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A CONJUGACY CRITERION FOR HALL SUBGROUPS IN FINITE GROUPS

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Abstract: A finite group G is said to satisfy C_π for a set of primes π, if G possesses exactly one class of conjugate π-Hall subgroups. We obtain a criterion for a finite group G to satisfy C_π in terms of a normal series of the group.

Keywords: Hall subgroup, conjugacy of Hall subgroups, C_π-property

Introduction

Let π be a set of primes. We denote by π' the set of all primes not in π and by $\pi(n)$ the set of prime divisors of a positive integer n, while for a finite group G we let $\pi(G)$ stand for $\pi(|G|)$. A positive integer n with $\pi(n) \subseteq \pi$ is called a *π-number*, while a group G with $\pi(G) \subseteq \pi$ is called a *π-group*. A subgroup H of G is called a *π-Hall subgroup*, if $\pi(H) \subseteq \pi$ and $\pi(|G : H|) \subseteq \pi'$. According to [1] we say that G *satisfies* E_π (or, briefly, $G \in E_\pi$), if G possesses a π-Hall subgroup. Moreover, if every two π-Hall subgroups are conjugate then we say that G *satisfies* C_π ($G \in C_\pi$). Further, if each π-subgroup of G lies in a π-Hall subgroup then we say that G *satisfies* D_π ($G \in D_\pi$). A group satisfying E_π (respectively, C_π, D_π) we also call an E_π- (respectively, C_π-, D_π-)*group*.

Let A, B, and H be subgroups of G such that $B \trianglelefteq A$. We denote $N_H(A) \cap N_H(B)$ by $N_H(A/B)$. Then every element $x \in N_H(A/B)$ induces an automorphism of A/B acting by $Ba \mapsto Bx^{-1}ax$. Thus, the homomorphism $N_H(A/B) \to \mathrm{Aut}(A/B)$ is defined. The image of the homomorphism is denoted by $\mathrm{Aut}_H(A/B)$ and is called the *group of H-induced automorphisms on A/B*, while the kernel is denoted by $C_H(A/B)$. If $B = 1$ then $\mathrm{Aut}_H(A/B)$ is denoted by $\mathrm{Aut}_H(A)$.

Assume that π is fixed. It is proved that the class of D_π-groups is closed under homomorphic images, normal subgroups (mod CFSG,[1)] [2, Theorem 7.7] or [3, Corollary 1.3]) and extensions (mod CFSG, [2, Theorem 7.7]). Thus, a finite group G satisfies D_π if and only if each composition factor S of G satisfies D_π. The class of E_π-groups is also known to be closed under normal subgroups and homomorphic images (see Lemma 4(1)) but not closed under extensions in general (see [4, Chapter V, Example 2]). In [5, Theorem 3.5] and [6, Corollary 6] it is proven that, if $1 = G_0 < G_1 < \dots < G_n = G$ is a composition series of G that is a refinement of a chief series, then G satisfies E_π if and only if $\mathrm{Aut}_G(G_i/G_{i-1})$ satisfies E_π for each $i = 1, \dots, n$.

The class of C_π-groups is closed under extensions (see Lemma 5) but not closed under normal subgroups in general (see the example below). Using the classification of finite simple groups, we in the present paper show that the class of C_π-groups is closed under homomorphic images (mod CFSG, see Lemma 9) and give a criterion for a finite group to satisfy C_π in terms of a normal series of the group.

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[†)] To Yu. L. Ershov on his seventieth birthday.

[1)](mod CFSG) in the paper means that the result is proven modulo the classification of finite simple groups.

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The main result is the following:

Theorem 1 (mod CFSG). *Let π be a set of primes, let H be a π-Hall subgroup, and let A be a normal subgroup of a C_π-group G. Then $HA \in C_\pi$.*

Corollary 2 (a conjugacy criterion for Hall subgroups, mod CFSG). *Let π be a set of primes and let A be a normal subgroup of G. Then $G \in C_\pi$ if and only if $G/A \in C_\pi$ and, for a π-Hall[2)] subgroup K/A of G/A its complete preimage K satisfies C_π. In particular, if $|G : A|$ is a π'-number then $G \in C_\pi$ if and only if $A \in C_\pi$.*

Using the corollary at the end of the paper, we give an algorithm that reduces the problem whether a finite group satisfies C_π to the same problem in some almost simple groups. In view of Theorem 1 notice that we do not know any counterexample to the following conjecture:

Conjecture 3. *Let π be a set of primes, and let A be a (not necessary normal) subgroup of a finite C_π-group G containing a π-Hall subgroup of G. Then $A \in C_\pi$.*

