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Large Normal Nilpotent Subgroups of Finite Groups

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Introduction. Let Ψ be some property of groups that is inherited by all subgroups (for example, commutativity, nilpotence, solvability, etc.). Then the natural question arises: How small is a normal Ψ-subgroup in an arbitrary finite group G? We state it more precisely as the following

Question. *Given a finite group G with a Ψ-subgroup of index n, is it true that G has a normal Ψ-subgroup whose index is bounded by some function $f(n)$?*

Clearly, for every property Ψ inherited by subgroups, it suffices to take the function $n!$ as $f(n)$. In the article [1], Babai, Goodman, and Pyber addressed this question in the case when the property Ψ is cyclicity or solvability. In particular, they proved, modulo the classification of finite simple groups, that if a finite group G has a solvable subgroup of index n then G has a normal solvable subgroup of index at most n^c for some absolute constant c [1, Theorem 2.13]. In the same article, they posed the question of validity of a similar assertion in the case when the property Ψ is commutativity or nilpotence.

In the present article, we give a positive answer to this question in the case when the property Ψ is nilpotence. Clearly, the theorem on existence of a large normal solvable subgroup in an arbitrary finite group reduces solution of a similar problem for nilpotent groups to finding a large normal nilpotent subgroup in an arbitrary finite solvable group. Therefore, throughout the article we mainly deal with solvable groups and only at the end we prove, as a corollary, the theorem for an arbitrary finite group.

The question of existence of a large normal subgroup in a solvable finite group was discussed in many articles. Probably, for the first time it was raised in the two articles [2, 3] by Burnside in which he proved solvability of a group of order $p^\alpha q^\beta$ (p and q are prime numbers) and existence in such a group (with minor exceptions) of a normal p-subgroup of order greater than $p^\alpha q^{-\beta}$. Much more later V. S. Monakhov [4] found a gap in Burnside's proof, repaired it, and clarified the statement, providing an exact description for all possible exceptions. In [4], Burnside's theorem was generalized to the case of a solvable group whose order is $p^\alpha m$ with $p^\alpha > m$ and GCD(p^α, m)=1.

¿From this viewpoint, the main theorem of the present article (as stated below) is a maximally possible generalization in this direction.

The notations and definitions we use in the article can be found in [5]. If G is a group then $H \leq G$ means that H is a subgroup of G and $H \trianglelefteq G$ means that H is a normal subgroup of G. The index of a subgroup H in G is denoted by $|G : H|$. If H is a normal subgroup of G then G/H is the factor-group of G by H. By $G = A \ltimes B$ we mean a semidirect product of groups A and B in which B is a normal subgroup. If M is a subset of a group G then $\langle M\rangle$ is the subgroup generated by M; we let $|M|$ stand for the cardinality of M (or the order of an element if we have a single element rather than a set). By $C_G(M)$ we mean the centralizer of a set M in a group G; $C_G(G) = \zeta(G)$ is the center of G. Conjugation of an element x by an element y in a group G is written as $x^y = y^{-1}xy$. The Fitting subgroup

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of a group G is denoted by $F(G)$ and the Frattini subgroup of G, by $\Phi(G)$. If x and y are two elements of a group G then $[x, y] = x^{-1}x^y$ is the commutator of x and y; $[G, G] = G'$ is the derived subgroup of G. The exponent of a group G is denoted by $\exp(G)$.

Given a finite group G, we denote by $O_p(G)$ the largest normal p-subgroup of G and by $i_p(G)$, the minimal number k such that the intersection of k Sylow p-subgroups of G equals $O_p(G)$.

Given a vector space V, we denote by $GL(V)$ the group of all invertible transformations of V; $GL(n, q)$ is the group of all invertible matrices over a finite filed F_q of order q. A subgroup G of the group $GL(n, q)$ is called *semisimple* if its order is coprime to $q = p^\alpha$ and *unipotent* if its order is a power of p.

If φ is a homomorphism of a group G and g is an element of G then G^φ and g^φ are the images of G and g under φ. The automorphism group of a group G is denoted by $\mathrm{Aut}(G)$.

The Main Theorem *Let G be a nontrivial finite solvable group. If G has a nilpotent subgroup of index n then $|G : F(G)| < n5$.*

1. Available results.

Lemma 1 [5, Theorem 5.3.3] *If G is a group of order p^m and $|G : \Phi(G)| = p^r$ then the order of $C_{\mathrm{Aut}(G)}(G/\Phi(G))$ divides $p^{(m-r)r}$.*

Corollary *Let G be a finite p-group. If the order of an automorphism α of G does not divide p and α acts trivially on $G/\Phi(G)$ then α centralizes G.*

Proof Let α be a nontrivial automorphism whose order does not divide p. Then by Lemma 1 α does not belong to $C_{\mathrm{Aut}(G)}(G/\Phi(G))$.

