NORMALIZERS OF SUBSYSTEM SUBGROUPS IN FINITE GROUPS OF LIE TYPE[1]

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ABSTRACT. In the present paper normalizers of subsystem subgroups of finite groups of Lie type are found in terms of groups of their induced automorphisms.

1 Introduction

Finite groups of Lie type form the main part of known finite simple groups. There are many papers dedicated to their subgroup structure. One of the most important subgroups are, so-called, reductive subgroups of Lie type. They appear as Levi factors of parabolic subgroups and centralizers of semisimple elements, and also as subgroups containing a maximal torus. More over reductive subgroups of maximal rank play an important role in the inductive investigation of subgroup structure in finite groups of Lie type. However some important problems about internal structure of reductive subgroups of maximal rank are remaining unsolved. In particular, possible quasisimple groups, that can occur as central multipliers of semisimple part of a reductive groups of maximal rank, are known, but the structure of their normalizers is not known. The present paper is dedicated to this problem.

Our notation is standard. If G is a finite group, denote by $\mathbf{P}G$ the factor group $G/Z(G) \simeq \mathrm{Inn}(G)$. A central product of G and H is denoted by $G \circ H$. If G is a group, A, B, H are subgroups of G and B is normal in A ($B \trianglelefteq A$), then we denote $N_H(A/B) = N_H(A) \cap N_H(B)$. If $x \in N_H(A/B)$, then x induces an automorphism $Ba \mapsto Bx^{-1}ax$ of A/B. Thus there exists a homomorphism of $N_H(A/B)$ into $\mathrm{Aut}(A/B)$. The image of the homomorphism is denoted by $\mathrm{Aut}_H(A/B)$, while it image is denoted by $C_H(A/B)$. In particular, if $S = A/B$ is a composition factor of G, then for every subgroup $H \leq G$ a group $\mathrm{Aut}_H(S)$ is defined. If A, H are subgroups of G then $\mathrm{Aut}_H(A) = \mathrm{Aut}_H(A/\{e\})$ is a group of induced automorphisms by definition.

2 Preliminary results for groups of Lie type

Our notation for groups of Lie type coincides with that of [1], while for linear algebraic groups coincides with that of [2]. If G is a canonical adjoint finite group of Lie type (the definition can be found below), then $\widehat{G}$ denotes the group of inner-diagonal automorphisms of G. In view of [3, 3.2], $\mathrm{Aut}(G)$ is generated by inner-diagonal, field, and graph automorphisms. Since we assume that $Z(G)$ is trivial, we obtain that $G \simeq \mathrm{Inn}(G)$ and so we may assume that $G \leq \widehat{G} \leq \mathrm{Aut}(G)$.

Let $\overline{G}$ be a simple connected linear algebraic group over an algebraic closure $\overline{\mathbb{F}}_p$ of a finite field $\mathbb{F}_p$ of positive characteristic p. Here $Z(\overline{G})$ can be nontrivial. An endomorphism σ of $\overline{G}$ is called a *Frobenius map*, if the subgroup of its stable points $\overline{G}_\sigma$ is finite, and σ is an automorphism of $\overline{G}$ as an abstract group. Groups $O^{p'}(\overline{G}_\sigma)$ are called *finite canonical groups of Lie type*, and each group G satisfying $O^{p'}(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma$, is called a *finite group of Lie type.* If $\overline{G}$ is a simple algebraic group of adjoint (respectively simply connected) type, then we shall say that G has adjoint (respectively universal) type as well. Note that if G is a canonical adjoint (respectively universal) group of Lie type, then there exists an adjoint (respectively simply

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connected) simple linear algebraic group $\overline{G}$ and a Frobenius map σ such that $O^{p'}(\overline{G}_\sigma)=G$ (see [4, 1.19] and [5, Corollary 12.6], for example). Note that in [1] only groups $O^{p'}(\overline{G}_\sigma)$ are called groups of Lie type. But in [4] of the same author the term "finite group of Lie type" is used also for every group $\overline{G}_\sigma$, where $\overline{G}$ is a connected reductive group. More over in [6] and [7] without any explanation every group G satisfying $O^{p'}(\overline{G}_\sigma)\leq G\leq \overline{G}_\sigma$, is called a finite group of Lie type. Thus above given definition of finite groups of Lie type and canonical finite groups of Lie type unifies the terminology. By Φ or $\Phi(\overline{G})$ is denoted a root system of $\overline{G}$, and by $\Phi(G)$ is denoted a root system of $O^{p'}(G)$. By $\Delta(\overline{G})$ a fundamental group of $\overline{G}$ is denoted, and by $\Delta(\Phi)$ a factor group of the lattice generated by fundamental weights of Φ by a lattice generated by Φ. Note that $\Delta(\overline{G})$ is always a factor group of $\Delta(\Phi(\overline{G}))$ and for each root system Φ, different from D_{2n}, $\Delta(\Phi)$ is cyclic, while $\Delta(D_{2n})$ is elementary Abelian of order 4. The Weyl group of $\overline{G}$ is denoted by $W(\overline{G})$, the Weyl group of Φ is denoted by $W(\Phi)$. If $W(\Phi)$ is a Weyl group of Φ then by w_0 a unique element of $W(\overline{G})$ mapping all positive roots onto negative is denoted.

We say that a group of Lie type G with $O^{p'}(G)$ isomorphic with one of the groups ${}^2A_n(q^2)$, ${}^2D_n(q^2)$, ${}^2E_6(q^2)$ are defined over $GF(q^2)$, a group of Lie type ${}^3D_4(q^3)$ is defined over $GF(q^3)$ and the remaining groups of Lie type are defined over $GF(q)$. A field $GF(q)$ in all cases is called a *base field*. In view of [8, Lemma 2.5.8] if $\overline{G}$ is of adjoint type, then $\overline{G}_\sigma$ is a group of inner-diagonal automorphisms of $O^{p'}(\overline{G}_\sigma)$. If $\overline{G}$ is simply connected $\overline{G}_\sigma=O^{p'}(\overline{G}_\sigma)$ (see [5, 12.4]). In any case in view of [8, Theorem 2.2.6(g)], $\overline{G}_\sigma=\overline{T}_\sigma O^{p'}(\overline{G}_\sigma)$ for any σ-stable maximal torus $\overline{T}$ of $\overline{G}$. In general, for given finite group of Lie type G (if we consider it as an abstract group) corresponding algebraic group is not uniquely defined. For example, if $G=\mathbf{P}SL_2(5)\simeq SL_2(4)$, then G can be obtained either as $(SL_2(\overline{\mathbb{F}}_2))_\sigma$, or as $O^{5'}((\mathbf{P}SL_2(\overline{\mathbb{F}}_5))_\sigma)$ (for suitable maps σ). Hence, for every finite group of Lie type G we fix (in a some way) corresponding algebraic group $\overline{G}$ and a Frobenius map σ such that $O^{p'}(\overline{G}_\sigma)\leq G\leq \overline{G}_\sigma$. Let $U=\langle X_r\mid r\in\Phi(\overline{G})^+\rangle$ be a maximal unipotent subgroup of G. If we fix an order on $\Phi(\overline{G})$, then each $u\in U$ can be uniquely written as

$$u=\prod_{r\in\Phi^+}x_r(t_r),\tag{1}$$

where roots are taken in given order and t_r are in the field of definition of G. If $O^{p'}(G)$ is equal to one of the groups ${}^2A_n(q^2)$, ${}^2B_2(2^{2n+1})$, ${}^2D_n(q^2)$, ${}^3D_4(q^3)$, ${}^2E_6(q^2)$, ${}^2G_2(3^{2n+1})$, or ${}^2F_4(2^{2n+1})$, then we shall say that G is *twisted*, in the remaining cases G is called *split*. If $O^{p'}(\overline{G}_\sigma)\leq G\leq\overline{G}_\sigma$ is a twisted group of Lie type and $r\in\Phi(\overline{G})$, then by $\bar{r}$ we always denote the image of r under a root system symmetry, corresponding to a graph automorphism used for construction of G. Sometimes we shall use notation $\Phi^\varepsilon(q)$, where $\varepsilon\in\{+,-\}$, and $\Phi^+(q)=\Phi(q)$ is a split canonical group of Lie type with the base field $GF(q)$, $\Phi^-(q)={}^2\Phi(q^2)$ is a twisted canonical group of Lie type defined over $GF(q^2)$ (with the base field $GF(q)$).

Now let $\overline{R}$ be a closed σ-stable subgroup of $\overline{G}$. Let $R=G\cap\overline{R}$. The group R is called a *maximal torus* (respectively a *reductive subgroup of maximal rank*) if $\overline{R}$ is a maximal torus (respectively a reductive subgroup of maximal rank) of $\overline{G}$. A maximal σ-stable torus $\overline{T}$ such that $\overline{T}_\sigma$ is a Cartan subgroup of $\overline{G}_\sigma$ is called a *maximal split torus*.

