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HALL SUBGROUPS OF ODD ORDER IN FINITE GROUPS

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We complete the description of Hall subgroups of odd order in finite simple groups initiated by F. Gross, and as a consequence, bring to a close the study of odd order Hall subgroups in all finite groups modulo classification of finite simple groups. In addition, it is proved that for every set π of primes, an extension of an arbitrary D_π-group by a D_π-group is again a D_π-group. This result gives a partial answer to Question 3.62 posed by L. A. Shemetkov in the "Kourovka Notebook."

INTRODUCTION

Ph. Hall's definition maintains that a finite group possesses property E_π (in other words, it is an E_π-group) for some set π of primes if it contains a π-subgroup whose index is not divisible by the primes in π. (Such is conventionally called a Hall π-subgroup.) If a group G contains a Hall π-subgroup, and all of its Hall π-subgroups are conjugate, then G is called a C_π-group. Finally, if G is a C_π-group, and its π-subgroups each is contained in some Hall π-subgroup, then G is said to possess property D_π.

It is well known that, for every set π of primes, the class of all E_π-groups is closed under normal subgroups and homomorphic images. Consequently, if a finite group possesses a Hall π-subgroup then its composition factors each also possess one. Traditionally, therefore, the problem of describing Hall subgroups of finite groups which are close to simple ones takes center-stage in their study. For finite Chevalley groups, a similar problem is pointed out, in particular, in [1].

We also know that property C_π is inherited under extensions. The question if property D_π is inherited under extensions remains open. (Shemetkov brought it in [2] as Question 3.62.)

Descriptions of Hall subgroups of the symmetric groups are obtained in Hall [3] and in Thompson [4]. Hall subgroups of the sporadic groups and of the finite Lie-type groups, for the case where the characteristic of a base field belongs to π, are described in Gross [5, 6] and in Revin [7, 8]. Due to Gross is also the description of Hall π-subgroups of the classical groups of arbitrary characteristic p, if $2, p \notin \pi$ (cf. [9]), and of the groups $GL_n(p^s)$ and $Sp_{2n}(p^s)$ if $3, p \notin \pi$ (cf. [6]).

The problem of describing finite groups with property E_π and their Hall π-subgroups for the case where $2 \notin \pi$ is of special interest, for the class of all E_π-groups will be closed under extensions, as follows from the main result of [10], which uses the classification of finite simple groups. In particular, Gross proved that properties E_π and C_π are equivalent if $2 \notin \pi$. Therefore, for an arbitrary finite group to contain a Hall

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π-subgroup given some set π of odd primes, it is necessary and sufficient that its composition factors each contain a Hall π-subgroup.

In the present article, we aim at providing a consistent description of Hall π-subgroups in all exceptional groups of Lie type for the case where π does not contain 2 and the base field characteristic. Specifically, we prove a theorem (cf. below) which generalizes Gross' result in [9, Thms. 4.6 and 4.8], proved for the classical groups.

THEOREM 1. Let G be a Lie-type group over a field of characteristic p, H be its Hall π-subgroup, where $2, p \notin \pi$, r be the least prime in $\pi \cap \pi(G)$, and $\tau = \pi \setminus \{r\}$. Then H contains a normal Abelian Hall τ-subgroup.

With due regard for the results laid out above, the present inquiry completes the description of Hall subgroups of odd order in finite simple groups, and as a consequence, brings to a close the study of all finite E_π-groups for any set π of odd primes modulo classification of finite simple groups.

In the final Sec. 5 we give a partial answer to Shemetkov's question as to whether property D_π is inherited under extensions. Namely, we prove the following:

THEOREM 2. Let π be some set of odd primes. Then an extension of a D_π-group by a D_π-group is again a D_π-group.

1. THE BASIC NOTATION

The notation and definitions used in the article can be found in [11-13]. If G is a group then the expressions $H \leq G$ and $H \trianglelefteq G$ mean that H is, respectively, a subgroup and a normal subgroup of G. An index of H in G is denoted by $|G : H|$, and the normalizer of H in G — by $N_G(H)$. If the subgroup H is normal in G then G/H is a factor group of G w.r.t. H. If M is a subset of the group G then $\langle M \rangle$ is a subgroup generated by the set M, and $|M|$ is the cardinality of M (or an element order, if one element is taken instead of the set). Denote by $C_G(M)$ the centralizer of M in G, and by $Z(G)$ the center of G. Conjugation of an element x by an element y is labelled $x^y = y^{-1}xy$. By $[x, y] = x^{-1}x^y$ we denote a commutator of x and y, and by $[A, B]$ the mutual commutant of subgroups A and B in G. For the groups A and B, $A \times B$, $A * B$, and $A \leftthreetimes B$ stand for, respectively, a direct product, a central product, and a semidirect product of A and B, where B is a normal subgroup. If A and B are subgroups of G such that $A \trianglelefteq B$ then the factor B/A is called a section of the group G.

If n is some natural number then $\pi(n)$ denotes a set of prime divisors of n. For a finite group G, we put $\pi(G) = \pi(|G|)$ by definition. A set of Sylow p-subgroups of the finite group G is denoted $Syl_p(G)$. For a p-group G, $\Omega_k(G)$ stands for the group $\langle x \mid x^{p^k} = e \rangle$.

Let n be a natural number and ω some set of primes. Write ω' to denote a set of all primes which do not belong to ω. Write n_ω for a greatest integer t which divides n and is such that $\pi(t) \subseteq \omega$. If ω consists of one prime p, we use the expressions p' and n_p instead of ω' and n_ω, respectively. Letting r be a prime, q a natural number, and $(r, q) = 1$, we define $e(r, q)$ to be the least natural n such that $q^n \equiv 1 \pmod{r}$. If φ is an homomorphism of the group G and g its element then G^φ and g^φ are the respective images of G and g under φ. G_φ denotes a fixed-point set of G under the endomorphism φ. Let $\operatorname{Aut} G$ be an automorphism group of G. Denote by $\Phi_i(t)$ an ith cyclotomic polynomial. Recall that $\Phi_1(t) = t - 1$ and $\Phi_n(t) = (t^n - 1)/(\prod\limits_{i<n,\, i|n} \Phi_i(t))$.