The condition of the conjecture that A includes a π-Hall subgroup of G cannot be weaken by the condition that the index of A is a π'-number. Indeed, consider $B_3(q) \simeq \mathrm{P}\Omega_7(q)$, where $q-1$ is divisible by 12 and is not divisible by 8 and 9. In view of [2, Lemma 6.2], $\mathrm{P}\Omega_7(q)$ is a $C_{\{2,3\}}$-group and its $\{2,3\}$-Hall subgroup lies in a monomial subgroup. On the other hand, $\Omega_7(2)$ is known to be isomorphic to a subgroup of $\mathrm{P}\Omega_7(q)$ and, under the above conditions on q, its index is not divisible by 2 and 3. However $\Omega_7(2)$ does not possess a $\{2,3\}$-Hall subgroup; i.e., it is not even an $E_{\{2,3\}}$-group.

1. Notation and Preliminary Results

By π we always denote a set of primes, and the term "group" always means a finite group.

The lemmas below are known and their proof do not refer to the classification of finite simple groups.

Lemma 4 [4, Chapter IV, (5.11), Chapter V, Theorem 3.7]. *Let A be a normal subgroup of G. Then the following hold:*

(1) *If H is a π-Hall subgroup of G then $H \cap A$ is a π-Hall subgroup of A, while HA/A is a π-Hall subgroup of G/A.*

(2) *If all factors of a subnormal series of G are either π- or π'-groups then $G \in D_\pi$.*

Notice that (2) of Lemma 4 follows from the celebrated Chunikhin's Theorem on π-solvable groups and the Feit–Thompson Odd Order Theorem.

Lemma 5 (Chunikhin; also see [1, Theorems C1 and C2] or [4, Chapter V, (3.12)]). *Let A be a normal subgroup of G. If both A and G/A satisfy C_π then $G \in C_\pi$.*

Lemma 6 [3, Lemma 2.1(e)]. *Let A be a normal subgroup of G such that G/A is a π-group, and let M be a π-Hall subgroup of A. Then a π-Hall subgroup H of G with $H \cap A = M$ exists if and only if G acting by conjugation leaves $\{M^a \mid a \in A\}$ invariant.*

Example. Suppose that $\pi = \{2,3\}$. Let $G = \mathrm{GL}_5(2) = \mathrm{SL}_5(2)$ be a group of order $99999360 = 2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$. Assume then that $\iota : x \in G \mapsto (x^t)^{-1}$ and $\widehat{G} = G \leftthreetimes \langle \iota \rangle$ is a natural semidirect product. By [7, Theorem 1.2], G possesses π-Hall subgroups and each π-Hall subgroup is the stabilizer of a series of subspaces $V = V_0 < V_1 < V_2 < V_3 = V$, where V is a natural module of G and $\dim V_k/V_{k-1} \in \{1,2\}$ for every $k = 1,2,3$. Hence G possesses exactly three classes of conjugate π-Hall subgroups with

2) Since $G/A \in C_\pi$ the phrase "for a π-Hall subgroup K/A of G/A" in the statement can be interpreted as "for every" and "for some," and both of them are correct.

the representatives

$$H_1 = \begin{pmatrix} \boxed{\mathrm{GL}_2(2)} & & * \\ & \boxed{1} & \\ 0 & & \boxed{\mathrm{GL}_2(2)} \end{pmatrix}$$

$$H_2 = \begin{pmatrix} \boxed{1} & & * \\ & \boxed{\mathrm{GL}_2(2)} & \\ 0 & & \boxed{\mathrm{GL}_2(2)} \end{pmatrix}, \quad H_3 = \begin{pmatrix} \boxed{\mathrm{GL}_2(2)} & & * \\ & \boxed{\mathrm{GL}_2(2)} & \\ 0 & & \boxed{1} \end{pmatrix}$$

Notice that $N_G(H_k) = H_k$, $k = 1, 2, 3$, since H_k is parabolic. By Lemma 4(1), for each π-Hall subgroup H of $\widehat{G}$ the subgroup $H \cap G$ is conjugate to one of the subgroups H_1, H_2, and H_3. The class containing H_1 is invariant under ι. Hence by Lemma 6 there exists a π-Hall subgroup H of $\widehat{G}$ with $H \cap G = H_1$. Moreover, $H = N_{\widehat{G}}(H_1)$. The classes containing H_2 and H_3 are permuted by ι. So from Lemmas 4(1) and 6 it follows that these subgroups do not lie in π-Hall subgroups of $\widehat{G}$. Thus, $\widehat{G}$ has exactly one class of conjugate π-Hall subgroups, and so satisfies C_π in contrast to its normal subgroup G.