Below we denote by R a reductive subgroup of maximal rank of G and by $\overline{R}$ a corresponding connected σ-stable reductive subgroup of maximal rank of $\overline{G}$. For each connected reductive subgroup of maximal rank $\overline{R}$ of $\overline{G}$ the equality $\overline{R}=\overline{G}_1\circ\ldots\circ\overline{G}_k\circ\overline{Z}$ holds, where $\overline{G}_i$ are simple connected linear algebraic groups and $\overline{Z}=Z(\overline{R})^0$ (see [2, Theorem 27.5]). More over, if $\Phi_1,\ldots,\Phi_k$ are root systems of $\overline{G}_1,\ldots,\overline{G}_k$ respectively, then $\Phi_1\cup\ldots\cup\Phi_k$ is a subsystem of $\Phi(\overline{G})$. There exists a nice algorithm due to Borel and de Siebental [9] and, independently, to Dynkin [10] of determining all subsystems of an irreducible root system Φ. One need to extend the Dynkin diagram of Φ to the extended Dynkin diagram, remove some vertices from the extended Dynkin diagram of Φ and repeat the procedure for remaining connected components. Connected components obtained in this way are Dynkin diagrams of irreducible subsystems and Dynkin diagram of any subsystem can be obtained in this way.

In view of [5, 10.10] there exists a σ-stable maximal torus $\overline{T}$ of $\overline{R}$. Let $\overline{G}_{i_1},\ldots,\overline{G}_{i_{j_i}}$ be a σ-orbit of $\overline{G}_{i_1}$. Consider the induced action of σ on

$$(\overline{G}_{i_1}\circ\ldots\circ\overline{G}_{i_{j_i}})/Z(\overline{G}_{i_1}\circ\ldots\circ\overline{G}_{i_{j_i}})\simeq \mathbf{P}\overline{G}_{i_1}\times\ldots\times\mathbf{P}\overline{G}_{i_{j_i}}.$$

Since $\mathbf{P}\overline{G}_{i_1}\simeq\ldots\simeq\mathbf{P}\overline{G}_{i_{j_i}}$ are simple (as abstract groups), then σ induces a cyclic permutation on $\mathbf{P}\overline{G}_{i_1},\ldots,\mathbf{P}\overline{G}_{i_{j_i}}$, and we may assume that the numbers are chosen so that $\mathbf{P}\overline{G}_{i_1}^{\sigma}=\mathbf{P}\overline{G}_{i_2}$, $\ldots$, $\mathbf{P}\overline{G}_{i_{j_i}}^{\sigma}=\mathbf{P}\overline{G}_{i_1}$. Thus the equality

$$(\mathbf{P}\overline{G}_{i_1}\times\ldots\times\mathbf{P}\overline{G}_{i_{j_i}})_\sigma = \{x\mid x=g\cdot g^{\sigma}\cdot\ldots\cdot g^{\sigma^{j_i-1}}\quad g\in\mathbf{P}\overline{G}_{i_1}\}_\sigma\simeq(\mathbf{P}\overline{G}_{i_1})_{\sigma^{j_i}}$$

holds. In view of [5, 10.15] the group $\mathbf{P}\overline{G}_{\sigma^{j_i}}$ is finite, hence $O^{p'}((\mathbf{P}\overline{G}_{i_1})_{\sigma^{j_i}})$ is a canonical finite group of Lie type, probably with the base field larger, than that of $O^{p'}(\overline{G}_\sigma)$.

Let $\overline{B}_{i_1}$ be a preimage of a σ^{j_i}-stable Borel subgroup of $\mathbf{P}\overline{G}_{i_1}$ in $\overline{G}_{i_1}$ under the natural epimorphism, and $\overline{T}_{i_1}$ be a σ^{j_i}-stable maximal torus of $\overline{G}_{i_1}$, contained in $\overline{B}_{i_1}$ (the existence of these subgroups follows from [5, 10.10]). Then from the note in the beginning of section 11 of [5] it follows that subgroups $\overline{U}_{i_1}$ and $\overline{U}_{i_1}^-$, generated by $\overline{T}_{i_1}$-invariant root subgroups taken over all positive and negative root respectively, are also σ^{j_i}-stable. Since $\overline{G}_{i_1}$ is a simple algebraic group, then $\overline{G}_{i_1}$ is generated by subgroups $\overline{U}_{i_1}$ and $\overline{U}_{i_1}^-$. Now $Z(\overline{G}_{i_1}\circ\ldots\circ\overline{G}_{i_{j_i}})$ consists of semisimple elements, hence the restrictions of the natural epimorphism $\overline{G}_{i_1}\to\mathbf{P}\overline{G}_{i_1}$ on $\overline{U}_{i_1}$ and $\overline{U}_{i_1}^-$ are isomorphisms. Therefore, for every k, subgroups $(\overline{U}_{i_1}^-)^{\sigma^k}$ are maximal σ^{j_i}-stable connected unipotent subgroups of $\overline{G}_{i_k}$ and generate $\overline{G}_{i_k}$.

Thus $\overline{U}_{i_1}\times(\overline{U}_{i_1})^{\sigma}\times\ldots\times(\overline{U}_{i_1})^{\sigma^{j_i-1}}$ and $\overline{U}_{i_1}^-\times(\overline{U}_{i_1}^-)^{\sigma}\times\ldots\times(\overline{U}_{i_1}^-)^{\sigma^{j_i-1}}$ are maximal σ-stable connected unipotent subgroups of $\overline{G}_{i_1}\circ\ldots\circ\overline{G}_{i_{j_i}}$ and generate $\overline{G}_{i_1}\circ\ldots\circ\overline{G}_{i_{j_i}}$. In view of [5, Corollary 12.3(a)], we have

$$\begin{aligned}O^{p'}((\overline{G}_{i_1}\circ\ldots\circ\overline{G}_{i_{j_i}})_\sigma) &= \\ &\langle(\overline{U}_{i_1}\times(\overline{U}_{i_1})^{\sigma}\times\ldots\times(\overline{U}_{i_1})^{\sigma^{j_i-1}})_\sigma,(\overline{U}_{i_1}^-\times(\overline{U}_{i_1}^-)^{\sigma}\times\ldots\times(\overline{U}_{i_1}^-)^{\sigma^{j_i-1}})_\sigma\rangle\simeq\\ &\langle(\overline{U}_{i_1})_{\sigma^{j_i}},(\overline{U}_{i_1}^-)_{\sigma^{j_i}}\rangle=O^{p'}((\overline{G}_{i_1})_{\sigma^{j_i}}).\end{aligned}$$

By [5, 11.6 and Corollary 12.3], the group $\langle(\overline{U}_{i_1})_{\sigma^{j_i}},(\overline{U}_{i_1}^-)_{\sigma^{j_i}}\rangle$ is a canonical finite group of Lie type. More over from the above arguments it follows that $\langle(\overline{U}_{i_1})_{\sigma^{j_i}},(\overline{U}_{i_1}^-)_{\sigma^{j_i}}\rangle/Z(\langle(\overline{U}_{i_1})_{\sigma^{j_i}},(\overline{U}_{i_1}^-)_{\sigma^{j_i}}\rangle)$ and $O^{p'}((\mathbf{P}\overline{G}_{i_1})_{\sigma^{j_i}})$ are isomorphic. Denoting $O^{p'}((\overline{G}_{i_1}\circ\ldots\circ\overline{G}_{i_{j_i}})_\sigma)$ by G_i, we obtain that G_i is a finite group of Lie type for all i. Subgroups G_i of $O^{p'}(\overline{G}_\sigma)$, appearing in this way, are called *subsystem subgroups* of $O^{p'}(\overline{G}_\sigma)$.

Since $\overline{G}_{i_1}\circ\ldots\circ\overline{G}_{i_{j_i}}$ is σ-stable, then $\overline{G}_{i_1}\circ\ldots\circ\overline{G}_{i_{j_i}}\cap\overline{T}$ is a σ-stable maximal torus of $\overline{G}_{i_1}\circ\ldots\circ\overline{G}_{i_{j_i}}$. Therefore we may assume that for every σ-orbit $\{\overline{G}_{i_1},\ldots,\overline{G}_{i_{j_i}}\}$, the intersection $\overline{T}\cap(\overline{G}_{i_1}\circ\ldots\circ\overline{G}_{i_{j_i}})$ is a maximal σ-torus of $\overline{G}_{i_1}\circ\ldots\circ\overline{G}_{i_{j_i}}$. Then $\overline{R}_\sigma=\overline{T}_\sigma(G_1\circ\ldots\circ G_m)$, where m is the number of σ-orbits and $\overline{T}_\sigma$ normalizes each G_i.

Consider $\mathrm{Aut}_{\overline{R}_\sigma}(G_i)$. Since $G_1 \circ \ldots \circ G_{i-1} \circ G_{i+1} \circ \ldots \circ G_k \circ \overline{Z}_\sigma \le C_{\overline{R}_\sigma}(G_i)$, we have that $\mathrm{Aut}_{\overline{R}_\sigma}(G_i) \simeq \left(\overline{T}_\sigma G_i\right)/Z\left(\overline{T}_\sigma G_i\right)$. By [8, Proposition 2.6.2] it follows that automorphisms, induced by $\overline{T}_\sigma$ on G_i, are diagonal. Therefore the inclusions $\mathbf{P}G_i \le \mathrm{Aut}_{\overline{R}_\sigma}(G_i) \le \widehat{\mathbf{P}G_i}$ hold, in particular $\mathrm{Aut}_{\overline{R}_\sigma}(G_i)$ is a finite group of Lie type.