The notation related to finite groups of Lie type is the same as in [12]. By a Chevalley group or Lie-type group, unless specified otherwise, we mean both the universal Chevalley group and every one of its factors w.r.t. a subgroup in the center. In dealing with Chevalley groups, we denote by $GF(q)$ a field of order q, by $p = \operatorname{char} GF(q)$ its characteristic, and by $GF(q)^*$ the multiplicative group of $GF(q)$. A Chevalley group G which corresponds to a root system Φ over $GF(q)$ is denoted $\Phi(q)$. And we call $GF(q)$ the *base* field of G. A Weyl group corresponding to Φ is denoted by $W(\Phi)$, and a Weyl group of a Lie-type group G — by $W(G)$. Twisted groups are denoted ${}^2A_n(q^2)$, ${}^2D_n(q^2)$, ${}^2E_6(q^2)$, ${}^3D_4(q^3)$, ${}^2B_2(q)$, ${}^2G_2(q)$, and ${}^2F_4(q)$, and we say that the base field for G is $GF(q)$, $GF(q^2)$, or $GF(q^3)$, which depends on the degree at q in a group's labelling. For all groups of Lie type, $GF(q)$ is called a *definition field*. We say that an element x in the Chevalley group $\Phi(q)$ is semisimple, if its order is coprime with p, and is unipotent if its order is the degree of p. Similarly, a semisimple and a unipotent subgroups of $\Phi(q)$ are, respectively, one whose order is coprime with p (p'-subgroups) and one whose order is the degree of p.

For a finite Lie-type group G, the group $\mathbf{W}(G)$ is defined as follows. If G is of normal type then we put $\mathbf{W}(G) = W(G)$. If G is twisted and G_1 is a normal-type group defining G, then we put $\mathbf{W}(G) = W(G_1)$ (cf. [12]).

If G is a finite Lie-type group with a definition field $GF(q)$ of characteristic p, and G is not a Suzuki or Ree group, then a p'-part of the order of G may be represented as $f(q)$ for some polynomial $f(t) \in \mathbb{Z}[t]$. Decompose the latter into irreducible (over $\mathbb{Z}$) divisors $f_1(t), \ldots, f_k(t)$, which are "almost coprime," in a sense that values of different polynomials $f_i(q)$ and $f_j(q)$ share few prime divisors in common. Moreover, these polynomials are all cyclotomic.

We also need data on the orders of exceptional universal groups of Lie type and on the possible common divisors of the values for irreducible polynomials in q, into which the p'-part of the order of a Lie-type group splits. A prime number r is said to be *small* for a finite Lie-type group G if there exist irreducible polynomials $f_i(t) \neq f_j(t)$, involved in the splitting of the p'-part of G, such that r divides $(f_i(q), f_j(q))$; otherwise, we say that r is *large*. Prime divisors of the order of the group $\mathbf{W}(G)$ (denoted W_0 in [5]) are called *singular* primes. Prime numbers are said to be *non-singular* if they are not singular. Necessary information on the order of a p'-part, and also on small and singular primes for the exceptional Chevalley groups which are not Suzuki or Ree, is contained in Table 1 (cf. below).

Note that every small prime is singular, whereas a singular prime may well be large. For instance, 5 for the group $E_6(q)$ is singular, but if $q-1$ is not divided by 5, then 5 will be large for $E_6(q)$. Further, for a finite Lie-type group G and for some set π of primes, by $\omega(G)$ we denote a set of singular primes of G, by $\tau(G)$ the set $\pi \setminus \omega(G)$, by $\rho(G)$ a set of small primes of G, and by $\theta(G)$ the set $\pi \setminus \rho(G)$. Sometimes we merely use the symbols ω, τ, ρ, and θ, provided that there is clarity as to which group G is implied.

The main definitions and the basic results concerning linear algebraic groups are contained in [13]. Since below we confine ourselves to linear algebraic groups, the word 'linear', for brevity, will be omitted. If G is an algebraic group, G^0 denotes a component of the unity of G. We say that an algebraic group is semisimple, if its radical is trivial, and is reductive if its unipotent radical is trivial. (In either case, an algebraic group is not assumed connected.) It is well known that a connected semisimple algebraic group is a central product of connected simple algebraic ones, and that a connected reductive algebraic group G is the product of a torus S and a semisimple group M; moreover, $S = Z(G)^0$, $M = [G,G]$, and $S \cap M$ is a finite group (cf. [13]). A torus is a connected diagonable (d-) group. The rank of a connected algebraic group is the dimension of its maximal torus.

We recall how finite Lie-type groups and simple algebraic groups are connected. Let G be a connected simple algebraic group, defined over an algebraically closed field of characteristic $p > 0$, and σ be an endomorphism of G such that its fixed-point set G_σ is finite. Below, the endomorphism σ with this condition

TABLE 1

Group	$p'(\lvert G\rvert)$	$\lvert\mathbf{W}(G)\rvert$	Possible small numbers
$G_2(q)$	$(q-1)^2(q+1)^2(q^2-q+1)(q^2+q+1)$	12	$2,3$
$F_4(q)$	$(q-1)^4(q+1)^4(q^2+1)^2(q^2-q+1)^2\times$ $(q^2+q+1)^2(q^4+1)(q^4-q^2+1)$	$2^7\cdot 3^2$	$2,3$
$E_6(q)$	$(q-1)^6(q+1)^4(q^2+1)^2(q^2+q+1)^3\times$ $(q^2-q+1)^2(q^4+q^3+q^2+q+1)\times$ $(q^4+1)(q^4-q^2+1)(q^6+q^3+1)$	$2^7\cdot 3^4\cdot 5$	$2,3,5$
$E_7(q)$	$(q-1)^7(q+1)^7(q^2+1)^2(q^2+q+1)^3\times$ $(q^2-q+1)^3(q^4+1)\times$ $(q^4+q^3+q^2+q+1)\times$ $(q^4-q^3+q^2-q+1)(q^4-q^2+1)\times$ $(q^6-q^3+1)(q^6+q^3+1)\times$ $(q^6+q^5+q^4+q^3+q^2+q+1)\times$ $(q^6-q^5+q^4-q^3+q^2-q+1)$	$2^{10}\cdot 3^4\cdot 5\cdot 7$	$2,3,5,7$
$E_8(q)$	$(q-1)^8(q+1)^8(q^2+1)^4(q^2+q+1)^4\times$ $(q^2-q+1)^4(q^4+1)^2(q^4-q^2+1)^2\times$ $(q^4+q^3+q^2+q+1)^2\times$ $(q^4-q^3+q^2-q+1)^2(q^6+q^3+1)\times$ $(q^6+q^5+q^4+q^3+q^2+q+1)\times$ $(q^6-q^3+1)\times$ $(q^6-q^5+q^4-q^3+q^2-q+1)\times$ $(q^8-q^4+1)(q^8-q^6+q^4-q^2+1)\times$ $(q^8-q^7+q^5-q^4+q^3-q+1)\times$ $(q^8+q^7-q^5-q^4-q^3+q+1)$	$2^{14}\cdot 3^5\cdot 5^2\cdot 7$	2, 3, 5, 7
${}^3D_4(q^3)$	$(q-1)^2(q+1)^2(q^2-q+1)^2\times$ $(q^2+q+1)^2(q^4-q^2+1)$	$2^6\cdot 3$	2, 3
${}^2E_6(q^2)$	$(q-1)^4(q+1)^6(q^2+q+1)^2(q^2+1)^2\times$ $(q^2-q+1)^3(q^4+1)\times$ $(q^4-q^3+q^2-q+1)\times$ $(q^4-q^2+1)(q^6-q^3+1)$	$2^7\cdot 3^4\cdot 5$	2, 3, 5

