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On the Intersections of Solvable Hall Subgroups in Finite Groups

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Abstract—The following conjecture is considered: if a finite group G possesses a solvable π-Hall subgroup H, then there exist elements $x, y, z, t \in G$ such that the identity $H \cap H^x \cap H^y \cap H^z \cap H^t = O_\pi(G)$ holds. Under additional conditions on G and H, it is shown that a minimal counterexample to this conjecture must be an almost simple group of Lie type.
Keywords: solvable Hall subgroup, finite simple group, π-radical.

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INTRODUCTION

The notation in this paper is standard and agrees with that of [1]. Since we consider finite groups only, the term "group" always means "finite group." By π, we always denote some set of primes, and π' is its complement in the set of all primes. For a positive integer n, we denote by $\pi(n)$ the set of all its prime divisors; the number n is called a π-*number* if $\pi(n) \subseteq \pi$. For a finite group G, we set $\pi(G) := \pi(|G|)$; the group G is called a π-*group* if $\pi(G) \subseteq \pi$. The maximal normal π-subgroup of G is denoted by $O_\pi(G)$. A subgroup H of G is called a π-*Hall subgroup in* G if $\pi(H) \subseteq \pi$ and $\pi(|G : H|) \subseteq \pi'$. The set of all π-Hall subgroups of a finite group G is denoted by $\mathrm{Hall}_\pi(G)$, and the set of all Sylow p-subgroups of G is denoted by $\mathrm{Syl}_p(G)$. A group G is called π-*solvable* if it possesses a subnormal series $1 = G_0 < G_1 < G_2 < \ldots < G_{n-1} < G_n = G$ such that all its factors are either π'-groups or solvable groups. Further, Sym_n is the symmetric group of order n. If G is a group and S is a subgroup of Sym_n, then we denote by $G \wr S$ the permutation wreath product of the group G and the subgroup S.

After Hall [2], we say that a group G *satisfies the property* E_π (or, shortly, $G \in E_\pi$) if G possesses a π-Hall subgroup. If $G \in E_\pi$ and any two π-Hall subgroups of G are conjugate, then we say that G *satisfies the property* C_π ($G \in C_\pi$). If $G \in C_\pi$ and every π-subgroup of G is embedded in a π-Hall subgroup of G, then we say that G *satisfies the property* D_π ($G \in D_\pi$).

Let A, B, and H be subgroups of a group G such that $B \trianglelefteq A$. Then, $N_H(A/B) := N_H(A) \cap N_H(B)$ is called the *normalizer of the section* A/B in H. If $x \in N_H(A/B)$, then x induces an automorphism of A/B acting by the rule $Ba \mapsto Bx^{-1}ax$. Thus, there exists a homomorphism

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$N_H(A/B) \to \mathrm{Aut}(A/B)$. The image of the normalizer $N_H(A/B)$ under this homomorphism is denoted by $\mathrm{Aut}_H(A/B)$ and is called the *group of induced automorphisms* of the section A/B in H. Note that, for composition factors of the group G that are isomorphic to A/B, the group of induced automorphisms $\mathrm{Aut}_G(A/B)$ depends on the choice of the subgroups A and B, i.e., on the choice of the composition series. If $A \le G$, then $\mathrm{Aut}_G(A) := \mathrm{Aut}_G(A/1)$.

Let G be a subgroup of the group Sym_n. A partition $\{P_1, P_2, \dots P_m\}$ of the set $\{1, \dots, n\}$ is called an *asymmetric partition* for the group G if only the identity element of G fixes this partition, i.e., if the relations $P_j x = P_j$ for $j = 1, \dots, m$ imply that $x = 1$. It is clear that the partition $\{\{1\}, \{2\}, \dots, \{n\}\}$ is asymmetric for any group G.

In this paper, we consider the following conjecture.

Conjecture [3]. *Let G be a finite group with a solvable π-Hall subgroup H. Then, there exist elements $x, y, z, t \in G$ such that*

$$H \cap H^x \cap H^y \cap H^z \cap H^t = O_\pi(G). \tag{1}$$

Under some additional conditions on G and H, we will show that a minimal counterexample to this conjecture is an almost simple group; we will also verify the conjecture for an almost simple group G whose socle is an alternating or sporadic group or the Tits group.

Proposition. *Let G be a finite group with a solvable π-Hall subgroup H. If the conjecture is true for G and H, then $|H/O_\pi(G)| \le |G : H|^4$.*

Proof. Let $\tilde{\ } : G \to G/O_\pi(G)$ be the natural homomorphism. Using the known identity $|A \cdot B| = \dfrac{|A| \cdot |B|}{|A \cap B|}$, where A and B are subgroups of G, we conclude that, if the conjecture is true, then the solvable π-Hall subgroup H satisfies the series of inequalities

$$|\tilde{G}| \ge |\tilde{H} \cdot \tilde{H}^x| = \frac{|\tilde{H}| \cdot |\tilde{H}^x|}{|\tilde{H} \cap \tilde{H}^x|} = \frac{|\tilde{H}| \cdot |\tilde{H}^x|}{|\tilde{H} \cap \tilde{H}^x|} \cdot \frac{|\tilde{H} \cap \tilde{H}^x| \cdot |\tilde{H}^y|}{|\tilde{H} \cap \tilde{H}^x \cap \tilde{H}^y| \cdot |(\tilde{H} \cap \tilde{H}^x)\tilde{H}^y|}$$

$$\ge \frac{|\tilde{H}| \cdot |\tilde{H}^x| \cdot |\tilde{H}^y|}{|\tilde{H} \cap \tilde{H}^x \cap \tilde{H}^y| \cdot |\tilde{G}|} \ge \dots \ge \frac{|\tilde{H}|^5}{|\tilde{G}|^3},$$

which implies the required inequality. The proposition is proved.