Let $\overline{R}$ be a σ-stable connected reductive subgroup of maximal rank (in particular, $\overline{R}$ can be a maximal torus) of G. Since $N_{\overline{G}}(\overline{R})/\overline{R}$ and $N_W(W_{\overline{R}})/W_{\overline{R}}$ are isomorphic, where W is the Weyl group of $\overline{G}$, $W_{\overline{R}}$ is the Weyl group of $\overline{R}$ (and it is a subgroup of W), we obtain an induced action of σ on $N_W(W_{\overline{R}})/W_{\overline{R}}$ and $w_1 \equiv w_2$ for $w_1, w_2 \in N_W(W_{\overline{R}})/W_{\overline{R}}$, if there exists an element $w \in N_W(W_{\overline{R}})/W_{\overline{R}}$ with $w_1 = w^{-1}w_2w^\sigma$. Let $Cl(\overline{G}_\sigma, \overline{R})$ be the set of $\overline{G}_\sigma$-conjugate classes of σ-stable subgroups $\overline{R}^g$, where $g \in \overline{G}$. Then $Cl(\overline{G}_\sigma, \overline{R})$ is in 1-1 correspondence with the set of σ-conjugate classes $Cl(N_W(W_{\overline{R}})/W_{\overline{R}}, \sigma)$. If w is an element of $N_W(W_{\overline{R}})/W_{\overline{R}}$ and $(\overline{R}^g)_\sigma$ corresponds to the σ-conjugate class of w, then we say that $(\overline{R}^g)_\sigma$ is obtained by "twisting" of $\overline{R}$ by $w\sigma$. Here $(\overline{R}^g)_\sigma \simeq \overline{R}_{\sigma w}$. Detailed information on twisting can be found in [11].

It is enough to find the structure of $\mathrm{Aut}_R(G_i)$ in order to investigate the structure of $N_R(G_i)$. Recall that $\mathbf{P}G_i \le \mathrm{Aut}_R(G_i) \le \widehat{\mathbf{P}G_i}$. If $G_i \not\simeq D_{2n}(q)$, then the factor group $\widehat{\mathbf{P}G_i}/\mathbf{P}G_i$ is cyclic and the structure of $\mathrm{Aut}_R(G_i)$ is completely determined by the index $|\mathrm{Aut}_R(G_i) : \mathbf{P}G_i|$. The index $|\mathrm{Aut}_R(G_i) : \mathbf{P}G_i|$ is found in the paper. If $G_i \simeq D_{2n}(q)$, then $\widehat{\mathbf{P}G_i}/\mathbf{P}G_i$ is either trivial or elementary Abelian of order 4, so contains 3 distinct subgroups of order 2. In this case, if $|\mathrm{Aut}_R(G_i) : \mathbf{P}G_i|$ is equal to 2, we shall not specify, which subgroup really appear. Since elements of distinct G_i, G_j commute, it is enough to consider the case $m = 1$, i. e., we may assume that $\overline{G}_1 \circ \ldots \circ \overline{G}_k$ are in the same σ-orbit. Denote G_1 by L.

Now we assume that a finite group of Lie type G is universal, i. e., $\overline{G}$ is simply connected and $G = \overline{G}_\sigma$. Then $R = \overline{R}_\sigma = \overline{T}_\sigma L$ for any maximal σ-stable torus $\overline{T}$ of $\overline{R}$. Let $\overline{B}$ be a Borel subgroup of $\overline{G}$ and $\overline{T}$ be a maximal torus contained in $\overline{B}$. Denote by $\overline{B}^-$ a unique Borel subgroup with $\overline{B} \cap \overline{B}^- = \overline{T}$, and by $\{X_r \mid r \in \Phi(\overline{G})\}$ 1-dimensional $\overline{T}$-invariant unipotent subgroups of $\overline{B}$ and $\overline{B}^-$. Let $n_r(t) = x_r(t)x_{-r}(-t^{-1})x_r(t)$ and $\overline{N} = \langle n_r(t) \mid r \in \Phi, t \in \overline{\mathbb{F}}_p^* \rangle$. Let $h_r(t) = n_r(t)n_r(-1)$, $\overline{H}_r = \{h_r(t) \mid t \in \overline{\mathbb{F}}_p^*\}$, and $\overline{H} = \langle h_r(t) \mid r \in \Phi, t \in \overline{\mathbb{F}}_p^* \rangle$. In view of [1, Chapters 6 and 7] $\overline{H}$ is a maximal torus of $\overline{G}$, $\overline{H} \le \overline{B} \cap \overline{B}^-$, i. e., $\overline{T} = \overline{H}$ and $\overline{N} = N_{\overline{G}}(\overline{T})$. Let $\Pi = \{r_1, \ldots, r_n\}$ be a set of fundamental roots of $\Phi(\overline{G})$, then $\overline{T}$ is generated by $\overline{H}_{r_1}, \ldots, \overline{H}_{r_n}$. More over, since $\overline{G}$ is simply connected, then $\overline{T}$ is a direct product of $\overline{H}_{r_1}, \ldots, \overline{H}_{r_n}$ (see the note after the proof of [1, Theorem 12.1.1]). More over in the proof of [1, Theorem 12.1.1] the following equalities $h_r(t)x_s(u)h_r(t)^{-1} = x_s(t^{<r,s>}u) = x_s(t^{(\check{r},s)}u)$ were obtained, where $r, s \in \Phi$ and $\check{r} = \frac{2r}{(r,r)}$ is a coroot of r.

If L is isomorphic to ${}^3D_4(q^3)$ or ${}^2B_2(2^{2n+1})$, then L coincides with the group of inner-diagonal automorphisms, so $\mathrm{Aut}_R(L) = L$. Thus we may assume that L is either split, or twisted by a graph automorphism of order 2, i. e., in the above notation, $L \simeq \Psi^\varepsilon(q^k)$, where $\Psi = \Psi(\overline{G}_1)$ is a root system of $\overline{G}_1$ and $\mathbb{F}_q$ is a base field of G. Let $Q(\Psi)$ be a lattice generated by coroots of Ψ, and $Q(\Phi, \Psi)$ be the factor group of the lattice $Q(\Phi)$ generated by coroots of Φ, by the lattice $Q(\Psi^\perp)$ generated by coroots, that are orthogonal to all roots from Ψ. In just defined notation the following theorem holds.

Theorem 1. *Assume that G is not a Ree group. If $L \not\simeq {}^2D_{2m}(q^{2k})$, then the equality*

$$|\mathrm{Aut}_R(L) : \mathbf{P}L| = (|Q(\Phi, \Psi) : Q(\Psi)|, q^k - \varepsilon 1)$$

holds. If $L \simeq {}^2D_{2m}(q^{2k})$, then the equality

$$|\mathrm{Aut}_R(L) : \mathbf{P}L| = (|Q(\Phi, \Psi) : Q(\Psi)|, q^{2km} + 1)$$

holds.

Proof. Let Θ be a root system of $\overline{G}_1\circ\ldots\circ\overline{G}_k$, then $\Theta=\Psi_1\cup\ldots\cup\Psi_k$, where $\Psi_i\simeq\Psi$ for all i. Denote by Π_i a set of fundamental roots of Ψ_i, then $\Pi=\Pi_1\cup\ldots\cup\Pi_k$ is a fundamental set of Θ. Since $\overline{R}$ is σ-stable, then σ induces a permutation on Θ. Denote this permutation also by σ. Since Π is a fundamental set, then Π^σ is also a set of fundamental roots of Θ. In view of [1, Theorem 2.2.4] there exists an element $w\in W(\Theta)=W(\overline{R})\simeq N_{\overline{R}}(\overline{T})/\overline{T}$ such that $\Pi^\sigma=\Pi^{w^{-1}}$. By [11, Propositions 3 and 6] groups $\overline{R}_\sigma$ and $\overline{R}_{\sigma w}$ coincides, so we may assume that $\Pi^\sigma=\Pi$, i. e., that σ acts as a symmetry of the Dynkin diagram of Θ.

Let $\Pi_1=\{r_1,\ldots,r_m\}$. Then for all i-s the inclusions $r_i^\sigma\in\Pi_2,\ \ldots,\ r_i^{\sigma^{k-1}}\in\Pi_k$, $r_i^{\sigma^k}\in\Pi_1$ holds. Thus σ^k induces a symmetry τ of Ψ_1. Further since $L\simeq\Phi^\varepsilon(q^k)$, then $|\tau|\leqslant 2$. Multiplying τ by σ^{i-1}, for all i-s we obtain a symmetry of Ψ_i. To shorten notation we shall denote this symmetry by τ as well. More over set $r_{1,j}=r_j$ and $r_{i,j}=r_j^{\sigma^{i-1}}$. By construction for each j we have $(r_{1,j}^\tau)^{\sigma^{i-1}}=(r_{1,j}^{\sigma^{i-1}})^\tau=r_{i,j}^\tau$. Denote $r_{i,j}^\tau$ by $\bar r_{i,j}$ and, extending τ on Θ, denote r^τ by $\bar r$ for each $r\in\Theta$. Since $\overline{T}$ is σ-stable, then σ permutes $\overline{T}$-root subgroups, and we obtain an induced action of σ on Φ, $Q(\Phi)$, $Q(\Theta)$, and $Q(\Phi,\Theta)$, and also on the factor group $Q(\Phi,\Theta)/Q(\Theta)$.