is referred to as the Frobenius automorphism, even though it may not coincide with classical Frobenius'. Note that σ is an automorphism, if G is treated as an abstract group, and σ is an endomorphism if G is treated as an algebraic one. In the general case σ has the form $q\sigma_0$, where q denotes raising to the qth power, $q=p^\alpha$, and σ_0 is a graph automorphism of order 1, 2, or 3. $O^{p'}(G_\sigma)$ is then a Lie-type group over a finite field of characteristic p, and every group of Lie type (normal or twisted) can be obtained as stated. The group G_σ coincides with a group of inner-diagonal automorphisms of the group $K=O^{p'}(G_\sigma)$, which we denote by $\hat{K}$. If G is one-connected then $G_\sigma=O^{p'}(G_\sigma)$ is a universal group. The *rank* of a finite Lie-type group $O^{p'}(G_\sigma)$, where G is a connected simple algebraic group, is called the rank of G.

Let T be some σ-invariant torus of the connected simple algebraic group G. In what follows, by a *torus* of G_σ [resp., of $O^{p'}(G_\sigma)$] we mean a group T_σ [resp., $T_\sigma\cap O^{p'}(G_\sigma)$]. If T is maximal then T_σ [resp., $T_\sigma\cap O^{p'}(G_\sigma)$] is called a *maximal torus*. It is well known that there exists a one-to-one correspondence between different classes of maximal tori of G_σ (for groups of non-twisted type) and classes of conjugate elements of its Weyl group W (see, e.g., [14]). If $T_w(q)$ is a maximal torus corresponding to an element $w \in W$ then $|T_w(q)| = f(q)$, where $f(t)$ is a characteristic polynomial of w. (For detailed information on the classes of conjugate elements of the Weyl group, their characteristic polynomials, and on the centralizers of different elements in Weyl groups for all simple Lie algebras, we ask the reader to consult [15].)

Let G be some finite Lie-type group defined over a field $GF(q)$. Then there exists an algebraic group $\overline{G}$ such that $G = O^{p'}(\overline{G}_\sigma)$. If G is universal then $\overline{G}$ can be taken to be one-connected, and so the latter is unique up to isomorphism. If $R = \overline{R}_\sigma$, where $\overline{R}$ is some connected reductive σ-invariant subgroup of maximal rank in $\overline{G}$, then we define $N(R)$ to be $(N_{\overline{G}}(\overline{R}))_\sigma$. Note that generally $N(R) \neq N_G(R)$, and yet $N(R) \leq N_G(R)$. If T is a Cartan subgroup in G then $N(T) = N$ is a monomial subgroup of G. For the case where $T = T_w(q)$ is some maximal torus of G, the group $N(T)$ (denoted $N_w(T)$ in [15]) is isomorphic to a certain subgroup of N. Generally, $N(R)/R$ is isomorphic to a section of the Weyl group $W(G)$.

2. AUXILIARIES

The lemma below is well known.

LEMMA 1. Let G be a finite group and A its normal subgroup. If H is some Hall π-subgroup of G then $H \cap A$ is a Hall π-subgroup of A and HA/A is one in G/A. In particular, property E_π is inherited by normal subgroups also under homomorphisms.

From [5, 10], it follows that for the case where $2 \notin \pi$, properties E_π and C_π are equivalent and are inherited under extensions. Therefore, under Secs. 2 and 3 below, we assume that all the Lie-type groups under examination are universal.

LEMMA 2. Let G be a universal Lie-type group with a definition field $GF(q)$ of characteristic p and R be its Sylow r-subgroup, where $3 < r \neq p$. Then there exists a maximal torus H such that $R \leq N_G(H)$ and $Z(R) \leq H$.

Proof. The fact that $R \leq N_G(H)$ is well known (see, e.g., [1, Sec. 1.2a]). Moreover, from [1, Secs. 1.3b,e], it follows that $R \cap H$ is a maximal Abelian subgroup of R; consequently, $Z(R) \leq H$.

The next lemma is applicable to the case where π is freed of 2 and p.

LEMMA 3. Let $G = \overline{G}_\sigma$ be a finite universal group of Lie type over a field of characteristic p, π be a set of primes without 2 and p, $|\pi \cap \pi(G)| \geqslant 2$, H be a Hall π-subgroup of G, r the greatest prime in $\pi \cap \pi(G)$, R a Sylow r-subgroup of H, and $A = Z(R)$. Then:

(1) $A \trianglelefteq H$;

(2) $C_{\overline{G}}(A)$ is a connected reductive subgroup of maximal rank in $\overline{G}$;

(3) $C = (C_{\overline{G}}(A))_\sigma$ contains a Sylow r-subgroup of G, $H \leq N(C)$, and, moreover, $C_0 = O^{p'}(C)$ is an E_π-group;

(4) $H/C_H(A)$ is a section of the Weyl group $W(G)$.

Proof. (1) From [10, Thm. B], it follows that A is a normal Abelian r-subgroup of H.

(2) Since $|\pi \cap \pi(G)| \geqslant 2$ and r is greatest in $\pi \cap \pi(G)$, we have $r > 3$, and by Lemma 2, A is contained in some maximal torus $\overline{T}$ of the group $\overline{G}$. A proof that $C_{\overline{G}}(A)$ is a connected reductive subgroup of maximal rank in $\overline{G}$ is the same as in [16, Thms. 2.2 and 2.10]. Moreover, it is clear that the group $C_{\overline{G}}(A)$ is σ-invariant.