Lemma 7. *Let A be a normal subgroup and let H be a π-Hall subgroup of a C_π-group G. Then $N_G(HA)$ and $N_G(H \cap A)$ satisfy C_π.*

Proof. Note that both $N_G(HA)$ and $N_G(H \cap A)$ include H, and so they satisfy E_π. Let K be a π-Hall subgroup of $N_G(HA)$. Since $HA \trianglelefteq N_G(HA)$ and $|N_G(HA) : HA|$ is a π'-number, we have $K \leq HA$ and $KA = HA$. If $x \in G$ is chosen so that $K = H^x$ then $(HA)^x = H^x A = KA = HA$ and so $x \in N_G(HA)$. Therefore, $N_G(HA) \in C_\pi$.

Suppose that K is a π-Hall subgroup of $N_G(H \cap A)$. Then $K(H \cap A) = K$ and $K \cap A = H \cap A$. If $x \in G$ is chosen so that $K = H^x$ then $(H \cap A)^x = H^x \cap A = K \cap A = H \cap A$ and so $x \in N_G(H \cap A)$. Therefore $N_G(H \cap A) \in C_\pi$. $\square$

Lemma 8 ([6, Corollary 9], mod CFSG). *Each π-Hall group of a homomorphic image of an E_π-group G is the image of a π-Hall subgroup of G.*

By Lemma 8 it is immediate that C_π is preserved under homomorphisms.

Lemma 9 (mod CFSG). *Let A be a normal subgroup of a C_π-group G. Then $G/A \in C_\pi$.*

Proof. Put $G/A = \overline{G}$. Since all π-Hall subgroups of G are conjugate, it is enough to show that for every π-Hall subgroup $\overline{K}$ of $\overline{G}$ there exists a π-Hall subgroup U of G such that $UA/A = \overline{K}$. The existence of U follows from Lemma 8. $\square$

If G is a group then a *G-class of π-Hall subgroups* is a class of conjugate π-Hall subgroups of G. Let A be a subnormal subgroup of an E_π-group G. A subgroup $H \cap A$ of A, where H is a π-Hall subgroup of G, is called a *G-induced π-Hall subgroup* of A. Thus, $\{(H \cap A)^a \mid a \in A\}$, where H is a π-Hall subgroup of G, is called an *A-class of G-induced π-Hall subgroups*. We denote the number of all A-classes of G-induced π-Hall subgroups by $k_\pi^G(A)$. Let $k_\pi(G) = k_\pi^G(G)$ be the number of classes of π-Hall subgroups of G. Clearly, $k_\pi^G(A) \leq k_\pi(A)$.

Recall that a finite group G is called *almost simple*, if G possesses a unique minimal normal subgroup S and S is a nonabelian finite group (equivalently, up to isomorphism, $S \simeq \mathrm{Inn}(S) \leq G \leq \mathrm{Aut}(S)$

for a nonabelian finite simple group S). The proof of Theorem 1 uses the following statement on the number of classes of π-Hall subgroups in finite simple groups.

Theorem 10 ([3, Theorem 1.1], mod CFSG). *Let π be a set of primes and let G be an almost simple finite E_π-group with the (nonabelian simple) socle S. Then the following hold:*
(1) *If $2 \notin \pi$ then $k_\pi^G(S) = 1$.*
(2) *If $3 \notin \pi$ then $k_\pi^G(S) \in \{1, 2\}$.*
(3) *If $2, 3 \in \pi$ then $k_\pi^G(S) \in \{1, 2, 3, 4, 9\}$.*
In particular, $k_\pi^G(S)$ is a π-number.

Lemma 11. *Let H be a π-Hall subgroup, let A be a normal subgroup of G, and $HAC_G(A) \trianglelefteq G$ (this condition is satisfied, if $HA \trianglelefteq G$). Then an A-class of π-Hall subgroups is a class of G-induced π-Hall subgroups if and only if it is H-invariant.*

Proof. If K is a π-Hall subgroup of G then $K \leq HAC_G(A)$, and so $KAC_G(A) = HAC_G(A)$. Since the A-class $\{(K \cap A)^a \mid a \in A\}$ is K-invariant, it follows that it is invariant under $HAC_G(A) = KAC_G(A)$ and so under H.