Since we are not considering Suzuki and Ree groups, then σ preserves the scalar product, in particular, for every two roots $r,s\in\Phi$ the equalities

$$<r,s>=<r^\sigma,s^\sigma>=(\check r,s)=(\check r^\sigma,s^\sigma)\tag{2}$$

hold.

By definition

$$Q(\Phi,\Psi_i)/Q(\Psi_i)=\left(\left(Q(\Phi)/Q(\Psi_i^\perp)\right)/Q(\Psi_i)\right)\simeq Q(\Phi)/(Q(\Psi_i^\perp)\times Q(\Psi_i)),$$

$$Q(\Phi,\Theta)/Q(\Theta)=\left(\left(Q(\Phi)/Q(\Theta^\perp)\right)/Q(\Theta)\right)\simeq Q(\Phi)/(Q(\Theta^\perp)\times Q(\Theta)).$$

Further $Q(\Theta)=Q(\Psi_i)\oplus\left(\sum_{j\neq i}Q\left(\Psi_j\right)\right)$, $Q(\Theta^\perp)\leq Q(\Psi_i^\perp)$, so $(Q(\Theta^\perp)\times Q(\Theta))\leq(Q(\Psi_i^\perp)\times Q(\Psi_i))$. Thus there exists a homomorphism π_i being a composition an isomorphism

$$Q(\Phi,\Theta)/Q(\Theta)\to Q(\Phi)/(Q(\Theta^\perp)\times Q(\Theta)),$$

a homomorphism

$$Q(\Phi)/(Q(\Theta)\times Q(\Theta))\to Q(\Phi)/(Q(\Psi_i^\perp)\times Q(\Psi_i)),$$

and an isomorphism

$$Q(\Phi)/(Q(\Psi_i^\perp)\times Q(\Psi_i))\to Q(\Phi,\Psi_i)/Q(\Psi_i).$$

By Homomorphism Theorem, for each i the following diagram is commutative

$$\begin{tikzcd}
Q(\Phi) \arrow[r] \arrow[dr] & Q(\Phi,\Theta)/Q(\Theta) \arrow[d, "\pi_i"] \\
 & Q(\Phi,\Psi_i)/Q(\Psi_i)
\end{tikzcd}$$

Consider an epimorphism

$$\pi: Q(\Phi,\Theta)/Q(\Theta)\to Q(\Phi,\Psi_1)/Q(\Psi_1)\times\ldots\times Q(\Phi,\Psi_k)/Q(\Psi_k),$$

acting by $\pi : s \mapsto (s^{\pi_1}, \ldots, s^{\pi_k})$. If for each i the equality $s^{\pi_i} = 0$ holds (we shall use additive notation for lattices), then $s \in Q(\Psi_i) \times Q(\Psi_i^{\perp})$. Since subsystems Ψ_i are pairwise orthogonal, it follows that $s \in Q(\Psi_1) \times \ldots \times Q(\Psi_k) \times Q(\Theta^{\perp})$. Therefore, $s = 0$ and π is injective. i. e., it is an isomorphism. Since $\Psi_1^\sigma = \Psi_2, \ldots, \Psi_k^\sigma = \Psi_1$, then σ induces a cyclic permutation on $\{Q(\Phi, \Psi_1)/Q(\Psi_1), \ldots, Q(\Phi, \Psi_k)/Q(\Psi_k)\}$. More over, equalities (2) show that σ acts on the direct product $Q(\Phi, \Psi_1)/Q(\Psi_1) \times \ldots \times Q(\Phi, \Psi_k)/Q(\Psi_k)$ and π is permutable with the action of σ.

Assume that $|\tau| = 1$, i. e., for each $r \in \Theta$, $r = \bar{r}$ and L is split. Since $\overline{Z} = \langle h_s(t) \mid s \in \Theta^{\perp}, t \in \overline{\mathbb{F}}_p^*\rangle$, then [11, Proposition 8] implies that the factor group $\overline{T}_\sigma/\overline{Z}_\sigma$ is isomorphic to a group of $\mathbb{F}_{q^k}$-characters of $Q(\Phi, \Theta)_\sigma \simeq Q(\Phi, \Psi)$ (by construction $Q(\Phi, \Theta)_\sigma$ is isomorphic to a diagonal of the direct product, while the diagonal is isomorphic to $Q(\Phi, \Psi)$). By using an isomorphism $Z(S) \simeq \widehat{\mathbf{P}S}/\mathbf{P}S$, where S is a universal group of Lie type (see. [5, Corollary 12.6]), as in [1, 8.6] we obtain that $\mathrm{Aut}_R(\mathbf{P}L) : \mathbf{P}L$ is isomorphic to the group of $\mathbb{F}_{q^k}$-characters of $Q(\Phi, \Psi)/Q(\Psi)$, so the equality $|\mathrm{Aut}_R(\mathbf{P}L) : \mathbf{P}| = (|Q(\Phi, \Psi)/Q(\Psi)|, q^k - 1)$ holds.

Assume that $|\tau| = 2$. Since $\overline{Z} = \langle h_s(t) \mid s \in \Theta^{\perp}, t \in \overline{\mathbb{F}}_p^*\rangle$, then [11, Proposition 8] implies that $\overline{T}_\sigma/\overline{Z}_\sigma$ is isomorphic to $Q(\Phi, \Theta)/(\sigma - 1)Q(\Phi, \Theta) \simeq Q(\Phi, \Psi)/(q^k\tau - 1)Q(\Phi, \Psi)$ (the last isomorphism follows from the action of π on the direct product $Q(\Phi, \Psi_1)/Q(\Psi_1) \times \ldots \times Q(\Phi, \Psi_k)/Q(\Psi_k)$). By using an isomorphism $Z(S) \simeq \widehat{\mathbf{P}S}/\mathbf{P}S$, where S is a universal group of Lie type (see. [5, Corollary 12.6]), as in [1, 14.1, note on page 253] we obtain that $\mathrm{Aut}_R(\mathbf{P}L)/\mathbf{P}L$ is isomorphic to the group of homomorphisms of $Q(\Phi, \Psi)/Q(\Psi)$ into a subgroup of the multiplicative group of $\mathbb{F}_{q^{2k}}$, defined by $t^{q^k+1} = 1$. Therefore the equality $|\mathrm{Aut}_R(\mathbf{P}L) : \mathbf{P}L| = (|Q(\Phi, \Psi)/Q(\Psi)|, q^k + 1)$ holds (we have the equality $|\mathrm{Aut}_R(\mathbf{P}L) : \mathbf{P}L| = (|Q(\Phi, \Psi)/Q(\Psi)|, q^{2km} + 1)$, if $L \simeq {}^2D_{2m}(q^{2k})$), and the theorem follows. $\square$

3 Main theorem

In the next theorem $r_1, \ldots, r_n$ always denotes a set of fundamental roots of $\Phi = \Phi(\overline{G})$ (the numbering of fundamental roots for each rot system will be specified below in the proof), while r_0 denotes the positive root of maximum height.

Theorem 2. *Let $G = \overline{G}_\sigma$ be a finite universal group of Lie type, where $\overline{G}$ is a simple simply connected linear algebraic group and σ is a Frobenius map. Let R be a reductive subgroup of maximal rank of G and $L \leq R$ be a subsystem subgroup of G. Denote by Φ and Ψ root systems of $\overline{G}$ and $\overline{L}$ respectively. Set $\varepsilon = +$, if L is split, and $\varepsilon = -$, if L is one of ${}^2A_n(q^2)$, ${}^2D_n(q^2)$, ${}^2E_6(q^2)$. Denote by q the order of the base field of L (it can be larger than the order of the base field of G).*

Then one of the following statements hold:

(1) $\Phi = B_{2n}$, $\Psi = A_{2n-1}$, $|\mathrm{Aut}_R(L) : L/Z(L)| = (n, q - \varepsilon 1)$. *Here a subsystem A_1 of B_n is generated by a long root, while B_1 is generated by a short root.*

(2) $\Phi = B_n$, $\Psi = D_n$, $|\mathrm{Aut}_R(L) : L/Z(L)| = 1$.

(3) $\Phi = B_n$, $\Psi = D_k$, $3 \leq k < n$, $|\mathrm{Aut}_R(L) : L/Z(L)| = (2, q - \varepsilon 1)$. *Here subsystems D_3 and A_3 of B_n, up to the action of the Weyl group, are generated by $r_1, r_2, -r_0$ r_1, r_2, r_3 respectively.*

(4) $\Phi = C_n$, $\Psi = C_k$, $1 \leq k < n$, $|\mathrm{Aut}_R(L) : L/Z(L)| = 1$. *Here a subsystem A_1 of C_n is generated by a short root, while C_1 is generated by a long root.*

(5) $\Phi = D_{2n}$, $\Psi = A_{2n-1}$, $|\mathrm{Aut}_R(L) : L/Z(L)| = (n, q - \varepsilon 1)$.