(3) and (4) are obvious. The lemma is proved.

(Note that all possible structures of C in all finite Lie-type groups are described in [14, 17-19].)

In dealing with finite groups of Lie type we need some data on Hall π-subgroups in the Weyl groups.

LEMMA 4. Let Φ be a root system of exceptional type (i.e., E_6, E_7, E_8, F_4, or G_2) and let $2 \notin \pi$, where π is some set of primes. Then $W(\Phi)$ is an E_π-group if and only if $|\pi \cap \pi(W(\Phi))| \leqslant 1$.

Proof. It suffices to show that if $W(\Phi)$ is an E_π-group then $|\pi \cap \pi(W(\Phi))| \leqslant 1$. For the case where $\Phi = F_4$ or $\Phi = G_2$, the statement is obvious since 3 is a unique odd prime which divides $|W(\Phi)|$.

Let $\Phi = E_6$. It is known that the unique non-Abelian composition factor of $W(E_6)$ is isomorphic to $PSp_4(3)$. Since property E_π is inherited by all composition factors, the fact that $W(E_6)$ is an E_π-group implies that $PSp_4(3)$ is also an E_π-group. Obviously, $\pi(W(E_6)) = \pi(PSp_4(3))$; hence, $\pi \cap \pi(W(E_6)) = \pi \cap \pi(PSp_4(3))$. In view of [7], $PSp_4(3)$ is not an $E_{\{3,5\}}$-group. From which the conclusion of the lemma follows immediately for E_6.

The unique non-Abelian composition factor of $W(E_7)$ is isomorphic to $Sp_6(2)$, and the one of $W(E_8)$ to $P\Omega_8^+(2)$. Furthermore, $\pi(Sp_6(2)) = \pi(W(E_7))$ and $\pi(P\Omega_8^+(2)) = \pi(W(E_8))$. By [9], the groups $Sp_6(2)$ and $P\Omega_8^+(2)$ do not contain non-trivial Hall π-subgroups, yielding the result for the present case.

LEMMA 5 [5, Lemma 2.3(2)]. Let G be some Lie-type group with definition field $GF(q)$ and r be an odd prime dividing $q-1$. Then the monomial subgroup N of G contains a Sylow r-subgroup of G.

LEMMA 6. [1, (1.13)]. Let $G = \overline{G}_\sigma$ be some universal Lie-type group with definition field $GF(q)$, r be the rank of $\overline{G}$, G contain a maximal torus $T = \overline{T}_\sigma$ of order $(q+1)^r$, and s be an odd prime dividing $q+1$. Then $N(T)$ contains a Sylow s-subgroup of G.

3. HALL SUBGROUPS OF FINITE EXCEPTIONAL GROUPS OF LIE TYPE

In the present section we deal with Hall π-subgroups of all exceptional Lie-type groups, subject to the condition that $2, p \notin \pi$.

We adopt the notation of Lemma 3. Namely, let G be a finite exceptional universal group of Lie type over a field of characteristic p and let $|\pi \cap \pi(G)| \geqslant 2$ and $2, p \notin \pi$ for a set π of primes. The case where $|\pi \cap \pi(G)| \leqslant 1$ is trivial and so omitted. Assume that G possesses a non-trivial Hall π-subgroup H, r is the greatest prime in $\pi \cap \pi(G)$, R is a Sylow r-subgroup of H, and $A = Z(R)$. Let $\overline{C} = C_{\overline{G}}(A)$ be a connected σ-invariant reductive subgroup of maximal rank in $\overline{G}$, and $C = \overline{C}_\sigma$. By a *simple polynomial* we always mean some cyclotomic polynomial occurring in the decomposition of a p'-part of the order of G (cf. Table 1).

We start to consider the Hall subgroups in $G_2(q)$.

LEMMA 7. Let $G = G_2(q)$. Then $\tau = \pi \setminus \{3\}$ and G contains a Hall π-subgroup if and only if the set π satisfies one of the following conditions:

(1)* $\pi \cap \pi(G) \subseteq \pi(q \pm 1)$ (here if $3 \in \pi$ then the Hall π-subgroup H is representable as $S \leftthreetimes T$, where T is a Hall τ-subgroup of some maximal torus of order $(q \pm 1)^2$, S is a Sylow 3-subgroup of G normalizing this torus, and if $3 \notin \pi$ then $H = T$);

(2) $3 \notin \pi$ and $\pi \cap \pi(G) \subseteq \pi(q^2 \pm q + 1)$ (here, the Hall π-subgroup H is a Hall π-subgroup of some maximal torus of order $q^2 \pm q + 1$).

In either case, the Hall π-subgroup of G satisfies the conclusion of Theorem 1.

Proof. Using Lemmas 5 and 6 it is not hard to show that if π satisfies one of (1), (2) then G contains a Hall π-subgroup. We argue for the converse.

*Hereinafter, in every item we assume that if the sign + (−) is taken in place of ± then + (−) should be taken in place of ±, and − (+) — in place of ∓.

The unique singular prime for G is 3. Hence r is a non-singular prime. Consequently, the Sylow r-subgroup R of G is Abelian (cf. [5, Thm. 2.4]). By Lemma 2, therefore, the group $R = A$ lies in some maximal torus T of G. Moreover, r divides the value of some simple polynomial, that is, $q-1$, $q+1$, q^2-q+1, or q^2+q+1. Since $A = R$ belongs to $Z(C)$, from [17, Table 4, p. 138], it follows that C is a maximal torus of order $(q-1)^2$, $(q+1)^2$, q^2-q+1, or q^2+q+1, respectively.

First let $3 \in \pi$. Then either $q \equiv 1 \pmod 3$ or $q \equiv -1 \pmod 3$. If $q \equiv 1 \pmod 3$, then C is a Cartan subgroup, since H contains a Sylow 3-subgroup of G. If $s \in \tau \cap \pi(G)$, then $q \equiv 1 \pmod s$, since H contains a Sylow s-subgroup. Hence $\pi \cap \pi(G) \subseteq \pi(q-1)$. If $q \equiv -1 \pmod 3$ then C is a torus of order $(q+1)^2$. Again, for any prime $s \in \tau \cap \pi(G)$, the fact that H contains a Sylow s-subgroup of G implies that $q+1$ is divided by s. Hence $\pi \subseteq \pi(q+1)$.