Note that Passman proved in [4] that a p-solvable group always possesses three Sylow p-subgroups such that their intersection coincides with $O_p(G)$. Later, Zenkov proved the same statement for an arbitrary group (see [5, Corollary C]). Dolfi proved in [6] that, if $2 \notin \pi$, then every π-solvable group G possesses three conjugate π-Hall subgroups such that their intersection coincides with $O_\pi(G)$. Finally, Dolfi proved in [7] that, in every π-solvable group G, there exist elements $x, y \in G$ such that $H \cap H^x \cap H^y = O_\pi(G)$, where H is a π-Hall subgroup in G (see also [8]). Based on these facts, it is natural to conjecture that, in the case of an arbitrary finite group G and an arbitrary solvable π-Hall subgroup H from G, the intersection of four subgroups conjugate to H with respect to G is equal to $O_\pi(G)$. However, a counterexample to this conjecture was constructed in [3], in which the intersection of five subgroups conjugate to H with respect to G was equal to $O_\pi(G)$. As a consequence of that example, the above conjecture was proposed in [3]. On the other hand, the solvability condition for H in the conjecture is essential. Indeed, if p is a prime, then S_{p-1} is a p'-Hall subgroup of S_p but the intersection of any $p-2$ conjugate p'-Hall subgroups is not equal to 1.

1. PRELIMINARY RESULTS

The statements collected in the following lemma are well known.

Lemma 1. *Let A be a normal subgroup of a group G. Then, the following statements hold*:

(a) *if H is a π-Hall subgroup of G, then HA/A and $H \cap A$ are π-Hall subgroups of the groups G/A and A, respectively*;

(b) *if there exists a subnormal series of the group G such that all its factors are either π- or π'-groups, then $G \in D_\pi$*;

(c) *if G/A is a π-group and $H \in \mathrm{Hall}_\pi(A)$, then a π-Hall subgroup H_0 of the group G with $H_0 \cap A = H$ exists if and only if G, acting by conjugation, leaves the set $\{H^a \mid a \in A\}$ invariant.*

Proof. Statement (a) follows from the following facts: the indices $|(G/A) : (HA/A)|$ and $|A : (A \cap H)|$ divide the index $|G : H|$ and the orders $|HA/A|$ and $|A \cap H|$ divide the order $|H|$.

Statement (b) follows from the Schur–Zassenhaus theorem (see [2, Theorems D6, D7]) and from the solvability of groups of odd order.

Let us prove (c). The necessity is obvious, because $N_G(H) = H_0N_A(H)$ and $G = H_0A$. Let us prove the sufficiency. The group G leaves the set $\{H^a \mid a \in A\}$ invariant; hence, it follows from the Frattini argument that $G = AN_G(H)$. Further, the normalizer $N_G(H)$ possesses a normal series $1 \trianglelefteq H \trianglelefteq N_A(H) \trianglelefteq N_G(H)$, and all sections of this series are either π- or π'-groups. It follows from statement (b) that $N_G(H) \in D_\pi$. In particular, there exists a π-Hall subgroup H_0 of the group $N_G(H)$ and $H \leq H_0$. Since $\pi(G/A) \subseteq \pi$, $\pi(|A : H|) \subseteq \pi'$, and $G = AN_G(H)$, we conclude that $|G : N_G(H)|$ is a π'-number. Thus, $H_0 \in \mathrm{Hall}_\pi(G)$. The lemma is proved.

Let S be a nonabelian finite simple group, and let a group G be such that there exists a normal subgroup $T = S_1 \times \ldots \times S_n$ of G satisfying the following conditions:

(1) $S_1 \simeq \ldots \simeq S_k \simeq S$;

(2) the subgroups $S_1, \ldots, S_k$ are conjugate in G;

(3) $C_G(T) = 1$.

By condition (2), the subgroups $N_G(S_1), \ldots, N_G(S_k)$ are conjugate in G. Let $\rho : G \to \mathrm{Sym}_k$ be the permutation representation of the group G by right multiplication of G on the right cosets of the subgroup $N_G(S_1)$. Since the action corresponding to ρ is equivalent to the conjugation action of G on the set $\{S_1, \ldots, S_k\}$, we find that $G\rho$ is a transitive subgroup of Sym_k. By [9, Theorem IV.1.4], there exists a monomorphism

$$\varphi : G \to (N_G(S_1) \times \ldots \times N_G(S_k)) : (G\rho) = N_G(S_1) \wr (G\rho) := L.$$

Consider the natural homomorphism

$$\psi : L \to L/(C_G(S_1) \times \ldots \times C_G(S_k)).$$

Denoting the subgroup $\mathrm{Aut}_G(S_i) = N_G(S_i)/C_G(S_i)$ by A_i, we obtain

$$\varphi \circ \psi : G \to (A_1 \times \ldots \times A_k) : (G\rho)$$

is a homomorphic embedding of G in $(A_1 \times \ldots \times A_k) : (G\rho)$, which is isomorphic to $A_1 \wr (G\rho) =: \overline{G}$. The kernel of this homomorphism coincides with $C_G(S_1, \ldots, S_k) = 1$; i.e., $\varphi \circ \psi$ is a monomorphism, and we will identify G with the subgroup $G(\varphi \circ \psi)$ of the group $\overline{G}$.