(6) $\Phi = D_n$, $\Psi = D_k$, $k < n$, $|\mathrm{Aut}_R(L) : L/Z(L)| = (2, q - \varepsilon 1)$. *Here subsystems D_3 and A_3 of D_n, up to the action of the Weyl group, are generated by $r_1, r_2, -r_0$ r_1, r_2, r_3 respectively.*

(7) $\Phi = F_4$, $\Psi = B_4$, $|\mathrm{Aut}_R(L) : L/Z(L)| = 1$.

(8) $\Phi = F_4$, $\Psi = D_4$, $|\mathrm{Aut}_R(L) : L/Z(L)| = 1$.

(9) $\Phi = F_4$, $\Psi = A_3$, $|\mathrm{Aut}_R(L) : L/Z(L)| = (2, q - \varepsilon 1)$.

(10) $\Phi = E_8$, $\Psi = A_8$, $|\mathrm{Aut}_R(L) : L/Z(L)| = (3, q - \varepsilon 1)$.

(11) $\Phi = E_8$, $\Psi = D_8$, $|\mathrm{Aut}_R(L) : L/Z(L)| = (2, q - \varepsilon 1)$.

(12) $\Phi = E_7$, $\Psi = A_7$, $|\mathrm{Aut}_R(L) : L/Z(L)| = (2, q - \varepsilon 1)$.

(13) $\Phi = E_7$, $\Psi = D_6$, $|\mathrm{Aut}_R(L) : L/Z(L)| = (2, q - \varepsilon 1)$.

(14) $\Phi = E_6$, $\Psi = A_5$, $|\mathrm{Aut}_R(L) : L/Z(L)| = (2, q - \varepsilon 1)$.

(15) $\Phi = G_2$, $\Psi = A_2$, $|\mathrm{Aut}_R(L) : L/Z(L)| = 1$.

(16) $G = {}^2G_2(3^{2n+1})$, $R = L \simeq A_1(3^{4n+2})$, $|\mathrm{Aut}_R(L) : L/Z(L)| = 1$.

(17) *In the remaining cases $\mathrm{Aut}_R(L)$ coincides with the group of all inner-diagonal automorphisms of L.*

Note that we need not to separate the case $L \simeq {}^2D_{2m}(q^2)$ (cf. Theorem 1), since $(4, q^{2m} + 1) = (2, q + 1)$. ∎

Proof. Note that $Q(\Phi)$ (the lattice generated by coroots ofΦ) contains elements $s_1, \ldots, s_k$ such that for the set $r_1, \ldots, r_k$ of fundamental roots of Ψ the equality $(s_i, r_j) = \delta_{i,j}$ holds, where $\delta_{i,j}$ is a Kronecker symbol, if and only if $Q(\Phi, \Psi)$ is, by definition, a full weight lattice and $\mathrm{Aut}_R(L) = \widehat{L/Z(L)}$. Thus to find $|Q(\Phi, \Psi) : Q(\Psi)|$ we shall try to construct the set $s_1, \ldots, s_k$ first, and only if such a set does not exist, we shall calculate $|Q(\Phi, \Psi) : Q(\Psi)|$ by definition of $Q(\Phi, \Psi)$.

For systems $\Theta \leq \Psi \leq \Phi$ we have $Q(\Psi^{\perp}) \leq Q(\Theta^{\perp})$, hence there exists a homomorphism $Q(\Phi, \Psi) \to Q(\Phi, \Theta)$, making the diagram

$$\begin{tikzcd} Q(\Phi) \arrow[r] \arrow[dr] & Q(\Phi, \Psi) \arrow[d] \\ & Q(\Phi, \Theta) \end{tikzcd}$$

commutative. The next lemma follows from the commutativity of this diagram.

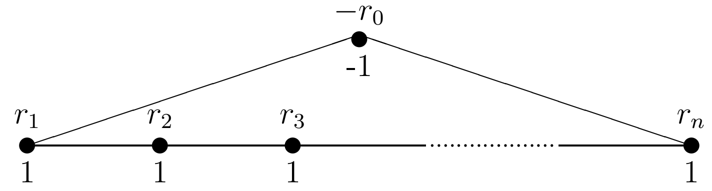
Lemma 3. *Let Φ, Ψ, Θ be root systems and $\Phi \geq \Psi \geq \Theta$. Then*

(1) *if $Q(\Psi, \Theta)$ is a full weight lattice, then $Q(\Phi, \Theta)$ is also a full weight lattice.*

(2) *if $Q(\Phi, \Psi) = Q(\Psi)$, then $Q(\Phi, \Theta) = Q(\Psi, \Theta)$.*

Consider all irreducible root systems. Note that groups ${}^2B_2(2^{2n+1})$ ${}^3D_4(q^3)$ coincide with their inner-diagonal automorphisms groups, so we may assume that L is not isomorphic to ${}^2B_2(2^{2n+1})$, ${}^3D_4(q^3)$.

Type A_n. Consider a Euclidean vector space of dimension $n+1$, let $e_1, \ldots, e_{n+1}$ be its orthonormal basis. Let V be a subspace, consisting of vectors with the sum of coordinates equals 0. Then a root system of type A_n can be chosen so that it generates V and its fundamental roots form a set $\{e_1 - e_2, \ldots, e_n - e_{n+1}\}$. Consider the extended Dynkin diagram of A_n,



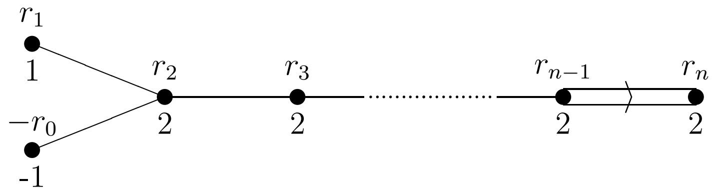
where r_0 is the positive root of maximal height and coefficients near r_i-s are defined by $r_0 = \sum \alpha_i r_i$. Clearly all maximal connected subsystems has type A_{n-1}. Up to the action of the Weyl group we may assume that $A_{n-1} = \langle r_1, \ldots, r_{n-1}\rangle$, where $r_i = e_i - e_{i+1}$, $i = 1, \ldots, n$. Since all roots of A_n has the same length, then in chosen basis coroots of A_n coincide with its root.

Set $s_k = r_1 + 2r_2 + \ldots + kr_k + kr_{k+1} + \ldots + kr_n$, $k = 1, \ldots, n-1$. Then $(s_k, r_j) = \delta_{k,j}$. Thus $Q(A_n, A_{n-1})$ is a full weight lattice.

For a subsystem A_k of A_n, the factor group $Q(A_k, A_{k+1})$ is a full weight lattice, and by Lemma 3(1) $Q(A_n, A_k)$ is a full weight lattice for each $k < n$.

Type B_n. Consider this case in detail, since we shall be able to demonstrate all methods of finding the index $|Q(\Phi, \Psi) : Q(\Psi)|$.

Let V be a Euclidean vector space of dimension n, and $e_1, \ldots, e_n$ be its orthonormal basis. Then a fundamental set of roots $\{r_1, \ldots, r_n\}$ in V can be chosen in the form $\{e_1 - e_2, \ldots, e_{n-1} - e_n, e_n\}$. Consider the extended Dynkin diagram for B_n



It follows from the diagram, that all maximal connected subsystems have types either B_{n-1}, or D_n. Up to the action of the Weyl group, $B_{n-1} = \langle r_2, \ldots, r_n\rangle$, $D_n = \langle -r_0, r_1, \ldots, r_{n-1}\rangle$.

(1) $B_{n-1} < B_n$. Show that $Q(B_n, B_{n-1})$ is a full weight lattice, i. e., there exists elements $s_2, \ldots, s_n$ of $Q(B_n)$ such that $(s_i, r_j) = \delta_{i,j}$, $i, j = 2, \ldots, n$.

Indeed, let $s_k = \alpha_1 \check{r}_1 + \ldots + \alpha_n \check{r}_n$ be an element of $Q(B_n)$ ($k = 2, \ldots, n$; $\check{r}_i$-s are coroots of B_n). For a root system B_n we have that $\check{r}_i = r_i$ for all $i \neq n$ and $\check{r}_n = 2r_n$. Then

$$\begin{aligned}
(s_k, r_2) &= -\alpha_1 + 2\alpha_2 - \alpha_3 = \delta_{k,2}, \\
(s_k, r_3) &= -\alpha_2 + 2\alpha_3 - \alpha_4 = \delta_{k,3}, \\
&\vdots \\
(s_k, r_{n-2}) &= -\alpha_{n-3} + 2\alpha_{n-2} - \alpha_{n-1} = \delta_{k,n-2}, \\
(s_k, r_{n-1}) &= -\alpha_{n-2} + 2\alpha_{n-1} - 2\alpha_n = \delta_{k,n-1}, \\
(s_k, r_n) &= -\alpha_{n-1} + 2\alpha_n = \delta_{k,n}.
\end{aligned}$$

This system of equations is solvable in $\mathbb{Z}$ for each k (starting from the last equation one can express all $\alpha_{n-1}, \dots, \alpha_1$ by using α_n), so $Q(B_n, B_{n-1})$ is a full weight lattice. By Lemma 3(1) $Q(B_n, B_k)$ is a full weight lattice for each $k < n$.