Lemma 2 [5, Theorem 5.3.2] *If G is a finite p-group then $\Phi(G) = [G, G]G^p$.*

Corollary *If G is a finite p-group then $G/\Phi(G)$ is an elementary abelian group.*

Proof Since $[G, G] \leq \Phi(G)$, the group $G/\Phi(G)$ is abelian. Moreover, each element g of G raised to the power p belongs to $\Phi(G)$.

Lemma 3 [5, Theorem 5.2.4] *If G is a finite group then the following properties are equivalent:*

(i) *the group G is nilpotent;*

(ii) *the group G is a direct product of its Sylow subgroups.*

Corollary *Let G be a finite group and B a normal p-subgroup of G. Suppose that G contains an element α whose order does not divide p and which does not centralize B. Then G is not nilpotent.*

Proof The group $\langle \alpha, B\rangle$ cannot be represented as a direct product of its Sylow subgroups; therefore, it is not nilpotent. In consequence, the whole group G is not nilpotent.

Lemma 4 [5, Theorem 5.4.4] *If G is a solvable group and F is the Fitting subgroup of G then $C_G(F) = \zeta(F)$.*

Lemma 5 [6, Theorem 1.6] *If G is a nilpotent subgroup of $GL(V)$ whose order is coprime to the characteristic of the field over which the finite vector space V is defined, then $|G| \leq |V|^\beta/2$, where $\beta = \log 32/\log 9$.*

Lemma 6 [7] *The inequality $i_p(G) \leq 3$ holds for every finite solvable group G.*

Corollary *Let P be a Sylow p-subgroup of a finite nontrivial solvable group G and $O_p(G) = \{e\}$. Then $|G : P|^2 > |P|$.*

Proof In view of Lemma 6, there are three Sylow p-subgroups P_1, P_2, and P_3 such that $P_1 \cap P_2 \cap P_3 = \{e\}$. Since G is not a p-group, the inequality $|G| > \frac{|P_1|\cdot|P_2|}{|P_1\cap P_2|}$ holds. Furthermore, since $P_1 \cap P_2 \cap P_3 = \{e\}$, we have $|P_1 \cap P_2| \cdot |P_3| < |G|$. Thus,

$$|G| > \frac{|P_1| \cdot |P_2|}{|P_1 \cap P_2|} > \frac{|P_1| \cdot |P_2 \cdot |P_3| \cdot |P_1 \cap P_2|}{|G| \cdot |P_1 \cap P_2|} = \frac{|P_1|^3}{|G|}.$$

Therefore, $|G|^2 > |P_1|^3$, and $|G : P_1|^2 > |P_1|$ by Lagrange's theorem [5, Theorem 1.3.6].
Remark The claim of Lemma 6 is proved in [8] for all finite groups. However, the proof of this fact in the general case essentially uses the classification of finite simple groups.

2. Proof of the main theorem. In this section we prove the main theorem we have stated in the Introduction.

Let H be a nilpotent subgroup of G such that $|G : H| = n$. Consider $N = F(G)$. In view of Lemma 3, $N = P_1 \times \cdots \times P_k$ where P_i are Sylow p_i-subgroups of N.

Consider the homomorphism $\varphi : G \to G/\Phi(N) = \overline{G}$, henceforth denoting the images of elements and sets under this homomorphism by overlining the letters signifying them.

Lagrange's theorem implies $|\overline{G} : \overline{N}| = |G : N|$ and $|\overline{G} : \overline{H}| \leq |G : H|$; therefore, to prove the main theorem, it suffices to show that $|\overline{G} : \overline{N}| < |\overline{G} : \overline{H}|^5$.