Lemma 2. *Assume that a nonabelian simple subgroup S of a group G and a π-Hall subgroup H of G, where $G = (S_1 \times \ldots \times S_k)H$, satisfy conditions* (1)–(3) *introduced above. Then, in the group $\overline{G}$ introduced before this lemma, there exists a π-Hall subgroup $\overline{H}$ such that $\overline{H} \cap G = H$.*

Proof. The factor group $\overline{G}/(S_1 \times \ldots \times S_k)$ is a π-group. By construction, $\mathrm{Aut}_{\overline{G}}(S_1) = \mathrm{Aut}_G(S_1)$ and $\mathrm{Aut}_{\overline{G}}(S_i) = \mathrm{Aut}_G(S_i) \simeq \mathrm{Aut}_G(S_1)$ for all i. By Lemma 1(c), a π-Hall subgroup M of the group $S_1 \times \ldots \times S_k$ is embedded in a π-Hall subgroup of the group G (respectively, of the group $\overline{G}$) if and only if the set $\{M^s \mid s \in S_1 \times \ldots \times S_k\}$ is invariant under the conjugation action of the group G (respectively, of the group $\overline{G}$). In order to complete the proof, it is sufficient to show that, if H is a π-Hall subgroup of G and $M = H \cap (S_1 \times \ldots \times S_k)$, then the class $\{M^s \mid s \in S_1 \times \ldots \times S_k\}$ is $\overline{G}$-invariant. Indeed, in this case, there exists a π-Hall subgroup $\overline{H}$ of $\overline{G}$ such that $\overline{H} \cap (S_1 \times \ldots \times S_k) = M$. Hence, $H, \overline{H} \leq N_{\overline{G}}(M)$. Since $\overline{G}/(S_1 \times \ldots \times S_k)$ is a π-group, Lemma 1(b) implies that $N_{\overline{G}}(M) \in D_\pi$; therefore, H is conjugate to a subgroup of the group $\overline{H}$. The assertion that the class $\{M^s \mid s \in S_1 \times \ldots \times S_k\}$ is $\overline{G}$-invariant follows easily from the construction of $\overline{G}$ and from the facts that G is embedded in it and that the class $\{M^s \mid s \in S_1 \times \ldots \times S_k\}$ is G-invariant by Lemma 1(c). Indeed, every element from $\overline{G}$ can be written as $(a_1, \ldots, a_k)g$, where $a_i \in A_i$ and $g \in G$. Since the set $\{M^s \mid s \in S_1 \times \ldots \times S_k\}$ is G-invariant, we may assume that $g = 1$. Moreover, for every i, the class $\{(M \cap S_i)^x \mid x \in S_i\}$ is A_i-invariant because $A_i = S_i \,\mathrm{Aut}_H(S_i)$, from which the assertion of the lemma follows.

Lemma 3 [10, Theorem 1]**.** *Let A be an abelian subgroup of a finite group G. Then, there exists $x \in G$ such that $A \cap A^x \leq F(G)$.*

Corollary 1. *Let $G = PR$, where P is a normal Sylow p-subgroup of the group G and R is an abelian p'-Hall subgroup of G such that $C_R(P) = 1$. Then, there exists $x \in P$ such that $R \cap R^x = 1$.*

Proof. By Lemma 3, there exists an element $x \in G$ such that $R \cap R^x \leq F(G)$. Hence, $[P, R \cap R^x] = 1$. Since $C_R(P) = 1$, we have $R \cap R^x = 1$. Since $G = PR$, it follows that $x = r \cdot p$, where $r \in R$ and $p \in P$. Therefore, $1 = R \cap R^x = R \cap R^{rp} = R \cap R^p$. The corollary is proved.

Lemma 4 [11, Theorem 1.2]**.** *Let G be a solvable permutation group on the set $\{1, 2, \ldots, n\}$. Then, there exists an asymmetric partition $\{P_1, P_2, \ldots, P_m\}$ of this set with $m \leq 5$.*

2. REDUCTION TO AN ALMOST SIMPLE GROUP

Definition 1. We say that a group G is an $\mathbf{(Orb)}_\pi$-group if it possesses a π-Hall subgroup, any π-Hall subgroup H satisfies the conclusion of the conjecture, and, under the conjugation action of H on the set of quadruples $\left(H^x, H^y, H^z, H^t\right)$, there exist at least five orbits.

Definition 2. We say that a group G is a $\mathbf{(CI)}_\pi$-group if, for any component $\overline{S}$ from $E(\overline{G})$, where $\overline{G} := G/S(G)$, every group L such that $\mathrm{Inn}(\overline{S}) \leq L \leq \mathrm{Aut}(\overline{S})$ is an $\mathbf{(Orb)}_\pi$-group.