(2) $D_n < B_n$. In this case, as we shall see later, the set $s_1, \dots, s_{n-1}$ does not exists, and we shall calculate $|Q(B_n, D_n) : Q(D_n)|$ by definition.

Since all roots of D_n have the same length, then $Q(D_n)$ is generated by $\{-r_0, r_1, \dots, r_n\}$, $Q(B_n)$ is generated by $\{\check{r}_1, \dots, \check{r}_n\}$. Since the dimensions of subsets, spanned by lattices $Q(B_n)$ and $(Q(D_n))$ are equal, then $Q(D_n^\perp) = \{0\}$. Hence the factor group $Q(B_n, D_n)$, obtained as a factor group of $Q(B_n)$ by $Q(D_n^\perp)$, coincides with $Q(B_n)$. An index $|Q(B_n, D_n) : Q(D_n)|$ is equal to the determinant of A, being a transfer matrix from $Q(B_n, D_n)$ to $Q(D_n)$. From the equations

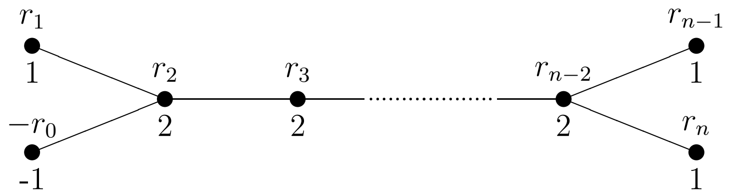
$$\begin{aligned} -r_0 &= -\check{r}_1 - 2\check{r}_2 - 2\check{r}_3 - \ldots - 2\check{r}_{n-1} - \check{r}_n, \\ r_1 &= \check{r}_1, \\ &\vdots \\ r_{n-1} &= \check{r}_{n-1} \end{aligned}$$

we have

$$A = \begin{pmatrix} -1 & -2 & \dots & -2 & -1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Therefore, $|Q(B_n, D_n) : Q(D_n)| = |\det A| = 1$.

Consider the extended Dynkin diagram for D_n



Clearly A_{n-1} and D_{n-1} are the maximal connected subsystems.

(a) $A_{n-1} < B_n$. Up to the action of the Weyl group and the symmetry of the Dynkin diagram we may assume that the set $\{r_1, \dots, r_{n-1}\}$ is a set of fundamental roots of A_{n-1} .

(a1) $n = 2m + 1$. Define $s_k \in Q(B_n)$ by:

$$s_k = \begin{cases} \check{r}_1 + 2\check{r}_2 + \ldots + k\check{r}_k + k\check{r}_{k+1} + \ldots + k\check{r}_{n-1} + l\check{r}_n, & k = 2l, \\ 2\check{r}_1 + 4\check{r}_2 + \ldots + 2k \cdot \check{r}_k + (2k+1)\check{r}_{k+1} + (2k+2)\check{r}_{k+2} + \\ \ldots + (2k + (n-k-1))\check{r}_{n-1} + (m+l+1)\check{r}_n, & k = 2l+1. \end{cases}$$

It can be directly checked that $(s_k, r_j) = \delta_{k,j}$ for $k, j = 1, \dots, n-1$. Thus $Q(B_n, A_{n-1})$ is a full weight lattice for n odd.

(a2) $n = 2m$. Since in a subsystem A_{n-1} all roots $r_1, \dots, r_{n-1}$ have the same length, then its coroots in chosen basis coincide with roots, while $Q(B_n)$ is generated by $\{r_1, \dots, r_{n-1}, 2r_n\} =$

$\{e_1 - e_2, \dots, e_{n-1} - e_n, 2e_n\}$. It follows that $Q(A_{n-1}^{\perp})$ is generated by coroots of B_n, that are orthogonal to all roots of A_{n-1}, and is generated by $e_1 + e_2 + \ldots + e_n$. Set

$$\begin{aligned} r'_1 &= e_1 + e_2 + \ldots + e_n = r_1 + 2r_2 + \ldots + (n-1)r_{n-1} + mr_n, \\ r'_2 &= r_2, \\ &\vdots \\ r'_n &= r_n. \end{aligned}$$

In view of equalities

$$\begin{aligned} r_1 &= r'_1 - 2r'_2 - 3r'_3 - \ldots - (n-1)r'_{n-1} + mr'_n, \\ r_2 &= r'_2, \\ &\vdots \\ r_n &= r'_n. \end{aligned}$$

$\{r'_1, \dots, r'_n\}$ form a basis of $Q(B_n)$, the factor group $Q(B_n, A_{n-1})$, obtained as a factor group of $Q(B_n)$ by $Q(A_{n-1}^{\perp})$, is obtained by eliminating of r'_1. A transfer matrix (for basis of corresponding vector space) from $Q(B_n, A_{n-1})$ to $Q(A_{n-1})$ has the following form

$$A = \begin{pmatrix} -2 & -3 & \ldots & -(n-1) & -m \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

Hence $|Q(B_n, A_{n-1}) : Q(A_{n-1})| = |\det A| = m$.

For a subsystem A_k in B_n, by Lemma 3(1) we obtain that $Q(B_n, A_k)$ is a full weight lattice for each $k < n-1$.

(b) $D_{n-1} < B_n$. We show that for a subsystem D_k with the set of fundamental root equal to $\{-r_0, r_1, \dots, r_{k-1}\}$ the index $|Q(B_n, D_k) : Q(D_k)|$ is equal to 2, for each $3 \le k < n$.

Since D_k is generated by roots $\{-e_1 - e_2, e_1 - e_2, \dots, e_{k-1} - e_k\}$, then we can take the lattice generated by $\{\check{r}_{k+1}, \dots, \check{r}_n\} = \{e_{k+1} - e_{k+2}, \dots, e_{n-1} - e_n, 2e_n\}$ as a lattice generated by coroots of B_n, orthogonal to all roots in D_k. Then, taking the factor by $Q(D_k^{\perp})$, we obtain

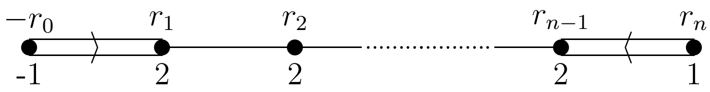
$$\begin{aligned} -r_0 &= -\check{r}_1 - 2\check{r}_2 - 2\check{r}_3 - \ldots - 2\check{r}_k, \\ r_1 &= \check{r}_1, \\ &\vdots \\ r_{k-1} &= \check{r}_{k-1}. \end{aligned}$$

Therefore,

$$A = \begin{pmatrix} -1 & -2 & \ldots & -2 & -2 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$$

and $|Q(B_n, D_k) : Q(D_k)| = |\det A| = 2$.

Type C_n. A set of fundamental roots of C_n can be chosen to be equal to $\{r_1, \ldots, r_n\} = \{e_1 - e_2, \ldots, e_{n-1} - e_n, 2e_n\}$. Consider the extended Dynkin diagram of C_n



A_{n-1} and C_{n-1} are all maximal connected subsystems of C_n. Up to the action of the Weyl group, we may assume that $A_{n-1} = \langle r_1, \ldots, r_{n-1}\rangle$, $C_{n-1} = \langle r_2, \ldots, r_n\rangle$.

(1) $C_{n-1} < C_n$. We shall show that in this case there do not exist $s_2, \ldots, s_n$ from $Q(C_n) = \langle e_1 - e_2, \ldots, e_{n-1} - e_n, e_n\rangle$ such that $(s_i, r_j) = \delta_{i,j}$ for $k, j = 2, \ldots, n$.

Let $s_n = \alpha_1 \check{r}_1 + \ldots + \alpha_n \check{r}_n$ be an element of $Q(C_n)$. Then

$$(s_n, r_n) = -2\alpha_{n-1} + 2\alpha_n = 1.$$

This equation is not solvable in $\mathbb{Z}$, so such a set $s_2, \ldots, s_n$ does not exist. Since $\Delta(C_{n-1}) = 2$ and $Q(C_n, C_{n-1})/Q(C_{n-1})$ is isomorphic to a subgroup of $\Delta(C_{n-1})$, then $|Q(C_n, C_{n-1}) : Q(C_{n-1})| = 1$. By Lemma 3(2) we obtain that $|Q(C_n, C_k) : Q(C_k)| = 1$ for each $2 \le k < n$.