By the Corollary to Lemma 2, the group $\overline{N}$ can be represented as $\overline{N} = \overline{P_1} \times \cdots \times \overline{P_k}$, where $|\overline{P_i}| = p_i^{n_i}$ and $\exp(\overline{P_i}) = p_i$, i.e., as a direct product of elementary abelian groups. Thus, each $\overline{P_i}$ may be regarded as a vector space of dimension n_i over the field F_{p_i}. Since $N \trianglelefteq G$, we may consider the homomorphisms $\varphi_i : G \to GL(n_i, p_i)$, $i = 1, \ldots, k$. These homomorphisms induce homomorphisms $\varphi_i : \overline{G} \to GL(n_i, p_i)$ and $\varphi_i : G/N \to GL(n_i, p_i)$ which we denote by the same letters to simplify notation. Let N_1 be a subgroup of N invariant under conjugation by some element x of G. The element x acts *unipotently on* N_1 if for every i the image of x under φ_i acts unipotently on $\overline{N_1} \cap \overline{P_i}$. If we take as N_1 the whole group N then we say that x acts *unipotently.* By analogy we define the notion of unipotent action on the subgroup $\overline{N_1}$ for the elements $\bar{x} \in \overline{G}$ and $x \in G/N$. A subgroup U of G acts unipotently on a subgroup N_1 of N if each element of U acts unipotently on N_1 (surely, N_1 is assumed invariant under conjugation by the group U). In the case when N_1 coincides with N, we say that the group U acts unipotently. We define unipotent action for subgroups of the groups $\overline{G}$ and G/N in the same way as for elements.

With the above notations, the following holds:

Lemma 7 *Let U be a normal subgroup of G which acts unipotently. Then $U \leq N$, and so $\overline{U} \leq \overline{N}$ and G/N lacks nontrivial normal subgroups that act unipotently.*

Proof We may assume that $N \leq U$: otherwise the group NU is normal in G and acts unipotently.

Suppose that $U \neq N$ and V/N is a minimal characteristic subgroup of U/N. Then $V \trianglelefteq G$ and V/N is a p-group. Let P be a Sylow p-subgroup of V. Then $V = P \cdot \prod_{p_i \neq p} P_i$. Since V acts unipotently, its image under each φ_i such that $p_i \neq p$ is the identity; hence, $\overline{P}$ centralizes every $\overline{P_i}$ for which $p_i \neq p$. In view of the Corollary to Lemma 1, the group P centralizes each P_i with $p_i \neq p$; i.e., it can be represented as a direct product of its Sylow subgroups and is nilpotent by Lemma 3. We thus obtain a normal nilpotent subgroup of G which does not lie in N. This contradicts the definition of N. The proof of the lemma is complete.

Note that as a straightforward consequence we have $C_{\overline{G}}(\overline{N}) = \zeta(\overline{N}) = \overline{N}$. Moreover, $\overline{N} = F(\overline{G})$. Indeed, $\overline{N}$ is a normal abelian subgroup. Since $F(\overline{G})$ is nilpotent, by the Corollary to Lemma 3 it acts unipotently on $\overline{N}$ and hence lies in $\overline{N}$. Therefore, we may assume that $G = \overline{G}$ and $F(G)$ is a product of elementary abelian groups. For this reason, to lighten notation we henceforth omit the overline. This consequence means in fact that the following holds:

Lemma 8 *If G is a finite solvable group then $F(G/\Phi(F(G))) = F(G)/\Phi(F(G))$.*

Lemma 9 *Let H be a nilpotent subgroup of G and let N_1 be a subgroup of N which is invariant under conjugation by H; moreover, the action of H on N_1 is not unipotent. Then the group $\langle H, N_1\rangle$ is not nilpotent.*

Proof Indeed, suppose that the group $\langle H, N_1\rangle$ is nilpotent. Then N_1 is its normal subgroup which is a direct product of elementary abelian groups. Therefore, the group $\langle H, N_1\rangle$ acts on N_1 unipotently (by the Corollary to Lemma 3), and so the group H acts on N_1 unipotently, which contradicts the hypothesis. The proof of the lemma is over.

We continue the proof of the theorem. Let H_1 be the subset of all elements of H that act unipotently. Then H_1 is a normal subgroup of H. Indeed, closure with respect to inversion and conjugation is obvious; therefore, it suffices to check closure with respect to multiplication. Let $x, y \in H_1$ be arbitrary two elements. Then $|x^{\varphi_i}| = p_i^m$ and $|y^{\varphi_i}| = p_i^l$ for all $i = 1, \dots, k$. Since H^{φ_i} is a nilpotent group, it can be represented as a direct product of its Sylow subgroups. In particular, the product of any two p_i-elements is again a p_i-element; i.e., $|(xy)^{\varphi_i}| = p_i^n$; hence, the element xy acts unipotently for all i and belongs therefore to H_1.