If $3 \notin \pi$, then $H = C_H(A)$, since $H/C_H(A)$ is a section of the Weyl group by Lemma 3; hence, $H/C_H(A)$ is a trivial group. Consequently, H is contained in some maximal torus C of G. Thus $\pi \cap \pi(G) \subseteq \pi(C)$ for some maximal torus C whose order is equal to $(q \pm 1)^2$ or to $q^2 \pm q + 1$.

We continue to consider the Hall subgroups in $F_4(q)$.

LEMMA 8. Let $G = F_4(q)$. Then $\tau = \pi \setminus \{3\}$ and G contains a Hall π-subgroup if and only if the set π satisfies one of the following:

(1) $\pi \cap \pi(G) \subseteq \pi(q \pm 1)$ (here if $3 \in \pi$ then the Hall π-subgroup H is representable as $S \leftthreetimes T$, where T is a Hall τ-subgroup of some maximal torus of order $(q \pm 1)^4$, $S \in Syl_3(G)$, and if $3 \notin \pi$ then $H = T$);

(2) $3 \notin \pi$ and $\pi \cap \pi(G)$ is contained in one of the sets $\pi(q^2 \pm q+1)$, $\pi(q^2+1)$, $\pi(q^4+1)$, or $\pi(q^4-q^2+1)$ (here, the Hall π-subgroup H is Abelian and lies in some maximal torus of order $(q^2 \pm q+1)^2$, $(q^2+1)^2$, q^4+1, or q^4-q^2+1, respectively).

In either case, the Hall π-subgroup of G satisfies the conclusion of Theorem 1.

Proof. As in the case of $G_2(q)$, establishing the sufficiency of (1) and (2) is a simple matter. We argue for the necessity.

For G, there is a unique singular prime — 3. Consequently, r is a non-singular prime. Hence the Sylow r-subgroup R of G is Abelian and $R = A$. In view of Lemma 3, the factor group $H/C_H(A)$ is isomorphic to a section of the Weyl group; so, the index $|H : C_H(A)|$ is the degree of 3 and does not exceed 9. Moreover, r divides some simple polynomial of G, and $R = A \leq Z(C)$. From [17, Table 2, p. 133], it follows that C is a maximal torus of order $(q \pm 1)^4$, $(q^2 \pm q+1)^2$, q^4+1, or q^4-q^2+1.

If $3 \notin \pi$ then the Hall π-subgroup is Abelian and $\pi \cap \pi(G) \subseteq \pi(C)$ for some C.

Let $3 \in \pi$ and $q \equiv 1 \pmod 3$. Then the Hall π-subgroup H contains a Sylow 3-subgroup of G and belongs to $N(C)$. Among the above-mentioned maximal tori, there is a unique one for which $N(C)$ contains a Sylow 3-subgroup of G — this is a torus of order $(q-1)^4$ (a Cartan subgroup); consequently, $\pi \cap \pi(G) \subseteq \pi(q-1)$ and $H = S \leftthreetimes T$, where T is a Hall τ-subgroup of both the Cartan subgroup and the group G, and $S \in Syl_3(G)$.

If $3 \in \pi$ and $q \equiv -1 \pmod 3$, then the unique maximal torus C for which $N(C)$ contains a Sylow 3-subgroup of G is one of order $(q+1)^4$. Thus $\pi \cap \pi(G) \subseteq \pi(q+1)$ and $H = S \leftthreetimes T$, where T is a Hall τ-subgroup of both the torus at hand and the group G, and $S \in Syl_3(G)$.

We proceed to the Hall subgroups of $E_6(q)$.

LEMMA 9. Let $G = E_6(q)$. Then $\theta = \theta(G) = \pi \setminus \rho$ and G contains a Hall π-subgroup if and only if π satisfies one of the following:

(1) $\pi \cap \pi(G) \subseteq \pi(q-1)$ and $|\pi \cap \rho| \leqslant 1$ (here if $\pi \cap \rho = \{s\}$ then the Hall π-subgroup of G is represented as $S \leftthreetimes T$, where $S \in Syl_s(G)$, T is a Hall θ-subgroup of some maximal torus of order $(q-1)^6$, and if $\pi \cap \rho = \varnothing$ then $H = T$);

(2) $\pi \cap \pi(G) \subseteq \pi(q+1)$ (here if $3 \in \pi$ then the Hall π-subgroup of G is represented as $S \leftthreetimes T$, where T is a Hall θ-subgroup of a maximal torus of order $(q-1)^2(q+1)^4$, $S \in Syl_3(G)$, and if $3 \notin \pi$ then $H = T$);

(3) $\pi \cap \pi(G)$ is a subset in one of the sets $\pi(q^6+q^3+1)$, $\pi(q^2+1)$, $\pi(q^2 \pm q+1)$, $\pi(q^4-q^2+1)$, $\pi(q^4+1)$, or $\pi(q^4+q^3+q^2+q+1)$ and $\pi \cap \rho = \varnothing$ (here, the Hall π-subgroup of G is Abelian and lies in some torus).

In all of the three cases above, every Hall π-subgroup of G satisfies the conclusion of Theorem 1.

Proof. It is not hard to verify that if π satisfies one of (1), (2) then G contains a Hall π-subgroup. We argue for the converse.

By Lemma 3, the factor $H/C_H(A)$ is isomorphic to a section of the Weyl group. Since $2 \notin \pi$, $|H/C_H(A)|$ can be divided only by 3 or 5. First assume that $\pi \cap \pi(G)$ contains at least one non-singular prime r. Then r is non-singular, the Sylow r-subgroup R is Abelian, and $R = A \leqslant Z(C)$. By [17, pp. 111-126], one of the following conditions holds:

(1) C is a maximal torus of order q^6+q^3+1, $(q^2+1)^2(q-1)^2$, $(q^2-q+1)^2(q^2+q+1)$, $(q^2+q+1)^3$, $(q^4-q^2+1)(q-1)(q+1)$, $(q-1)^6$, $(q^4+1)(q-1)(q+1)$, $(q^4-q^2+1)(q^2+q+1)$, or $(q+1)^4(q-1)^2$;

(2) C contains a subgroup $S{*}A_1(q)$ of index at most 2, where S is a torus of order $(q-1)(q^4+q^3+q^2+q+1)$.