(2) $A_{n-1} < C_n$. Let $s_k = \alpha_1 \check{r}_1 + \ldots + \alpha_n \check{r}_n$ ($k = 1, \ldots, n-1$) be an element of $Q(C_n)$. Then

$$\begin{aligned}
(s_k, r_1) &= 2\alpha_1 - \alpha_2 = \delta_{k,1}, \\
(s_k, r_2) &= -\alpha_1 + 2\alpha_2 - \alpha_3 = \delta_{k,2}, \\
&\vdots \\
(s_k, r_{n-2}) &= -\alpha_{n-3} + 2\alpha_{n-2} - \alpha_{n-1} = \delta_{k,n-2}, \\
(s_k, r_{n-1}) &= -\alpha_{n-2} + 2\alpha_{n-1} - \alpha_n = \delta_{k,n-1}.
\end{aligned}$$

Clearly this system of equations is solvable in $\mathbb{Z}$ for all k (starting from the first equation, all coefficients $\alpha_2, \ldots, \alpha_n$ can be expressed by using α_1), and so $Q(C_n, A_{n-1})$ is a full weight lattice. For a subsystem A_k of C_n, by Lemma 3(1), $Q(C_n, A_k)$ is also a full weight lattice for each $k < n$.

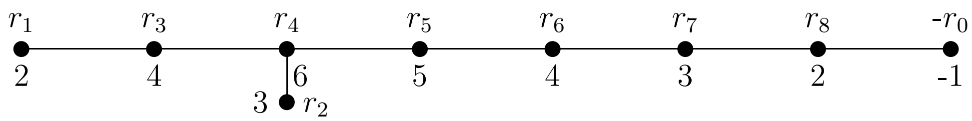
D_n. The results for D_n follow immediately fro Lemma 3(2) and the fact that $Q(B_n, D_n) = Q(D_n)$. Namely

$|Q(D_n, D_k) : Q(D_k)| = 2$, for each $3 \le k < n$,

$|Q(D_n, A_{n-1}) : Q(A_{n-1})| = m = \frac{n}{2}$, if $n = 2m$.

$|Q(D_n, A_k) : Q(A_k)| = k + 1$ in the remaining cases.

Type E_8. A set of fundamental roots of E_8 can be chosen in the form $r_1 = -\frac{1}{2}\sum_{i=1}^{8} e_i, r_2 = e_6 - e_7, r_3 = e_6 + e_7, r_4 = e_5 - e_6, r_5 = e_4 - e_5, r_6 = e_3 - e_4, r_7 = e_2 - e_3, r_8 = e_1 - e_2$, where $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$ is an orthonormal basis. The extended Dynkin diagram of E_8 has the form



All maximal connected subsystems has either type A_8, or type D_8, or type E_7. Up to the action of the Weyl group, we may assume that $A_8 = \langle -r_0, r_1, r_3, r_4, r_5, r_6, r_7, r_8\rangle$, $D_8 = \langle -r_0, r_2, r_3, r_4, r_5, r_6, r_7, r_8\rangle$, $E_7 = \langle r_1, r_2, r_3, r_4, r_5, r_6, r_7\rangle$. Note, that since roots of E_8 have the same length, then they are equal to coroots. This fact remains true for E_7, E_6, obtained as subsystems of E_8.

(1) $A_8 < E_8$. The lattice $Q(A_8^{\perp})$ generated by coroots of E_8, that are orthogonal to all roots of A_8, is equal to 0. Since

$$-r_0 = -2r_1 - 3r_2 - 4r_3 - 6r_4 - 5r_5 - 4r_6 - 3r_7 - 2r_8,$$

then

$$A = \begin{pmatrix} -2 & -3 & -4 & -6 & -5 & -4 & -3 & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore $|Q(E_8, A_8) : Q(A_8)| = |\det A| = 3$. By Lemma 3(1), for $k < 8$ the factor group $Q(E_8, A_k)$ is a full weight lattice.

(2) $D_8 < E_8$. As in point (1) we have $Q(D_8^{\perp}) = 0$ and

$$A = \begin{pmatrix} -2 & -3 & -4 & -6 & -5 & -4 & -3 & -2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

So $|Q(E_8, D_8) : Q(D_8)| = |\det A| = 2$.

A_7 and D_7 are maximal connected subsystems of D_8. By Lemma 3 $Q(E_8, A_7)$ is a full weight lattice.

$D_7 < E_8$. $D_7 = \langle r_2, r_3, r_4, r_5, r_6, r_7, r_8\rangle$. Define $s_2, \ldots, s_8$ from $Q(E_8)$ in the following way:

$$\begin{aligned} s_2 &= r_1 + 3r_2 + 3r_3 + 5r_4 + 4r_5 + 3r_6 + 2r_7 + r_8, \\ s_3 &= 3r_1 + 5r_2 + 7r_3 + 10r_4 + 8r_5 + 6r_6 + 4r_7 + 2r_8, \\ s_4 &= 2r_1 + 5r_2 + 6r_3 + 10r_4 + 8r_5 + 6r_6 + 4r_7 + 2r_8, \\ s_5 &= 2r_2 + 2r_3 + 4r_4 + 4r_5 + 3r_6 + 2r_7 + r_8, \\ s_6 &= 2r_1 + 4r_2 + 5r_3 + 8r_4 + 7r_5 + 6r_6 + 4r_7 + 2r_8, \\ s_7 &= r_2 + r_3 + 2r_4 + 2r_5 + 2r_6 + 2r_7 + r_8, \\ s_8 &= 2r_1 + 3r_2 + 4r_3 + 6r_4 + 5r_5 + 4r_6 + 3r_7 + 2r_8. \end{aligned}$$

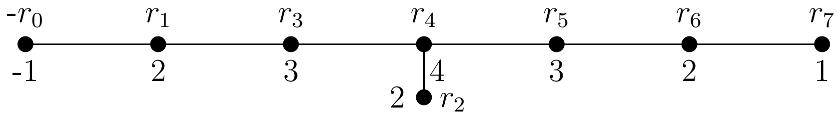
It is immediate that $(s_k, r_j) = \delta_{k,j}$ for $k, j = 2, \ldots, 8$; i. e., $Q(E_8, D_7)$ is a full weight lattice. Since D_k can be obtained from D_7 by eliminating some fundamental roots, then $Q(E_8, D_k)$ is a full weight lattice for each $k < 7$.

(3) $E_7 < E_8$. Set

$$\begin{aligned}
s_1 &= 2r_1 + 2r_2 + 3r_3 + 4r_4 + 3r_5 + 2r_6 + r_7, \\
s_2 &= r_1 + 2r_2 + 2r_3 + 3r_4 + 2r_5 + r_6 - r_8, \\
s_3 &= r_1 + r_2 + 2r_3 + 2r_4 + r_5 - r_7 - 2r_8, \\
s_4 &= -r_5 - 2r_6 - 3r_7 - 4r_8, \\
s_5 &= -r_6 - 2r_7 - 3r_8, \\
s_6 &= -r_7 - 2r_8, \\
s_7 &= -r_8.
\end{aligned}$$

Under this choice of $s_1, \dots, s_7$ we have $(s_k, r_j) = \delta_{k,j}$ for $k, j = 1, \dots, 7$. Therefore $Q(E_8, E_7)$ is a full weight lattice. Below we shall prove that $Q(E_7, E_6)$ is a full weight lattice, hence $Q(E_8, E_6)$ is a full weight lattice.

Type E_7. A set of fundamental roots of E_7 can be obtained by removing from E_8 the root r_8, i. e., a fundamental set of E_7 coincides with $\{r_1, r_2, r_3, r_4, r_5, r_6, r_7\} = \{-\frac{1}{2}\sum_{i=1}^{8} e_i, e_6 - e_7, e_6 + e_7, e_5 - e_6, e_4 - e_5, e_3 - e_4, e_2 - e_3\}$. The extended Dynkin diagram of E_7 has the form



It is clear from the diagram, that all maximal connected subsystems are either of type A_7, or of type D_6, or of type E_6. Up to the action of the Weyl group we may assume that for subsystems A_7, D_6, and E_6 fundamental sets coincide with $\{-r_0, r_1, r_3, r_4, r_5, r_6, r_7\}$, $\{r_2, r_3, r_4, r_5, r_6, r_7\}$, and $E_6 = \{r_1, r_2, r_3, r_4, r_5, r_6\}$.

(1) $A_7 < E_7$. The lattice $Q(A_7^{\perp})$, generated by coroots of E_7, that are orthogonal to all roots of A_7, is equal to 0, and $-r_0 = -2r_1 - 2r_2 - 3r_3 - 4r_4 - 3r_5 - 2r_6 - r_7$. Therefore the transfer matrix from the basic of E_7 to the basis of A_7 has the form

$$A = \begin{pmatrix} -2 & -2 & -3 & -4 & -3 & -2 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence $|Q(E_7, A_7) : Q(A_7)| = |\det A| = 2$. By Lemma 3(1) for $k < 7$ the factor group $Q(E_7, A_k)$ is a full weight lattice.