Conversely, without loss of generality we may assume that $G = HA$ and the claim follows from Lemma 6. □

Lemma 12. *Let H be a π-Hall subgroup, let A be a normal subgroup of G, and $HA \trianglelefteq G$. Then $k_\pi^G(A) = k_\pi^{HA}(A)$.*

Proof. Since HA is a normal subgroup of G, each π-Hall subgroup of G lies in HA. Hence, $k_\pi^G(A) = k_\pi^{HA}(A)$. □

Lemma 13. *Let H be a π-Hall subgroup, let A be a normal subgroup of G, and $HA \trianglelefteq G$. Then the following are equivalent:*
(1) *$k_\pi^G(A) = 1$.*
(2) *$HA \in C_\pi$.*
(3) *Every two π-Hall subgroups of G are conjugate by an element of A.*

Proof. (1) ⇒ (2) If K is a π-Hall subgroup of HA then by (1) $H \cap A$ and $K \cap A$ are conjugate in A. We may assume that $H \cap A = K \cap A$. Then H and K lie in $N_{HA}(H \cap A)$. By the Frattini argument, $HA = N_{HA}(H \cap A)A$. So,

$$N_{HA}(H \cap A)/N_A(H \cap A) = N_{HA}(H \cap A)/N_{HA}(H \cap A) \cap A \simeq N_{HA}(H \cap A)A/A = HA/A$$

is a π-group. Thus, $N_{HA}(H \cap A)$ possesses the normal series

$$N_{HA}(H \cap A) \geq N_A(H \cap A) \geq H \cap A \geq 1$$

such that every factor of the series is either a π- or π'-group, and, by Lemma 4(2), satisfies D_π. In particular, H and K are conjugate in $N_{HA}(H \cap A)$.

(2) ⇒ (3) and (3) ⇒ (1) are evident. □

Lemma 14. *Let H be a π-Hall subgroup, let $A = A_1 \times \cdots \times A_s$ be a normal subgroup of G, and $G = HAC_G(A)$. Then for every $i = 1, \ldots, s$ the following hold:*
(1) *$N_G(A_i) = N_H(A_i)AC_G(A)$.*
(2) *$N_H(A_i)$ is a π-Hall subgroup of $N_G(A_i)$.*
(3) *$k_\pi^{\mathrm{Aut}_G(A_i)}(\mathrm{Inn}(A_i)) = k_\pi^{N_G(A_i)}(A_i)$.*

Proof. (1) follows since $G = HAC_G(A)$ and $AC_G(A) \leq N_G(A_i)$. Using (1) and the identity $N_H(A_i) \cap AC_G(A) = H \cap AC_G(A)$, we see that $|N_G(A_i) : N_H(A_i)| = |AC_G(A) : (H \cap AC_G(A))|$ is a π'-number, whence (2).

Assume that $\rho : A_i \to \mathrm{Inn}(A_i)$ is the natural epimorphism. Since $\mathrm{Ker}(\rho) = Z(A_i)$ is an abelian group, the kernel of ρ possesses a unique π-Hall subgroup which lies in each π-Hall subgroup of A_i.

Hence the map $H \mapsto H\rho$ defines a bijection between the sets of π-Hall subgroups of A_i and $\mathrm{Inn}(A_i)$, and also induces a bijection (we denote it by the same symbol σ) between the sets Δ and Γ of A_i- and $\mathrm{Inn}(A_i)$-classes of π-Hall subgroups, respectively. We show that the restriction of σ on the set Δ_0 of all A_i-classes of $N_G(A_i)$-induced π-Hall subgroups is a bijective map from Δ_0 onto the set Γ_0 of all $\mathrm{Inn}(A_i)$-classes of $\mathrm{Aut}_G(A_i)$-induced π-Hall subgroups. Since $A = A_1 \times \dots \times A_s$; therefore, (1) implies $N_G(A_i) = N_H(A_i)A_iC_G(A_i)$. The normalizer $N_G(A_i)$ permutes elements from Δ acting by conjugation on the π-Hall subgroups of A_i. Thus, it acts on Δ. By Lemma 11, Δ_0 is the union of all one-element orbits under this action. By using σ, define an equivalent action of $N_G(A_i)$ on Γ. Since $C_G(A_i)$ lies in the kernel of both actions, the induced actions of $\mathrm{Aut}_G(A_i) = N_G(A_i)/C_G(A_i)$ on Δ and Γ are well-defined. It is easy to see that the action of $\mathrm{Aut}_G(A_i)$ on Γ defined in this way coincides with the natural action of the group on the set of $\mathrm{Inn}(A_i)$-classes of conjugate π-Hall subgroups. Since

$$\mathrm{Aut}_G(A_i)/\,\mathrm{Inn}(A_i) \simeq N_G(A_i)/A_iC_G(A_i) \simeq N_H(A_i)/(N_H(A_i) \cap A_iC_G(A_i))$$

is a π-group, by Lemma 11, Γ_0 coincides with the union of one-element orbits of $\mathrm{Aut}_G(A_i)$ on Γ. By the definition of the action, Γ_0 is the image of Δ_0 under σ. Since σ is a bijection,

$$k_\pi^{N_G(A_i)}(A_i) = |\Delta_0| = |\Gamma_0| = k_\pi^{\mathrm{Aut}_G(A_i)}(\mathrm{Inn}(A_i)).$$