Theorem 1. *Suppose that G is a finite $\mathbf{(CI)}_\pi$-group and H is a solvable π-Hall subgroup of G. Then, the group G contains elements x, y, z, t such that*

$$H \cap H^x \cap H^y \cap H^z \cap H^t = O_\pi(G).$$

Moreover, if G is not π-closed, then it is an $\mathbf{(Orb)}_\pi$-group.

Proof. Let G be a counterexample of minimal order to the statement of the theorem. If $O_\pi(G) \in \mathrm{Hall}_\pi(G)$, then there is nothing to prove. Therefore, we assume that $O_\pi(G) \notin \mathrm{Hall}_\pi(G)$; i.e., no π-Hall subgroup of the group G is normal.

Case 1. G is solvable.

By [7, Theorem 1.3] or [8, Theorem 1.3], there exist elements $x, y \in G$ such that $H \cap H^x \cap H^y = O_\pi(G)$. Since H is not a normal subgroup of G, we may assume that $H \neq H^x$ and $H \neq H^y$. Thus, $H \not\leq N_G(H^x)$ and $H \not\leq N_G(H^y)$, otherwise the subgroups HH^x and HH^y would be π-subgroups of G of order greater than $|H|$. Therefore, the quadruples

$$(H, H^x, H^y, H^x),\ (H^x, H, H^y, H^x),\ (H^x, H^y, H, H^x),\ (H^x, H^y, H^x, H),\ (H, H, H^x, H^y)$$

are in different orbits under the conjugation action of the subgroup H, and these quadruples satisfy (1). Thus, G is an $\mathbf{(Orb)}_\pi$-group.

Case 2. G is not solvable.

Let $\overline{G} = G/S(G)$, where $S(G)$ is the solvable radical of the group G. The solvability of the subgroup H implies that the subgroup $O_\pi(G)$ is also solvable; hence, $O_\pi(G) \leq S(G)$.

Assume that $S(G) \neq 1$. Then, $\overline{G}$ is a $\mathbf{(CI)}_\pi$-group; therefore, by induction and in view of the inequality $|\overline{G}| < |G|$, the group $\overline{G}$ is an $\mathbf{(Orb)}_\pi$-group. By Lemma 1(a), $H_1 := H \cap S(G)$ is a π-Hall subgroup of $S(G)$ and $|S(G)| < |G|$. Hence, $S(G)$ satisfies the conclusion of the theorem. Since $S(G)$ is solvable, it follows that all its π-Hall subgroups are conjugate; by the Frattini argument, $G = N_G(H_1)S(G)$. Choose elements $x_1, y_1, z_1, t_1 \in S(G)$ so that

$$H_1 \cap H_1^{x_1} \cap H_1^{y_1} \cap H_1^{z_1} \cap H_1^{t_1} = O_\pi(S(G)) = O_\pi(G).$$

Let us choose a π-Hall subgroup H in $N_G(H_1)$. We choose elements $x_2, y_2, z_2, t_2 \in G$ so that

$$\overline{H} \cap \overline{H}^{\bar{x}_2} \cap \overline{H}^{\bar{y}_2} \cap \overline{H}^{\bar{z}_2} \cap \overline{H}^{\bar{t}_2} = \bar{1}. \tag{2}$$

In $N_G(H_1^{x_1})$, $N_G(H_1^{y_1})$, $N_G(H_1^{z_1})$, and $N_G(H_1^{t_1})$, we choose π-Hall subgroups H^x, H^y, H^z, and H^t such that their images in $\overline{G}$ coincide with $\overline{H}^{\bar{x}_2}$, $\overline{H}^{\bar{y}_2}$, $\overline{H}^{\bar{z}_2}$, and $\overline{H}^{\bar{t}_2}$, respectively (such subgroups exist by Lemma 1(c)).

Let us show that $D = H \cap H^x \cap H^y \cap H^z \cap H^t = O_\pi(G)$. We have

$$D \cap S(G) = H \cap H^x \cap H^y \cap H^z \cap H^t \cap S(G)$$
$$= (H \cap S(G)) \cap (H^x \cap S(G)) \cap (H^y \cap S(G)) \cap (H^z \cap S(G)) \cap (H^t \cap S(G))$$
$$= H_1 \cap H_1^{x_1} \cap H_1^{y_1} \cap H_1^{z_1} \cap H_1^{t_1} = O_\pi(G);$$

i.e., $D \cap S(G) = O_\pi(G)$. Moreover, by equality (2),

$$\overline{D} = \overline{H} \cap \overline{H}^{\bar{x}_2} \cap \overline{H}^{\bar{y}_2} \cap \overline{H}^{\bar{z}_2} \cap \overline{H}^{\bar{t}_2} = \bar{1};$$

thus, $D = O_\pi(G)$. Since G is unsolvable, the factor group $\overline{G}$ is unsolvable as well. By induction, under the conjugation action of the subgroup $\overline{H}$, there exist at least five orbits of quadruples $(\overline{H}^{\bar{x}_2}, \overline{H}^{\bar{y}_2}, \overline{H}^{\bar{z}_2}, \overline{H}^{\bar{t}_2})$ satisfying equality (2). Consequently, there exist at least five orbits of quadruples (H^x, H^y, H^z, H^t) satisfying equality (1), because the full preimages of representatives of five orbits in the factor are in five different orbits in G. This contradicts the fact that G is a counterexample to the assertion of the theorem.