The subgroup $H \cap N$ is invariant under conjugation by H. Therefore, Lemma 9 implies that H acts on $H \cap N$ unipotently. Since N is an elementary abelian group, we may consider the factor-group $N/(N \cap H) = Q_1 \times \dots \times Q_k$, where $|Q_i| = p_i^{m_i}$ and $\exp(Q_i) = p_i$. By invariance of $N \cap H$ under conjugation by H, we may consider the induced homomorphisms $\phi_i : H \to GL(m_i, p_i) = GL(Q_i)$. For every i the group H^{ϕ_i} is nilpotent; therefore, it can be represented as $T_i \times U_i$, the direct product of its semisimple and unipotent parts. In view of Lemma 5, $|T_i| < |Q_i|^\beta$. Demonstrate that $|H/H_1| \leq \prod_i |T_i|$ and, in consequence,

$$|H/H_1| \leq |N/(N \cap H)|^\beta. (1)$$

Let x and y be two elements in H whose images in H/H_1 differ. Then there is an $i \in \{1, \dots, k\}$ such that $x^{\phi_i}U_i \neq y^{\phi_i}U_i$. Indeed, otherwise the element xy^{-1} acts unipotently on $N/(N \cap H)$. Since this element acts unipotently also on $N \cap H$, it acts unipotently on the whole group N and belongs therefore to H_1. This implies that the images of x and y in the group H/H_1 coincide, which contradicts the choice of these elements. To complete the proof of inequality (1), we need the following simple lemma.

Lemma 10 *Suppose that A is a finite set and $\psi_i : A \to A_i$ $(i = 1, \dots, n)$ are mappings such that, for arbitrary two distinct elements a and b in A, there is an i such that $a^{\psi_i} \neq b^{\psi_i}$. Then $|A| \leq |A_1| \cdot \ldots \cdot |A_n|$.*

Proof By the hypothesis of the lemma, we can arrange an injective embedding of A into the Cartesian product $A_1 \times \dots \times A_n$ by the following rule: $a \to (a^{\psi_1}, \dots, a^{\psi_n})$. It follows that $|A| \leq |A_1 \times \dots \times A_n| = |A_1| \cdot \ldots \cdot |A_N|$, which completes the proof of the lemma.

To finish the proof of inequality (1), observe that there are mappings of the elements of the group H/H_1 into the cosets of the subgroups U_i in the groups H^{ϕ_i} which satisfy the hypothesis of the lemma. Therefore, $|H/H_1| \leq \prod_i |H^{\phi_i} : U_i| = \prod_i |T_i|$, and inequality (1) is proven.

We now validate the inequality

$$|G/N : H_1N/N|^2 > |H_1N/N| = |H_1/(H \cap N)|. (2)$$

To this end, we consider the group $C_i = C_G(\prod_{j\neq i} P_j)/N$. Since $C_G(N) = N$, we have $C_i \cap \langle C_j | j \neq i\rangle = \{e\}$. Furthermore, it is clear that each group C_i is normal in G/N, and we can hence consider the subgroup $C = C_1 \times \cdots \times C_k$ of the group G/N. Since the group H_1 acts unipotently (and is itself nilpotent), the factor-group $H_1N/N \cong H_1/(H\cap N)$ (obviously, $H\cap N = H_1\cap N$) can be represented as a direct product of its Sylow p_i-subgroups: $H_1N/N = H_{p_1} \times \cdots \times H_{p_k}$. It follows from the proof of Lemma 7 that $H_{p_i} \leq C_i$.

Next, since $C_i \trianglelefteq G/N$, there are no nontrivial normal p_i-subgroups in C_i. Otherwise the largest of these subgroups is automorphism admissible and hence is a nontrivial normal subgroup of G/N acting unipotently. This contradicts Lemma 7. Thus, the Corollary to Lemma 6 implies that $|C_i : H_{p_i}|^2 > |H_{p_i}|$. Combining these inequalities for all i, we obtain

$$|G/N : H_1N/N|^2 \geq |C : H_1N/N|^2 > |H_1N/N| = |H_1/(H\cap N)|,$$

completing the proof of (2).

To finish the proof of the main theorem, we need two equalities that are easy consequences of Lagrange's theorem:

$$|G : H| = |G : HN| \cdot |HN : H| \xrightarrow{1}= |G/N : HN/N| \cdot |N/(N\cap H)|.(3)$$

Here in step 1 we use the fact that every element of HN can be written as $n \cdot h$, with $n \in N$ and $h \in H$. Therefore, every coset of H can be written as nH for some $n \in N$, and coincidence of two cosets n_1H and n_2H means that $n_2^{-1}n_1 \in H\cap N$. In consequence, $|NH : H| = |N : (N\cap H)| = |N/(N\cap H)|$ (the group N is abelian). Next,

$$|G/N : H_1N/N| = |G/N : HN/N| \cdot |HN/H_1N|$$

$$\xrightarrow{2}= |G/N : NH/N| \cdot |H/H_1| = |G/N : H_1N/N|.(4)$$

Here the proof of step 2 bases on the fact that $H\cap N = H_1\cap N$ and, in consequence, $|HN/H_1N| = |HN/N : H_1N/N| = |H/(H\cap N)|/|H_1/(H_1\cap N)| = |H|/|H_1| = |H/H_1|$.