(3) follows. □

Suppose that $A = A_1 \times \dots \times A_s$ and for every $i = 1, \dots, s$ by $\mathscr{K}_i$ we denote an A_i-class of π-Hall subgroups of A_i. The set

$$\mathscr{K}_1 \times \dots \times \mathscr{K}_s = \{\langle H_1, \dots, H_s\rangle \mid H_i \in \mathscr{K}_i,\ i = 1, \dots, s\}$$

is called the *product* of $\mathscr{K}_1, \dots, \mathscr{K}_s$. Clearly $\mathscr{K}_1 \times \dots \times \mathscr{K}_s$ is an A-class of π-Hall subgroups of A. It is also obvious that for a normal subgroup A of G every A-class of G-induced π-Hall subgroups is a product of some A_1-, …, A_s-classes of G-induced π-Hall subgroups. In particular, $k_\pi^G(A_i) \le k_\pi^G(A)$ for every $i = 1, \dots, s$. The reverse inequality fails in general.

Lemma 15. *Let H be a π-Hall subgroup and let $A = A_1 \times \dots \times A_s$ be a normal subgroup of G. Assume also that the subgroups $A_1, \dots, A_s$ are normal in G and $G = HAC_G(A)$. Then $k_\pi^G(A) = k_\pi^G(A_1) \cdot \ldots \cdot k_\pi^G(A_s)$.*

Proof. Two π-Hall subgroups P and Q of A are conjugate in A if and only if the π-Hall subgroups $P \cap A_i$ and $Q \cap A_i$ of A_i are conjugate in A_i for every $i = 1, \dots, s$. In order to prove the claim it is enough to show that the product of A_1-, …, A_s-classes of G-induced π-Hall subgroups is an A-class of G-induced π-Hall subgroups as well. Assume that $U_1, \dots, U_s$ are G-induced π-Hall subgroups of $A_1, \dots, A_s$, respectively. We show that

$$U = \langle U_1, \dots, U_s\rangle = U_1 \times \dots \times U_s$$

is a G-induced π-Hall subgroup of A. By Lemma 11 it is enough to show that for every $h \in H$ there exists $a \in A$ with $U^h = U^a$. Since $U_i = K_i \cap A_i$ for an appropriate π-Hall subgroup K_i of G, the set $\{U_i^{x_i} \mid x_i \in A_i\}$ is K_i-invariant, and so this set is invariant under $K_iA = HA$. In particular, $U_i^h = U_i^{a_i}$ for some $a_i \in A_i$. Thus,

$$U^h = U_1^h \times \dots \times U_s^h = U_1^{a_1} \times \dots \times U_s^{a_s} = U_1^a \times \dots \times U_s^a = U^a,$$

where $a = a_1 \dots a_s \in A$. □

Lemma 16. *Let H be a π-Hall subgroup, let $A = A_1 \times \dots \times A_s$ be a normal subgroup of G that acts transitively on the set $\{A_1, \dots, A_s\}$ by conjugation, and $G = HAC_G(A)$. Then $k_\pi^G(A) = k_\pi^G(A_i) = k_\pi^{N_G(A_i)}(A_i)$ for every $i = 1, \dots, s$.*

Proof. We show that each G-induced π-Hall subgroup of A_i is also an $N_G(A_i)$-induced subgroup. Indeed, if K is a π-Hall subgroup of G then $G = KAC_G(A)$ and, by Lemma 14, the identity $N_G(A_i) =$

$N_K(A_i)AC_G(A)$ holds. Moreover, $K \cap A_i = N_K(A_i) \cap A_i$ and $N_K(A_i)$ is a π-Hall subgroup of $N_G(A_i)$. So every G-induced π-Hall subgroup of A_i is also an $N_G(A_i)$-induced π-Hall subgroup, in particular, $k_\pi^G(A_i) \leq k_\pi^{N_G(A_i)}(A_i)$.