Thus, $S(G) = 1$. Consider the subgroup $G_1 = F^*(G)H$. Then, by [12, Theorem 9.8], $C_G(F^*(G)) \leq F^*(G)$ and, since $S(G) = 1$, we have $F(G) = 1$ and $F^*(G) = E(G) = S_1 \times \ldots \times S_k$, where $S_1, \ldots, S_k$ are nonabelian simple groups. Therefore, H acts by conjugation faithfully on $E(G)$. If $G \neq G_1$, then, by induction (clearly, G_1 is a $\mathbf{(CI)}_\pi$-group), G_1 satisfies the assertion of the theorem. Hence, G also satisfies the assertion of the theorem; i.e., G is not a counterexample. Therefore, $G = G_1 = E(G)H$. Since $S_1, \ldots, S_k$ are nonabelian simple groups, the group G, acting by conjugation, permutes the elements of $\{S_1, \ldots, S_k\}$.

Let $E_1 := \langle S_1^H \rangle$. Since $E(G) = S_1 \times \ldots \times S_k$, we have $E(G) = E_1 \times E_2$, where E_1 and E_2 are H-admissible subgroups. By Remak's theorem [13, Theorem 4.3.9], there exists a homomorphism $G \to G/C_G(E_1) \times G/C_G(E_2)$ such that it maps G to the subdirect product of the groups $G/C_G(E_1)$ and $G/C_G(E_2)$ and its kernel coincides with $C_G(E_1) \cap C_G(E_2) = C_G(E(G)) = 1$. Denote by π_1 and π_2 the projection mappings of the group G on $G/C_G(E_1)$ and $G/C_G(E_2)$, respectively. Since $G = E(G)H$, $E_1 \leq \mathrm{Ker}(\pi_2)$, and $E_2 \leq \mathrm{Ker}(\pi_1)$, we have $G\pi_1 = E_1(H\pi_1)$ and $G\pi_2 = E_2(H\pi_2)$ (we identify $E_i\pi_i$ and E_i in view of the isomorphism $E_i\pi_i \simeq E_i$).

Assume that $E_1 \neq E(G)$. Then, by induction, for every $i \in \{1, 2\}$, there exist elements x_i, y_i, z_i, t_i of the group $E_i(H\pi_i)$ such that

$$(H\pi_i) \cap (H\pi_i)^{x_i} \cap (H\pi_i)^{y_i} \cap (H\pi_i)^{z_i} \cap (H\pi_i)^{t_i} = 1. \tag{3}$$

Since $G\pi_i = E_i(H\pi_i)$, we may assume that the elements x_i, y_i, z_i, t_i are in E_i. Consider the elements $x = x_1x_2$, $y = y_1y_2$, $z = z_1z_2$, and $t = t_1t_2$. Since (3) is true for all i, it follows that the elements x, y, z, t satisfy (1). There exist at least five orbits of quadruples $((H\pi_1)^{x_1}, (H\pi_1)^{y_1}, (H\pi_1)^{z_1}, (H\pi_1)^{t_1})$ under the action of the group $H\pi_1$; hence, there exist at least five orbits of quadruples (H^x, H^y, H^z, H^t) under the action of H. Thus, in this case, G satisfies the assertion of the theorem; i.e., G is not a counterexample.

Thus, $E_1 = E(G)$ and H acts transitively on the set $\{S_1, \ldots, S_k\}$. Since $\mathrm{Aut}(S_1)$ is an $\mathbf{(Orb)}_\pi$-group, we may assume that $k > 1$. By Lemma 2, we also may assume that $G = (A_1 \times \ldots \times A_k) : L = A_1 \wr L$, where $A_i = \mathrm{Aut}_G(S_i)$ and $L = G\rho = H\rho \leq \mathrm{Sym}_k$ (in particular, L is a solvable π-group). Denote $H \cap A_i$ by H_i. By Lemma 1, the subgroup H_i is a solvable π-Hall subgroup of A_i. Since A_1 is an $\mathbf{(Orb)}_\pi$-group under the action of the group H_1, there exist at least five orbits of quadruples $(H_1^{x_1}, H_1^{y_1}, H_1^{z_1}, H_1^{t_1})$ satisfying the equality

$$H_1 \cap H_1^{x_1} \cap H_1^{y_1} \cap H_1^{z_1} \cap H_1^{t_1} = 1.$$

Let $(H_1^{x_{1,j}}, H_1^{y_{1,j}}, H_1^{z_{1,j}}, H_1^{t_{1,j}})$ be a representative of the jth orbit ($j = 1, 2, 3, 4, 5$). Let $h \in H$. Then, $S_1^h = S_i$ for some $i \in \{1, \ldots, k\}$; hence, $H_1^h = H_i$. Consider the representatives

$$(H_1^{x_{1,j}}, H_1^{y_{1,j}}, H_1^{z_{1,j}}, H_1^{t_{1,j}}) \text{ and } (H_1^{x_{1,l}}, H_1^{y_{1,l}}, H_1^{z_{1,l}}, H_1^{t_{1,l}})$$