(2) $D_6 < E_7$. The lattice $Q(D_6^{\perp})$, generated by coroots of $E_7 = \langle -\frac{1}{2}\sum_{i=1}^{8} e_i, e_6 - e_7, e_6 + e_7, e_5 - e_6, e_4 - e_5, e_3 - e_4, e_2 - e_3 \rangle$, that are orthogonal to all roots of $D_6 = \langle e_6 - e_7, e_6 + e_7, e_5 -$

$e_6, e_4 - e_5, e_3 - e_4, e_2 - e_3\rangle$, is generated by $e_1 + e_8$. Define a new basis $r'_1, \ldots, r'_7$ of E_7:

$$\begin{aligned} r'_1 &= r_1, \\ &\vdots \\ r'_6 &= r_6, \\ r'_7 &= -e_1 - e_8 = 2r_1 + 2r_2 + 3r_3 + 4r_4 + 3r_5 + 2r_6 + r_7. \end{aligned}$$

Then the transfer matrix A from $Q(E_7, D_6)$ to $Q(D_6)$ has the form

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -2 & -2 & -3 & -4 & -3 & -2 \end{pmatrix}.$$

and $|Q(E_7, D_6) : Q(D_6)| = |\det A| = 2$.

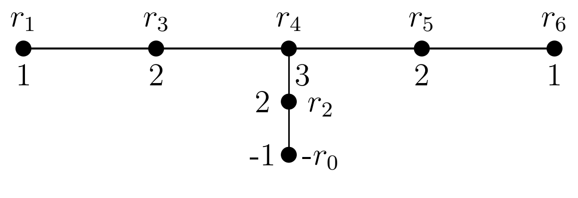
A subsystem $D_5 = \langle r_2, r_3, r_4, r_5, r_6\rangle$ is contained in E_6, and, as we shall show later, $Q(E_6, D_5)$ is a full weight lattice. Therefore, by Lemma 3(1) $Q(E_7, D_5)$ is a full weight lattice as well.

(2) $E_6 < E_7$. Set

$$\begin{aligned} s_1 &= -r_2 - r_3 - 2r_4 - 2r_5 - 2r_6 - 2r_7, \\ s_2 &= r_1 + 2r_2 + 2r_3 + 3r_4 + 2r_5 + r_6, \\ s_3 &= r_1 + r_2 + 2r_3 + 2r_4 + r_5 - r_7, \\ s_4 &= -r_5 - 2r_6 - 3r_7, \\ s_5 &= -r_6 - 2r_7, \\ s_6 &= -r_7. \end{aligned}$$

Then $(s_k, r_j) = \delta_{k,j}$ and $Q(E_7, E_6)$ is a full weight lattice.

Type E_6. A set of fundamental roots of E_6 can be obtained by removing roots r_7, r_8 from the system E_8, i. e., $E_6 = \langle r_1, r_2, r_3, r_4, r_5, r_6\rangle = \langle -\frac{1}{2}\sum_{i=1}^{8} e_i, e_6 - e_7, e_6 + e_7, e_5 - e_6, e_4 - e_5, e_3 - e_4\rangle$. The extended Dynkin diagram of E_6 has the form



Clearly all maximal connected subsystems are either of type A_5, or of type D_5. Up to the symmetry of the diagram and action of the Weyl group we may assume that $A_5 = \langle r_1, r_3, r_4, r_5, r_6\rangle$, $D_5 = \langle r_2, r_3, r_4, r_5, r_6\rangle$.

(1) $A_5 < E_6$. In this case $Q(A_5^{\perp})$ is generated by $r_1' = -e_1 - e_2 + e_3 + e_4 + e_5 + e_6 - e_7 - e_8$. Define

$$\begin{aligned}
r_1' &= r_1 + 2r_2 + 2r_3 + 3r_4 + 2r_5 + r_6, \\
r_2' &= r_2, \\
r_3' &= r_3, \\
r_4' &= r_4, \\
r_5' &= r_5, \\
r_6' &= r_6.
\end{aligned}$$

Taking the factor by $Q(A_5^{\perp})$ we obtain

$$A = \begin{pmatrix} -2 & -2 & -3 & -2 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

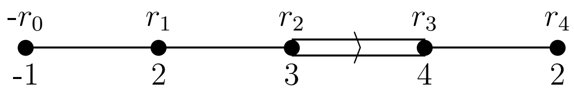
Therefore $|Q(E_6, A_5) : Q(A_5)| = 2$. By Lemma 3(1) for $k < 5$ the factor group $Q(E_6, A_k)$ is a full weight lattice.

(2) $D_5 < E_6$. Set

$$\begin{aligned}
s_2 &= r_1 + 2r_2 + 2r_3 + 3r_4 + 2r_5 + r_6, \\
s_3 &= 3r_1 + 3r_2 + 5r_3 + 6r_4 + 4r_5 + 2r_6, \\
s_4 &= 2r_1 + 3r_2 + 4r_3 + 6r_4 + 4r_5 + 2r_6, \\
s_5 &= r_2 + r_3 + 2r_4 + 2r_5 + r_6, \\
s_6 &= 2r_1 + 2r_2 + 3r_3 + 4r_4 + 3r_5 + 2r_6.
\end{aligned}$$

Then $(s_k, r_j) = \delta_{k,j}$ and $Q(E_6, D_5)$ is a full weight lattice. Therefore by Lemma 3(1) $Q(E_6, D_k)$ is a full weight lattice for $k < 5$.

Type F_4**.** A set of fundamental roots of F_4 can be chosen in the form $\{r_1, r_2, r_3, r_4\} = \{e_1 - e_2, e_2 - e_3, e_3, \frac{1}{2}(-e_1 - e_2 - e_3 + e_4)\}$. The extended Dynkin diagram of F_4 has the form



All maximal connected subsystems are either of type B_4, or of type C_3. Up to the action of the Weyl group $B_4 = \langle -r_0, r_1, r_2, r_3 \rangle$, $C_3 = \langle r_2, r_3, r_4 \rangle$.

(1) $B_4 < F_4$. Lattices generated by coroots has the form $Q(F_4) = \langle \check{r}_1, \check{r}_2, \check{r}_3, \check{r}_4 \rangle = \langle e_1 - e_2, e_2 - e_3, 2e_3, -e_1 - e_2 - e_3 + e_4 \rangle$ and $Q(B_4) = \langle -e_1 - e_4, e_1 - e_2, e_2 - e_3, 2e_3 \rangle$. Since $Q(B_4^{\perp}) = 0$ and $-r_0 = -2r_1 - 3r_2 - 4r_3 - 2r_4$, then

$$A = \begin{pmatrix} -2 & -3 & -2 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore $|Q(F_4, B_4) : Q(B_4)| = 1$.

By Lemma 3(2) from the equality $Q(F_4, B_4) = Q(B_4)$ we obtain that $Q(F_4, \Theta) = Q(B_4, \Theta)$ for each subsystem Θ of B_4. Thus, from already considered case B_n it is immediate that

$$|Q(F_4, D_4) : Q(D_4)| = 1,\ |Q(F_4, D_3) : Q(D_3)| = 2,\ |Q(F_4, A_3) : Q(A_3)| = 2.$$

For the remaining subsystems Ψ-s of F_4 the factor group $Q(F_4, \Psi)$ is a full weight lattice.

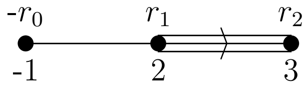
(2) $C_3 < F_4$.

Set

$$\begin{array}{rcl} s_2 &=& -\check{r}_1, \\ s_3 &=& -2\check{r}_1 - \check{r}_2, \\ s_4 &=& -2\check{r}_1 - 2\check{r}_2 - \check{r}_3. \end{array}$$

Then $(s_k, r_j) = \delta_{k,j}$ and $Q(F_4, C_3)$ is a full weight lattice.

Type G_2. The extended Dynkin diagram of G_2 has the form



A_2 (generated by long roots) and A_1 (generated by a shot root) are the maximal connected subsystems. Up to the action of the Weyl group $A_2 = \langle -r_0, r_1\rangle$ and $A_1 = \langle r_2\rangle$.

(1) $A_2 < G_2$. Since the root system is uniquely defined up to equivalence, set $(r_1, r_1) = 6$, then $(r_1, r_2) = -3$ and $(r_2, r_2) = 2$. Coroots of G_2 has the form $\check{r}_1 = \frac{2r_1}{(r_1,r_1)} = \frac{r_1}{3}$, $\check{r}_2 = \frac{2r_2}{(r_2,r_2)} = r_2$, and $-\check{r}_0 = \frac{-2r_0}{(r_0,r_0)} = \frac{1}{3}(-2r_1 - 3r_2) = \frac{1}{3}(-6\check{r}_1 - 3\check{r}_2) = -2\check{r}_1 - \check{r}_2$. Therefore the transfer matrix from $Q(G_2, A_2)$ to $Q(A_2)$ has the form

$$A = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}.$$

Thus, $|Q(G_2, A_2) : Q(A_2)| = 1$.

(2) $A_1 < G_2$. Set $s = \check{r}_1 + \check{r}_2$, then $(s, r_2) = (\check{r}_1 + \check{r}_2, r_2) = \frac{1}{3}(r_1, r_2) + (r_2, r_2) = 1$. So $Q(G_2, A_1)$ is a full weight lattice.

The case $G \simeq {}^2G_2(3^{2n+1})$ and $L \simeq A_1(3^{4n+2})$ follows from [7, Table 5, p. 139]. $\square$

Note that analogous results for reductive subgroups, containing a Cartan subgroup, of split groups were obtained by different methods by Nikolay A. Vavilov in an unpublished paper.

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