We show now that if $x, y \in G$ are in the same coset of G by $N_G(A_1)$, then for every G-induced π-Hall subgroup U_1 of A_1 the subgroups U_1^x and U_1^y are conjugate in $A_i = A_1^x = A_1^y$. It is enough to show that the subgroups U_1 and U_1^t, where $t = xy^{-1} \in N_G(A_1)$, are conjugate in A_1. Put $t = hac$, $a \in A$, $c \in C_G(A)$, $h \in H$. Since U_1^h and U_1^{hac} are conjugate in A_1, it is enough to show that U_1 and U_1^h are conjugate in A_1. Since $(ac)^{h^{-1}} \in AC_G(A) \leq N_G(A_1)$, the element h normalizes A_1 as well. Assume that $U_1 = U \cap A_1$ for a G-induced π-Hall subgroup U of A. Suppose that $\mathscr{K}$ is an A-class of π-Hall subgroups containing U, and let $\mathscr{K} = \mathscr{K}_1 \times \dots \times \mathscr{K}_s$, where $\mathscr{K}_i$ is an A_i-class of G-induced π-Hall subgroups. Clearly, $U_1 \in \mathscr{K}_1$. Since, by Lemma 11, $\mathscr{K}$ is H-invariant, H acts on the set $\{\mathscr{K}_1, \dots, \mathscr{K}_s\}$. The element h normalizes A_1, hence it fixes the A_1-class $\mathscr{K}_1$. In particular, the subgroups U_1 and U_1^h are in $\mathscr{K}_1$, and so they are conjugate in A_1.

Assume that $f \in H$ and $A_1^f = A_i$. For an A_1-class of π-Hall subgroups $\mathscr{K}_1$ we define the A_i-class $\mathscr{K}_1^f$ by $\mathscr{K}_1^f = \{U_1^f \mid U_1 \in \mathscr{K}_1\}$. As we noted above, $\mathscr{K}_1^f$ is an A_i-class of G-induced π-Hall subgroups.

Let $h_1 = 1, h_2, \dots, h_s$ be the right transversal of $N_H(A_1)$ in H. Since G acts transitively, up to renumbering we may assume that $A_i = A_1^{h_i}$ and $(N_G(A_1))^{h_i} = N_G(A_i)$. So $k_\pi^{N_G(A_1)}(A_1) = k_\pi^{N_G(A_i)}(A_i)$ for $i = 1, \dots, s$. Consider

$$\sigma : \mathscr{K}_1 \mapsto \mathscr{K}_1^{h_1} \times \dots \times \mathscr{K}_1^{h_s},$$

mapping an A_1-class of $N_G(A_1)$-induced π-Hall subgroups $\mathscr{K}_1$ to an A-class of π-Hall subgroups. Note that $\mathscr{K}_1^{h_1} \times \dots \times \mathscr{K}_1^{h_s}$ is always H-invariant and, by Lemma 11, it is an A-class of G-induced π-Hall subgroups. Notice also that σ is injective, and so $k_\pi^{N_G(A_1)}(A_1) \leq k_\pi^G(A)$. Consider the restriction τ of σ on the set of A_1-classes of G-induced π-Hall subgroups. We need to show that the image of τ coincides with the set of A-classes of G-induced π-Hall subgroups in order to complete the proof, since in this case we derive the inequality $k_\pi^G(A_1) \geq k_\pi^G(A)$.

Let $\mathscr{K} = \mathscr{K}_1 \times \dots \times \mathscr{K}_s$ be an A-class of G-induced π-Hall subgroups. It is enough to show that $\mathscr{K}_i = \mathscr{K}_1^{h_i}$ for every $i = 1, \dots, s$. Since G acts transitively on the set $\{A_1, \dots, A_s\}$, there exists an element $g \in G$ such that $A_1^g = A_i$. Let gct, where $a \in A$, $c \in C_G(A)$, and $t \in H$. Then $A_1^t = A_i$ and $t \in N_H(A_1)h_i$. As we already proved, $\mathscr{K}_1^t = \mathscr{K}_1^{h_i}$. By Lemma 11, $\mathscr{K}$ is H-invariant. Hence, $\mathscr{K}_i = \mathscr{K}_1^t = \mathscr{K}_1^{h_i}$ and $\mathscr{K} = \mathscr{K}_1^{h_1} \times \dots \times \mathscr{K}_1^{h_s} = \mathscr{K}_1\tau$. $\square$

2. A Conjugacy Criterion for Hall Subgroups

In this section we prove Theorem 1, Corollary 2, and provide an algorithm for determining whether G satisfies C_π by using a normal series of G.

Proof of Theorem 1. Assume that the claim is not true, and let G be a counterexample of minimal order. Then G possesses a π-Hall subgroup H and a normal subgroup A such that HA does not satisfy C_π. We choose A to be minimal. Let K be a π-Hall subgroup of HA that is not conjugate with H in HA. We divide into several steps the process of canceling the group G.

Clearly,

(1) $HA = KA$.