of different orbits of quadruples under the action of H_1. It is clear that

$$\big((H_1^{x_{1,j}})^h, (H_1^{y_{1,j}})^h, (H_1^{z_{1,j}})^h, (H_1^{t_{1,j}})^h\big) \text{ and } \big((H_1^{x_{1,l}})^h, (H_1^{y_{1,l}})^h, (H_1^{z_{1,l}})^h, (H_1^{t_{1,l}})^h\big)$$

are also representatives of different orbits of quadruples under the action of H_i. Moreover, the set

$$\Big\{\big((H_1^{x_{1,j}})^h \cap S_i, (H_1^{y_{1,j}})^h \cap S_i, (H_1^{z_{1,j}})^h \cap S_i, (H_1^{t_{1,j}})^h \cap S_i\big) \mid h \in H,\ S_1^h = S_i\Big\}$$

is an orbit under the action of H_i. For each of these H_i-orbits, we choose one representative. Then, each of these representatives can be written in the form

$$(H_i^{x_{i,j}}, H_i^{y_{i,j}}, H_i^{z_{i,j}}, H_i^{t_{i,j}})$$

for appropriate $x_{i,j}, y_{i,j}, z_{i,j}, t_{i,j} \in S_i$, where $j \in \{1,2,3,4,5\}$. Note also that the subgroup $H_{1,j_1} \times \ldots \times H_{k,j_k}$, where $H_{i,j_i} \in \{H_i, H_i^{x_{i,j}}, H_i^{y_{i,j}}, H_i^{z_{i,j}}, H_i^{t_{i,j}} \mid j = 1,2,3,4,5\}$, is a π-Hall subgroup of $S_1 \times \ldots \times S_k$ (see Lemma 1(a)) and is contained in some π-Hall subgroup of G (see Lemma 1(c)).

By Lemma 4, there exists an asymmetric partition $\{P_1, P_2, P_3, P_4, P_5\}$ of the set $\{1, \ldots, k\}$ (some of these subsets may be empty) such that only the identity element of L ($\simeq G/(A_1 \times \ldots \times A_k)$) stabilizes this partition. Set $j(i) = m$ for $i \in P_m$ and consider the subgroups

$$M_1 = H_1 \times \ldots \times H_k, \quad M_2 = H_1^{x_{1,j(1)}} \times \ldots \times H_k^{x_{k,j(k)}}, \quad M_3 = H_1^{y_{1,j(1)}} \times \ldots \times H_k^{y_{k,j(k)}},$$
$$M_4 = H_1^{z_{1,j(1)}} \times \ldots \times H_k^{z_{k,j(k)}}, \quad M_5 = H_1^{t_{1,j(1)}} \times \ldots \times H_k^{t_{k,j(k)}}.$$

By construction, there exist elements $x, y, z, t \in S_1 \times \ldots \times S_k$ such that $M_2 = M_1^x$, $M_3 = M_1^y$, $M_4 = M_1^z$, $M_5 = M_1^t$. We will show that $H \cap H^x \cap H^y \cap H^z \cap H^t = 1$. Note that, by construction, $H \cap (S_1 \times \ldots \times S_k) = M_1$, $H^x \cap (S_1 \times \ldots \times S_k) = M_2$, $H^y \cap (S_1 \times \ldots \times S_k) = M_3$, $H^z \cap (S_1 \times \ldots \times S_k) = M_4$, $H^t \cap (S_1 \times \ldots \times S_k) = M_5$; hence, if $h \in H \cap H^x \cap H^y \cap H^z \cap H^t$, then h normalizes the subgroups M_1, M_2, M_3, M_4, M_5. Assume that $S_i^h = S_{h(i)}$ for some element $h(i) \in \{1, \ldots, k\}$. Then, $H_i^h = H_{h(i)}$ and the tuples

$$\big((H_i^{x_{i,j(i)}})^h, (H_i^{y_{i,j(i)}})^h, (H_i^{z_{i,j(i)}})^h, (H_i^{t_{i,j(i)}})^h\big), \big(H_{h(i)}^{x_{h(i),j(h(i))}}, H_{h(i)}^{y_{h(i),j(h(i))}}, H_{h(i)}^{z_{h(i),j(h(i))}}, H_{h(i)}^{t_{h(i),j(h(i))}}\big)$$

are in the same $H_{h(i)}$-orbit. Hence, $j(i)$ and $j(h(i))$ are in the same set P_j for $j = 1,2,3,4,5$. Therefore, the element h stabilizes the partition $\{P_1, P_2, P_3, P_4, P_5\}$ of the set $\{1, \ldots, k\}$ and, hence, its image in L equals 1. Therefore, $h \in M_1 \cap M_2 \cap M_3 \cap M_4 \cap M_5 = 1$.

Choosing M_2, M_3, M_4, M_5 to be equal to the groups $H_1^{x_{1,j(1)+1}} \times \ldots \times H_k^{x_{k,j(k)+1}}$, $H_1^{y_{1,j(1)+1}} \times \ldots \times H_k^{y_{k,j(k)+1}}$, $H_1^{z_{1,j(1)+1}} \times \ldots \times H_k^{z_{k,j(k)+1}}$, $H_1^{t_{1,j(1)+1}} \times \ldots \times H_k^{t_{k,j(k)+1}}, \ldots,$ $H_1^{x_{1,j(1)+4}} \times \ldots \times H_k^{x_{k,j(k)+4}}$, $H_1^{y_{1,j(1)+4}} \times \ldots \times H_k^{y_{k,j(k)+4}}$, $H_1^{z_{1,j(1)+4}} \times \ldots \times H_k^{z_{k,j(k)+4}}$, $H_1^{t_{1,j(1)+4}} \times \ldots \times H_k^{t_{k,j(k)+4}}$, we obtain at least five orbits of quadruples under the action of H satisfying (1). The theorem is proved.