Let $3 \in \pi$. Then H contains a Sylow 3-subgroup of G, in which case $q \equiv 1 \pmod 3$ or $q \equiv -1 \pmod 3$.

First let $q \equiv 1 \pmod 3$. The unique maximal torus C for which $N(C)$ contains a Sylow 3-subgroup is a Cartan subgroup. Thus $|C| = (q-1)^6$ holds. Then $H/C_H(R)$ is a subgroup of the Weyl group $W(G)$, and every prime $s \in \pi \cap \pi(G)$ divides $q-1$. If $3 \in \pi$, $5 \notin \pi$, and G contains a Hall π-subgroup then $\pi \cap \pi(G) \subseteq \pi(q-1)$. If $5 \in \pi$, then 5 divides $q-1$, since otherwise H would not contain a Sylow 5-subgroup of G. On the other hand, if $3, 5 \in \pi$ and $\pi \cap \pi(G) \subseteq \pi(q-1)$ then $H/C_H(R)$ is a Hall $\{3,5\}$-subgroup of $W(E_6)$, which is a contradiction with Lemma 4. Hence if $3, 5 \in \pi$ and $\pi \cap \pi(G) \subseteq \pi(q-1)$ then G is not an E_π-group.

Next let $q \equiv -1 \pmod 3$. Since H contains a Sylow 3-subgroup, the group $C_H(R)$ lies in a torus of order $(q-1)^2(q+1)^4$. And there exists a subgroup M of G which contains a subgroup $A_2(q^2) * {}^2A_2(q^2) * S$ of index 3, where S is a torus of order $(q-1)^2$. In addition, M contains a Sylow 3-subgroup of G and a maximal torus of order $(q-1)^2(q+1)^4$. If s is some non-singular prime in $\pi \cap \pi(G)$, then s divides $q+1$, since H is a Hall group. Lastly, if $5 \in \pi \cap \pi(G)$, then the Sylow 5-subgroup of G lies in H, in which case 5 divides $q+1$. Moreover, 5 is a large prime and C contains a Sylow 5-subgroup of G; therefore, G contains a Hall π-subgroup. Thus if $3 \in \pi$, $q \equiv -1 \pmod 3$, and G contains a Hall π-subgroup, then $\pi \cap \pi(G) \subseteq \pi(q+1)$.

At the moment assume that $3 \notin \pi$ but $5 \in \pi$. Then either $q \equiv 1 \pmod 5$, or $q \equiv -1 \pmod 5$, or $q^2 \equiv -1 (\mathrm{mod}\ 5)$. An argument similar to the above shows that G contains a Hall π-subgroup iff: $\pi \cap \pi(G) \subseteq (q-1)$ in the first case, $\pi \cap \pi(G) \subseteq (q+1)$ in the second case, and $\pi \cap \pi(G) \subseteq (q^2+1)$ in the third. Again we note that H satisfies the conclusion of Theorem 1 in all of the cases envisaged.

If $3, 5 \notin \pi$ then the Hall π-subgroup H is Abelian and lies either in one of the above-mentioned maximal tori or in a torus S of order $(q-1)(q^4+q^3+q^2+q+1)$, whence the lemma.

Suppose that $\pi \cap \pi(G)$ is freed of non-singular primes. Hence the Hall π-subgroup H is a $\{3,5\}$-subgroup. By Lemma 3, C contains a Sylow 5-subgroup of G. There are three cases to consider.

Case 1. Let $q \equiv 1 \pmod 5$. Then C contains a subgroup $C_0 = O^{p'}(C)$ which is isomorphic either to $A_5(q)$ or to $D_5(q)$. By Lemma 3, $H \cap C_0$ is a Hall $\{3,5\}$-subgroup of C_0. From [9, Thms. 4.1 and 4.4], it follows that C_0 does not contain a $\{3,5\}$-subgroup.

Case 2. Let $q \equiv -1 \pmod 5$. Then the Sylow 5-subgroup is Abelian and lies in a maximal torus C of order $(q+1)^4(q-1)^2$. Consequently, the Sylow 3-subgroup belongs to $N(C)$, and we are back in the case treated above.

Case 3. Let $q^2 \equiv -1 \pmod 5$. Then the Sylow 5-subgroup is again Abelian and lies in a maximal torus C of order $(q^2+1)^2(q-1)^2$. Consequently, the Sylow 3-subgroup belongs to $N(C)$, which is false.

We pass to the Hall π-subgroups of $E_7(q)$.

LEMMA 10. Let $G = E_7(q)$. Then $\theta = \theta(G) = \pi \setminus \rho$ and G contains a Hall π-subgroup if and only if π satisfies one of the following:

(1) $\pi \cap \pi(G) \subseteq \pi(q \pm 1)$ and $|\pi \cap \rho| \leqslant 1$ (here if $\pi \cap \rho = \{s\}$ then the Hall π-subgroup of G is represented as $S \leftthreetimes T$, where $S \in Syl_s(G)$, T is a Hall θ-subgroup of some maximal torus of order $(q \pm 1)^7$, and if $\pi \cap \rho = \varnothing$ then $H = T$);

(2) $\pi \cap \pi(G)$ is a subset in one of the sets $\pi(q^2+1)$, $\pi(q^2 \pm q+1)$, $\pi(q^4-q^2+1)$, $\pi(q^4+1)$, $\pi(q^4 \pm q^3 + q^2 \pm q + 1)$, $\pi(q^6 \pm q^3+1)$, or $\pi(q^6 \pm q^5 + q^4 \pm q^3 + q^2 \pm q + 1)$ and $\pi \cap \rho = \varnothing$ (here, the Hall π-subgroup of G is Abelian and lies in some torus).

In either case, the Hall π-subgroup of G satisfies the conclusion of Theorem 1.

Proof. Assume that π satisfies one of (1), (2). Using Lemmas 5 and 6, it is not hard to show that G contains a Hall π-subgroup. Therefore it is sufficient to prove the converse statement.