(2) *A is a minimal normal subgroup of G.*

Otherwise assume that M is a nontrivial normal subgroup of G that is contained properly in A. Put $\overline{G} = G/M$ and, given a subgroup B of G, denote BM/M by $\overline{B}$. By Lemma 9, $\overline{G}$ satisfies C_π $\overline{H}$, and $\overline{K}$ are π-Hall subgroups of $\overline{G}$, $\overline{A}$ is a normal subgroup of $\overline{G}$, $\overline{HA} = \overline{KA}$, and $|\overline{G}| < |G|$. In view of the minimality of G, the group $\overline{HA}$ satisfies C_π. So $\overline{H}$ and $\overline{K}$ are conjugate by an element of $\overline{A}$. Hence, the subgroups HM and KM are conjugate by an element of A. Without loss of generality, we may assume that $HM = KM$. In view of the choice of A, the group HM satisfies C_π. Hence H and K are conjugate by an element of $M \leq A$; a contradiction.

(3) $A \notin C_\pi$. *In particular, A is not solvable.*

Otherwise, by Lemma 5, the group HA satisfies C_π as an extension of a C_π-group by a π-group.

(4) *HA is a normal subgroup of G.*

Otherwise $N_G(HA)$ is a proper subgroup of G and, by Lemma 7, we have $N_G(HA) \in C_\pi$. In view of the minimality of G, it follows that $HA \in C_\pi$; a contradiction.

In view of (2) and (3)

(5) *A is a direct product of nonabelian simple groups $S_1, \dots, S_m$. The group G acts transitively on $\Omega = \{S_1, \dots, S_m\}$ by conjugation.*

Let $\Delta_1, \dots, \Delta_s$ be the orbits of HA on Ω, and put $T_j = \langle \Delta_j \rangle$ for every $j = 1, \dots, s$. In view of (4) and (5)

(6) *G acts transitively on $\{T_1, \dots, T_s\}$ by conjugation. The subgroup A is a direct product of $T_1, \dots, T_s$ and each of these subgroups is normal in HA.*

Assume that $S \in \Omega$, and let T be a subgroup, generated by the orbit from the set $\{\Delta_1, \dots, \Delta_s\}$ that contains S. By Lemma 16,

(7) $k_\pi^{HA}(T) = k_\pi^{HA}(S)$.

By (7) and Lemmas 12 and 15

(8) $k_\pi^G(A) = k_\pi^{HA}(A)\big(k_\pi^{HA}(T)\big)^s = \big(k_\pi^{HA}(S)\big)^s$.

By (8), Theorem 10, and Lemmas 14 and 16

(9) *$k_\pi^G(A)$ is a π-number.*

By Lemma 11

(10) *HA fixes every A-class of G-induced π-Hall subgroups.*

Since $G \in C_\pi$,

(11) *G acts transitively on the set of A-classes of G-induced π-Hall subgroups.*

In view of (10), the subgroup HA lies in the kernel of this action. Now, by (11)

(12) *$k_\pi^G(A)$ is a π'-number.*

By (9) and (12)

(13) $k_\pi^G(A) = 1$.

Now by Lemma 13

(14) *$HA \in C_\pi$; a contradiction.* □

Proof of Corollary 2. *Necessity.* If $G \in C_\pi$ then, by Lemma 9, $G/A \in C_\pi$ as well. Let K/A be a π-Hall subgroup of G/A. By Lemma 8, there exists a π-Hall subgroup H of G such that $K = HA$. By Theorem 1, the group $K = HA$ satisfies C_π.

Sufficiency. Let H be a π-Hall subgroup of K. Since K/A is a π-Hall subgroup of G/A, we have $|G : H| = |G : K| \cdot |K : H|$ is a π'-number, and so H is a π-Hall subgroup of G. In particular, $G \in E_\pi$. Let H_1 and H_2 be π-Hall subgroups of G. Since $G/A \in C_\pi$, the subgroups H_1A/A and H_2A/A are conjugate in G/A, and we may assume that $H_1A = H_2A$. However, $H_1A \in C_\pi$, and so H_1 and H_2 are conjugate. Thus, $G \in C_\pi$. □

Lemma 17. *Assume that $G = HA$, where H is a π-Hall subgroup, A is a normal subgroup of G, and $A = S_1 \times \dots \times S_k$ is a direct product of simple groups. Then $G \in C_\pi$ if and only if* $\mathrm{Aut}_G(S_i) \in C_\pi$ *for every $i = 1, \dots, k$.*