3. INTERSECTION OF SOLVABLE HALL SUBGROUPS IN ALMOST SIMPLE GROUPS

In this section, we prove that an almost simple group with a simple socle isomorphic to an alternating group, a sporadic group, or the Tits group is an $\mathbf{(Orb)}_\pi$-group.

Theorem 2. *Let H be a solvable π-Hall subgroup of an almost simple group G. Assume also that the subgroup $F^*(G)$ is isomorphic to an alternating group, a sporadic group, or the Tits group. Then, there exist elements x, y, z, t of the group G that satisfy equality* (1). *Moreover, G is an $\mathbf{(Orb)}_\pi$-group.*

Proof. We will show first that, if G contains elements x, y, z such that $H \cap H^x \cap H^y \cap H^z = 1$ and $H \notin \{H^x, H^y, H^z\}$, then, under the conjugation action of H, there exist at least five orbits of quadruples of subgroups conjugate to H that satisfy equality (1). Indeed, if $H \cap H^x \cap H^y \cap H^z = 1$, then the quadruples

$$(H, H^x, H^y, H^z), (H^x, H, H^y, H^z), (H^x, H^y, H, H^z), (H^x, H^y, H^z, H), (H^x, H^x, H^y, H^z)$$

are in different H-orbits.

We will continue the proof by successively investigating different almost simple groups. For a given almost simple group, we will first try to find elements x, y, z for which $H \cap H^x \cap H^y \cap H^z = 1$ (in this case, as we noted above, the condition on the number of H-orbits is satisfied automatically) and only if there are no such elements will we specify four elements x, y, z, t and prove that there exist at least five H-orbits. Set $S = F^*(G)$ to be the simple socle of G.

Let $S \simeq \mathrm{Alt}_n$, $n \geqslant 5$. It follows from [14, Theorem 4.3] that, in the group S, the property E_π implies the property C_π. Then, by Lemma 1(c), every π-Hall subgroup of S is contained in some π-Hall subgroup of $\mathrm{Aut}(S)$. Therefore, it follows from [2, Theorem A4] that either $|\pi \cap \pi(S)| = 1$ and H is a Sylow subgroup of S (hence, of G) or $n = 5$, 7, or 8 and $\pi \cap \pi(S) = \{2, 3\}$. In the first case, by [5, Corollary C], there exist elements $x, y \in G$ such that $H \cap H^x \cap H^y = 1$. Hence, we may assume that H is a $\{2, 3\}$-Hall subgroup of G and $n = 5$, 7, or 8. In this case, $\mathrm{Aut}(\mathrm{Alt}_n) = \mathrm{Sym}_n$ and, since $G = HS$, we need to prove the assertion of Theorem 2 only in the case $G = \mathrm{Aut}(S) = \mathrm{Sym}_n$. Consider all three possibilities for n separately, taking [1] into account.

Let $G = \mathrm{Sym}_5$. Then, $H = \mathrm{Sym}_4$ is a point stabilizer for the group G in its natural permutation representation. Since the intersection of any four point stabilizers in G is trivial and all point stabilizers in a transitive permutation group are conjugate, there exist elements $x, y, z \in G$ such that $H \cap H^x \cap H^y \cap H^z = 1$.

Let $G = \mathrm{Sym}_7$. Then, $H = \mathrm{Sym}_3 \times \mathrm{Sym}_4$. Up to conjugation in G, the sets $\{1, 2, 3\}$ and $\{4, 5, 6, 7\}$ are H-orbits. Direct calculations show that the elements $x = (2, 4)(3, 5)$ and $y = (1, 2, 4)(3, 6, 5)$ satisfy the equality $H \cap H^x \cap H^y = 1$.

Let $G = \mathrm{Sym}_8$. Then, $H = \mathrm{Sym}_4 \wr \mathrm{Sym}_2$. Up to conjugation in G, we may assume that H is generated by the elements $(1, 2, 3, 4)$, $(1, 2)$, $(1, 5)$, $(2, 6)$, $(3, 7)$, and $(4, 8)$. In this case, there are no elements $x, y, z \in G$ such that $H \cap H^x \cap H^y \cap H^z = 1$. Indeed, the right multiplication action of G on the set $G : H$ of right cosets of H in G gives an embedding of G in Sym_{35} (and the image of G under this embedding is primitive). Using [15], it is easy to verify that the Cartesian power $(G : H)^4$ is divided under this action into 152 orbits of order at most $20\,160 = |G|/2$; i.e., the stabilizer of any four points is nontrivial. This means that the intersection of any four subgroups conjugate to H (by construction, H is a point stabilizer) is nontrivial.