By Lemma 3, the group $H/C_H(A)$ is isomorphic to a section of the Weyl group; consequently, $|H/C_H(A)|$ is divisible only by 3, 5, or 7. First assume that $\pi \cap \pi(G)$ contains at least one non-singular prime. Then r is non-singular, $R = A$, and $Z(C)$ contains a Sylow r-subgroup of G. By [15; 19, Table 1], for C, one of the following cases is realized:

(1) C is a maximal torus of order $(q-1)^7$, $(q+1)^7$, $(q+1)(q^6-q^3+1)$, $(q+1)(q^6-q^5+q^4-q^3+q^2-q+1)$, $(q-1)(q^6+q^3+1)$, $(q-1)(q^6+q^5+q^4+q^3+q^2+q+1)$, $(q-1)(q^2+q+1)^3$, $(q+1)(q^2-q+1)^3$, $(q+1)(q^2-q+1)(q^4-q^2+1)$, $(q-1)(q^2+q+1)(q^4-q^2+1)$, or $(q+1)(q^2-q+1)(q^4-q^3+q^2-q+1)$;

(2) C contains a subgroup $A_1(q) * S$ of index at most 2, where S is a torus of order $(q^2 \pm 1)(q^4+1)$ or $(q+1)^2(q^4-q^3+q^2-q+1)$;

(3) C contains a subgroup $A_2(q) * S$ whose index is equal to 1 or to 3, where S is a torus of order $(q-1)(q^4+q^3+q^2+q+1)$;

(4) C contains a subgroup $A_1(q) * A_1(q) * S$ whose index is even and does not exceed 4, where S is a torus of order $(q-1)(q^2+1)^2$.

We treat all of these four cases singly. If the set $\pi \cap \pi(G)$ does not contain any singular prime then H is Abelian and $\pi \cap \pi(G) \subseteq \pi(T)$ for some torus T of G.

Suppose $3 \in \pi$. Then either $q \equiv 1 \pmod 3$ or $q \equiv -1 \pmod 3$. First let $q \equiv 1 \pmod 3$. Then C is a Cartan subgroup, and every s, $s \in \pi \cap \pi(G)$, divides $q-1$. $H/C_H(R)$ is a Hall π-subgroup in $W(G)$; consequently, $5, 7 \notin \pi$ by Lemma 4. Thus $\pi \cap \pi(G) \subseteq \pi(q-1)$, and the numbers 5 and 7 do not belong to π.

Next let $q \equiv -1 \pmod 3$. Then C is a maximal torus of order $(q+1)^7$, and every s such that $s \in \pi \cap \pi(G)$ divides $q+1$. By Lemma 6, the group $N(C)$ contains a Sylow 3-subgroup of G of order $3^4 \cdot (q+1)_3^7$. Hence the order of a Sylow 3-subgroup in the factor $N(C)/C$ is equal to 3^4. Since $N(C)/C$ is isomorphically embedded in the Weyl group and the order of a Sylow 3-subgroup in $W(E_7)$ is 3^4, the Sylow 3-subgroup of $N(C)/C$ coincides with one in $W(E_7)$. If 5 or 7 belongs to π then $N(C)$ contains a Sylow 5- or 7-subgroup of G. The group $H/C_H(R)$ is a Hall π-subgroup in $N(C)/C$. As in the third case, the Sylow 5- and 7-subgroups of $N(C)/C$ coincide with those in $W(E_7)$. Then $N(C)/C$ and hence $W(E_7)$ will possess a Hall π-subgroup for the set π such that $|\pi \cap \pi(W(E_7))| \geqslant 2$, a contradiction with Lemma 4. Thus $\pi \cap \pi(G) \subseteq (q+1)$ and $5, 7 \notin \pi$.

Now let $5 \in \pi$ but $3 \notin \pi$. Then one of the following cases holds: $q \equiv 1 \pmod 5$, $q \equiv -1 \pmod 5$, or

$q^2 \equiv -1 \pmod 5$. The first two cases are treated in the same way as the above case where $3 \in \pi$. Let $q^2 \equiv -1 \pmod 5$. Then the Sylow 5-subgroup of G is Abelian and $C_H(A)$ lies in the group $A_1(q) * A_1(q) * S$, where S is a torus of order $(q-1)(q^2+1)^2$. Consequently, $\pi \cap \pi(G) \subseteq \pi(q^2+1)$.

Lastly let $3, 5 \notin \pi$ but $7 \in \pi$. Then one of the following congruences is effected: $q \equiv 1 \pmod 7$, $q \equiv -1 \pmod 7$, $q^2 - q \equiv -1 \pmod 7$, or $q^2 + q \equiv -1 \pmod 7$. All of the three cases can be treated similarly to the previous two — where 3 or 5 belongs to π.

At the moment assume that $\pi \cap \pi(G)$ is freed of non-singular primes. Then one of the sets $\{3, 5\}$, $\{3, 7\}$, or $\{5, 7\}$ is contained in π.

Suppose $3, 5 \in \pi$. If $q^2 \equiv -1 \pmod 5$ then the Sylow 5-subgroup of G is Abelian, and we are back in the case treated above. Assume $q \equiv 1 \pmod 5$. By Lemma 3, C contains a Sylow 5-subgroup of G. Therefore C contains a subgroup $C_0 = O^{p'}(C)$, which is isomorphic either to $A_{7-k}(q)$, where $k \leqslant 3$, or to D_{7-k}, where $k \leqslant 2$. By Lemma 3, $H \cap C_0$ is a Hall π-subgroup in the group C_0. Theorem 4.9 in [9] implies that C_0 should be an $E_{\{3,5\}}$-group. Theorems 4.1 and 4.4 in [9] hold that C_0 cannot be an $E_{\{3,5\}}$-group in any of the above-envisaged cases. Consequently, G is not an E_π-group.

Let $q \equiv -1 \pmod 5$. Again, by Lemma 3, C contains a Sylow 5-subgroup of G, but Table 1 in [19] shows that such a subgroup C does not exist. Hence in this case, too, G does not enjoy property E_π.

The remaining two cases can be treated similarly.

We embark on the Hall π-subgroups in $E_8(q)$.

LEMMA 11. Let $G = E_8(q)$. Then $\theta = \theta(G) = \pi \setminus \rho$ and G contains a Hall π-subgroup if and only if π satisfies one of the following:

(1) $\pi \cap \pi(G) \subseteq \pi(q \pm 1)$ and $|\pi \cap \rho| \leqslant 1$ (here if $\pi \cap \rho = \{s\}$ then the Hall π-subgroup of G is represented as $S \leftthreetimes T$, where $S \in Syl_s(G)$, T is a Hall θ-subgroup of some maximal torus of order $(q \pm 1)^8$, and if $\pi \cap \rho = \varnothing$ then $H = T$);

(2) $\pi \cap \pi(G)$ is a subset of one of the sets $\pi(q^2+1)$, $\pi(q^2 \pm q+1)$, $\pi(q^4 - q^2+1)$, $\pi(q^4+1)$, $\pi(q^4 \pm q^3 + q^2 \pm q+1)$, $\pi(q^6 \pm q^3+1)$, $\pi(q^6 \pm q^5 + q^4 \pm q^3 + q^2 \pm q + 1$, $\pi(q^8 - q^4+1)$, $\pi(q^8 - q^6 + q^4 - q^2+1)$, or $\pi(q^6 \pm q^7 \mp q^5 - q^4 \pm q^3 \mp q+1)$ and $\pi \cap \rho = \varnothing$ (here, the Hall π-subgroup of G is Abelian and lies in some torus).

