, $(n,q) \neq (4,2),(6,2),(7,2)$	$\left[\frac{3n+4}{4}\right]$	$\{r_{2i} \mid \left[\frac{n}{2}\right] \leqslant i \leqslant n\} \cup \cup \{r_i \mid \left[\frac{n}{2}\right] < i < n, i \equiv 1 \pmod 2\}$

.

Lastly, [1, Table 9] must be remedied as follows:
the line

$E_6(q)$	$q = 2$	5	$\{5, 12, 17, 19, 31\}$
	$q > 2$	6	$\{r_4, r_5, r_6, r_8, r_9, r_{12}\}$

should be replaced by

$E_6(q)$	none	5	$\{r_4, r_5, r_8, r_9, r_{12}\}$

,

the line

$E_7(q)$	none	7	$\{r_7, r_8, r_9, r_{10}, r_{12}, r_{14}, r_{18}\}$

by

$E_7(q)$	none	8	$\{r_5, r_7, r_8, r_9, r_{10}, r_{12}, r_{14}, r_{18}\}$

,

and the line

$E_8(q)$	none	11	$\{r_7, r_8, r_9, r_{10}, r_{12}, r_{14}, r_{15}, r_{18}, r_{20}, r_{24}, r_{30}\}$

by

$E_8(q)$	none	12	$\{r_5, r_7, r_8, r_9, r_{10}, r_{12}, r_{14}, r_{15}, r_{18}, r_{20}, r_{24} r_{30}\}$

.

A revised version of [1] can be found at http://arxiv.org/abs/math/0506294.

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