The elements $x = (4, 5)$, $y = (3, 5)(4, 6)$, $z = (4, 7, 6, 5)$, and $t = (2, 3, 4, 5)$ satisfy the equality $H \cap H^x \cap H^y \cap H^z \cap H^t = 1$. Let us estimate the number of orbits of H on the set

$$X := \{(H^x, H^y, H^z, H^t) \mid H \cap H^x \cap H^y \cap H^z \cap H^t = 1\}.$$

Since $N_G(H) = H$, the element $g \in G$ leaves the tuple (H^x, H^y, H^z, H^t) invariant if and only if $g \in H^x \cap H^y \cap H^z \cap H^t$. Thus, the action of H on X is regular; therefore, the size of every H-orbit on X is $|H| = 1152$. The intersection $H^x \cap H^y \cap H^z \cap H^t$ is the group of order 2 generated by the element $a = (1, 8)(2, 4)(3, 6)(5, 7)$. The equality $H \cap (H^x)^g \cap (H^y)^g \cap (H^z)^g \cap (H^t)^g = 1$ is true if and only if $a^g \notin H$. Moreover, the equality $N_G(H) = H$ implies that the quadruples

$$((H^x)^{g_1}, (H^y)^{g_1}, (H^z)^{g_1}, (H^t)^{g_1}), ((H^x)^{g_2}, (H^y)^{g_2}, (H^z)^{g_2}, (H^t)^{g_2})$$

coincide if and only if $g_1 g_2^{-1} \in \langle a \rangle$. It is easy to check by direct calculations (or using [15]) that G contains 72 elements conjugate to a and lying outside H. In addition, $C_G(a) \simeq 2 \wr \mathrm{Sym}_4$ and the order $|C_G(a)|$ is 384. Therefore, the cardinality of X is $72 \times 192 = 13\,824$. Thus, the group H has at least $13\,824/1152 = 12$ orbits on X.

Assume that S is a sporadic group or the Tits group. If H is a Sylow subgroup of G, then, by [5, Corollary C], there exist $x, y \in G$ such that $H \cap H^x \cap H^y = 1$. Hence, we may assume that the order of H is divisible by at least two primes.

Assume first that the order of the group H is divisible by precisely two primes r, p and $H = R : P$, where $R \in \mathrm{Syl}_r(H)$, $P \in \mathrm{Syl}_p(H)$, and $O_p(H) = 1$. Assume also that either r is odd or $S = \mathrm{Aut}(S)$. For any sporadic group and for the Tits group, the groups of their outer automorphisms are 2-groups; therefore, under our assumptions, $R \leq S$. By [16], there exists $x \in S$ such that $R \cap R^x = 1$. Hence, up to conjugation in H, we may assume that $H \cap H^x \leq P$. It follows from [4] that there exist $y, z \in R$ such that $P \cap P^y \cap P^z = 1$. Consequently, $(H \cap H^x) \cap (H \cap H^x)^y \cap (H \cap H^x)^z = H \cap H^x \cap H^{xy} \cap H^{xz} = 1$.

If the order of H is divisible by two primes r, p only and $H = R \times P$, where $R \in \mathrm{Syl}_r(H)$ and $P \in \mathrm{Syl}_p(H)$, then one of these primes, say r, is odd and, again, $R \leq S$. By [16], there exists an element $x \in S$ such that $R \cap R^x = 1$. Moreover, by [5, Corollary C], there exist $y, z \in G$ such that $P \cap P^y \cap P^z = 1$. Hence, $H \cap H^x \cap H^y \cap H^z = 1$.

By [17, Theorem 6.14 and Table III] and [18, Theorem 4.1], a π-Hall subgroup H of a sporadic group S with $|\pi| \geq 2$ such that its structure differs from the structure of the groups considered above exists only if either $S \simeq M_{23}$ and $H \simeq 2^4 : (3 \times \mathrm{Alt}_4) : 2 \simeq (2^4 : 2^2) : 3^2 : 2$ or $S \simeq J_1$ and $H \simeq 2^3 : 7 : 3$. In both cases, we have $\mathrm{Aut}(S) \simeq S = G$.

Assume first that $S \simeq M_{23}$ and $H = R : P : Q$, where $R \simeq (2^4 : 2^2)$, $P \simeq 3^2$, and $Q \simeq 2$. By [16], there exists $x \in G$ such that $R \cap R^x = 1$; therefore, we may assume that $H \cap H^x \leq PQ$. By Corollary 1, there exists $y \in R$ such that $(H \cap H^x) \cap (H \cap H^x)^y = H \cap H^x \cap H^{xy}$ and $|H \cap H^x \cap H^{xy}| \leqslant 2$. Using Corollary 1 once more, we find an element $z \in H \cap H^x$ such that

$$(H \cap H^x \cap H^{xy}) \cap (H \cap H^x \cap H^{xy})^z = H \cap H^x \cap H^{xy} \cap H^{xyz} = 1.$$

Finally, assume that $S = J_1 \simeq G$ and $H = R : P$, where $P \simeq 2^3$ and $P \simeq 7 : 3$. Then, $R = O_2(H) \in \mathrm{Syl}_2(H)$ and $P \in \mathrm{Hall}_{\{3,7\}}(H)$; moreover, $O_{\{3,7\}}(H) = 1$. By [16], there exists $x \in S$ such that $R \cap R^x = 1$; therefore, up to conjugation in H, we may assume that $H \cap H^x \leq P$. By [6], there exist $y, z \in R$ such that $P \cap P^y \cap P^z = 1$. Hence, $H \cap H^x \cap H^{xy} \cap H^{xz} = 1$. The theorem is proved.

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