Now, we derive the final estimate:

$$|G : N| = |G/N : HN/N| \cdot |HN/N : H_1N/N| \cdot |H_1N/N|$$

$$= |G/N : HN/N| \cdot |H/H_1| \cdot |H_1N/N|$$

$$\xrightarrow{3}< |G/N : HN/N| \cdot |N/(N\cap H)|^\beta \cdot |G/N : H_1N/N|^2$$

$$\xrightarrow{4}= |G/N : HN/N| \cdot |N/(N\cap H)|^\beta \cdot |G/N : HN/N|^2 \cdot |H/H_1|^2$$

$$\xrightarrow{5}< |G/N : HN/N|^3 \cdot |N/(N\cap H)|^\beta \cdot |N/(N\cap H)|^{2\beta}$$

$$\xrightarrow{6}\leq |G/N : HN/N|^3 \cdot |N/(N\cap H)|^{3\beta} < |G : H|^5.$$

Here step 3 is obtained by applying (1) and (2) to the second and third factors respectively. Step 4 results from applying (4) to the last factor. Step 5 follows again from (1). Finally, step 6 ensues from (3) and the inequality $|G/N : HN/N| \geq 1$ which follows from the fact that $3 < 3\beta < 5$. The proof of the theorem is over.

3. Corollary. As a corollary to the main theorem, we obtain a general answer to the question we raised in the beginning of the article.

Theorem 1 *Let G be a finite group. If G has a nilpotent subgroup of index n then it has a normal nilpotent subgroup of index at most n^c for some absolute constant c.*

Proof By [1, Theorem 2.13] the group G has a normal solvable subgroup R of index at most n^{c_1} for some absolute constant c_1. Let H be a nilpotent subgroup of index n appearing in the hypothesis of the theorem. Then $R \cap H$ is a nilpotent subgroup of index at most n in R. By the Main Theorem, $|R : F(R)| < n^5$. Since the Fitting subgroup is characteristic, it is normal in G and $|G : F(R)| < n^{c_1+5}$. The proof of the theorem is over.

Remark We proved the Main Theorem without appealing to the classification of finite simple groups. The proof of Theorem 2.13 in [1] leans essentially on the theorem of classification of finite simple groups. Using the proof of Theorem 2.13 in [1], we can obtain an estimate for the constant c_1 (in the proof of Theorem 1):

$$c_1 \leq \frac{\beta+1}{1-\alpha} + \frac{2}{(1-\alpha)\log_2 60}.$$

Here the constants α and β are defined as follows:

$\alpha < 1$ is an absolute constant such that the inequality $|N| \leq |G|^\alpha$ holds for every finite nonabelian simple group G and every nilpotent subgroup N of G;

β is an absolute constant such that the inequality $|\operatorname{Out}(G)| \leq |G|^\beta$ holds for every finite nonabelian simple group G.

It was shown in [9] that we can take $\frac{1}{2}$ as α; β can be taken to be $\frac{1}{2}$ as well. Thus, the constant c in Theorem 1 is at most 9.

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1. Babai L., Goodman A. J., and Pyber L. *Groups without faithful transitive permutation representations of small degree* J. Algebra, 1997, 195, 1–29

2. Burnside W. *On groups of order $p^\alpha q^\beta$* Proc. London Math. Soc. 1904 2 388–392.

3. Burnside W. *On groups of order $p^\alpha q^\beta$ (Second paper)* Proc. London Math. Soc. 1905 2 432–437

4. Monakhov V. S. *Invariant subgroups of biprimary groups* Mat. Zametki 1975 18 6 877–887

5. Robinson D. J. S. *A Course in the Theory of Groups* 1996 Springer-Verlag New York

6. Wolf T. R. *Solvable and nilpotent subgroups of $GL_n(q^m)$* Canad. J. Math. 1982 34 1097–1111

7 Passman D. S. *Groups with normal solvable Hall p-subgroups* Trans. Amer. Math. Soc. 1966 123 99–111

8 Zenkov V. I. *Intersection of nilpotent subgroups in finite groups* Fund. i Appl. Mat. 1996 2 1–91

7. Vdovin E. P., *Large nilpotent subgroups of finite simple groups,* Algebra and Logic, 39, N5 (2000), p.301–312.