Proof. By the Hall theorem and Lemma 5 we may assume that $S_1, \dots, S_k$ are nonabelian simple groups, and so the set $\{S_1, \dots, S_k\}$ is invariant under the action of G by conjugation. Moreover, this set is partitioned into the orbits $\Omega_1, \dots, \Omega_m$. Denote an element of Ω_j by S_{i_j}. By Lemmas 15 and 16, we obtain

$$k_\pi^G(A) = k_\pi^G(S_{i_1}) \cdot \ldots \cdot k_\pi^G(S_{i_m}). \tag{1}$$

Lemmas 12 and 13 imply that $HA \in C_\pi$ if and only if $k_\pi^G(A) = 1$, and in view of (1) if and only if $k_\pi^G(S_i) = 1$ for every i. By Lemmas 14 and 16, $k_\pi^G(S_i) = k_\pi^{N_G(S_i)}(S_i) = k_\pi^{\mathrm{Aut}_G(S_i)}(S_i)$ for every i. Moreover, since $|\operatorname{Aut}_G(S_i) : S_i|$ is a π-number, in view of Lemma 13, $k_\pi^{\operatorname{Aut}_G(S_i)}(S_i) = 1$ if and only if $\operatorname{Aut}_G(S_i) \in C_\pi$. Therefore, $k_\pi^G(S_i) = 1$ for every i if and only if $\operatorname{Aut}_G(S_i) \in C_\pi$. $\square$

Now we are able to give an algorithm reducing the problem, whether a finite group satisfies C_π, to the check of C_π-property in some almost simple groups. Assume that

$$G = G_0 > G_1 > \cdots > G_n = 1 \tag{2}$$

is a chief series of G. Put $H_1 = G = G_0$. Suppose that for some $i = 1, \ldots, n$ the group H_i is constructed so that $G_{i-1} \leq H_i$ and H_i/G_{i-1} is a π-Hall subgroup of G/G_{i-1}. Since (2) is a chief series,

$$G_{i-1}/G_i = S_1^i \times \cdots \times S_{k_i}^i,$$

where $S_1^i, \ldots, S_{k_i}^i$ are simple groups. We check whether

$$\operatorname{Aut}_{H_i}(S_1^i) \in C_\pi, \ldots, \operatorname{Aut}_{H_i}(S_{k_i}^i) \in C_\pi.$$

If so then by Lemma 17, $H_i/G_i \in C_\pi$ and we can take the full preimage of a π-Hall subgroup of H_i/G_i to be equal to H_{i+1}. Otherwise, by Corollary 2, we deduce that $G \notin C_\pi$ and stop the process. From Corollary 2 it follows that G satisfies C_π if and only if the group H_{n+1} can be constructed. Notice that in this case H_{n+1} is a π-Hall subgroup of G.

Corollary 18. *If either $2 \notin \pi$ or $3 \notin \pi$ then $G \in C_\pi$ if and only if every nonabelian composition factor of G satisfies C_π.*

PROOF. The *sufficiency* follows from Lemma 5. We prove the *necessity*. By the above algorithm, we may assume that $S \leq G \leq \operatorname{Aut}(S)$ for a nonabelian finite simple group S, and G/S is a π-number. We need to show that $S \in C_\pi$. Assume the contrary. Then Lemmas 11 and 13 imply that G stabilizes precisely one class of π-Hall subgroups of S. Therefore, S possesses at least three classes of π-Hall subgroups. On the other hand, since either $2 \notin \pi$ or $3 \notin \pi$, by Theorem 10 the number of classes of π-Hall subgroups in S is not greater than 2; a contradiction. $\square$

Notice that in the case $2 \notin \pi$ the corollary is immediate from [8, Theorem A] and Lemma 5.

References

1. *Hall P.*, "Theorems like Sylow′s," Proc. London Math. Soc., **6**, No. 22, 286–304 (1956).
2. *Revin D. O. and Vdovin E. P.*, "Hall subgroups of finite groups," Contemp. Math., No. 402, 229–263 (2006).
3. *Revin D. O. and Vdovin E. P.*, "On the number of classes of conjugate Hall subgroups in finite simple groups," J. Algebra (to appear). Available at http://arxiv.org/abs/0912.1922.
4. *Suzuki M.*, Group Theory. II, Springer-Verlag, New York (1986).
5. *Gross F.*, "On the existence of Hall subgroups," J. Algebra, **98**, No. 1, 1–13 (1986).
6. *Revin D. O. and Vdovin E. P.*, "Existence criterion for Hall subgroups of finite groups," J. Group Theory (to appear). Available at http://arxiv.org/abs/0803.3868.
7. *Revin D. O.*, "Hall π-subgroups of finite Chevalley groups whose characteristic belongs to π," Siberian Adv. Math., **9**, No. 2, 25–71 (1999).
8. *Gross F.*, "Conjugacy of odd order Hall subgroups," Bull. London Math. Soc., **19**, No. 79, 311–319 (1987).

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