In both cases, the Hall π-subgroup of G satisfies the conclusion of Theorem 1.

Proof. If π satisfies one of (1), (2) then it is not hard to see that G contains a Hall π-subgroup, and so we need only prove the converse statement.

First assume that $\pi \cap \pi(G)$ contains at least one non-singular prime. Then r is non-singular and $R = A \leq Z(C)$. In view of [15; 19, Table 1], one of the following statements holds:

(1) C is a maximal torus of order $(q-1)^8$, $(q+1)^8$, $(q^2+1)^4$, $(q^2+q+1)^4$, $(q^2-q+1)^4$, $(q^4+1)^2$, $(q^4-q^2+1)^2$, $(q^4+q^3+q^2+q+1)^2$, $(q^4-q^3+q^2-q+1)^2$, q^8-q^4+1, $q^8-q^6+q^4-q^2+1$, $q^8-q^7+q^5-q^4+q^3-q+1$, $q^8+q^7-q^5-q^4-q^3+q+1$, $(q-1)(q+1)(q^6-q^3+1)$, $(q^2-q+1)(q^6-q^3+1)$, or $(q-1)(q+1)(q^6-q^5+q^4-q^3+q^2-q+1)$.

(2) C contains a subgroup $A_1(q) * S$ of index at most 2, where S is a torus of order $(q-1)(q^6+q^3+1)$ or $(q-1)(q^6+q^5+q^4+q^3+q^2+q+1)$.

If the set $\pi \cap \pi(G)$ is freed of singular primes then the Hall π-subgroup H is Abelian and lies in one of the above-mentioned tori.

Next suppose $\pi \cap \omega(G) \neq \varnothing$. All instances admit a uniform treatment, and so we dwell on just the case where $3 \in \pi$ and $q \equiv -1 \pmod 3$. Here, C is a maximal torus of order $(q+1)^8$; consequently, $\pi \cap \pi(G) \subseteq \pi(q+1)$. If 5 or 7 belongs to π then $N(C)$ will contain a Sylow 5- or 7-subgroup of G by Lemma 6. Therefore a Sylow 5- or 7-subgroup of $N(C)/C$ coincides with one in $W(E_8)$. Which is also true for a Sylow 3-subgroup of $N(C)/C$. $H/C_H(R)$ is a Hall π-subgroup of $N(C)/C$, and hence of $W(E_8)$. The latter enjoys property E_π iff $|\pi \cap \pi(W(E_8))| \leqslant 1$, and so G is not an E_π-group.

The case where $\pi \cap \pi(G)$ is freed of non-singular primes can be treated in the same way as was the relevant case for $E_7(q)$.

We pass to the Hall π-subgroups in ${}^3D_4(q^3)$.

LEMMA 12. Let $G = {}^3D_4(q^3)$. Then $\theta = \theta(G) = \pi \setminus \{3\}$ and G contains a Hall π-subgroup if and only if π satisfies one of the following conditions:

(1) $\pi \cap \pi(G) \subseteq \pi(q \pm 1)$; here if $3 \in \pi$ then the Hall π-subgroup of G is represented as $S \leftthreetimes T$, where T is a Hall θ-subgroup of some maximal torus of order $(q \pm 1)^2(q^2 \mp q + 1)$, $S \in Syl_3(G)$, and if $3 \notin \pi$ then $H = T$;

(2) $\pi \cap \pi(G)$ is a subset of one of the sets $\pi(q^2 \pm q + 1)$ or $\pi(q^4 - q^2 + 1)$ and $3 \notin \pi$; here, the Hall π-subgroup of G is Abelian and lies in some torus.

In both cases, the Hall π-subgroup of G satisfies the conclusion of Theorem 1.

Proof. Let $\pi \cap \pi(G) \subseteq \pi(q - 1)$. Since N contains a Sylow 3-subgroup of G, G enjoys property E_π by Lemma 5 in this instance.

Let $\pi \cap \pi(G) \subseteq \pi(q+1)$. There exists a subgroup M of G which contains a subgroup of index 3 equal to $S * {}^2A_2(q^2)$, where S is a torus of order $q^2 - q + 1$. Let C be a maximal torus of order $(q + 1)^2(q^2 - q + 1)$. It is clear that $C \leq M$. Moreover, the Sylow 3-subgroup of $O^{p'}(M) = {}^2A_2(q^2)$ lies in $N(C)$ by Lemma 6. Therefore the Sylow 3-subgroup of M also lies in $N(C)$. On the other hand, the Sylow 3-subgroup of M coincides with one in G, and $N(C)$ then contains a Sylow 3-subgroup of G. Thus $N(C)$ contains a Hall π-subgroup of G. It is not hard to verify that if π satisfies one of (1), (2) then G also contains a Hall π-subgroup. We argue for the converse.

Since 3 is the unique singular prime for G, r is a non-singular prime. Consequently, $R = A \leq Z(C)$. Table 7 in [17, p. 140] shows that C coincides with a maximal torus of order $(q-1)^2(q^2+q+1)$, $(q+1)^2(q^2-q+1)$, $(q^2-q+1)^2$, $(q^2+q+1)^2$, or q^4-q^2+1. If $3 \notin \pi$ then the Hall π-subgroup is Abelian and lies in some maximal torus, mentioned above. If $3 \in \pi$ then either $q \equiv 1 \,(\mathrm{mod}\, 3)$ or $q \equiv -1 \,(\mathrm{mod}\, 3)$. Let $q \equiv 1 \,(\mathrm{mod}\, 3)$. Then C is a Cartan subgroup, its order is equal to $(q - 1)^2(q^2 + q + 1)$, and $\pi \cap \pi(G) \subseteq \pi(q - 1)$. Let $q \equiv -1 \,(\mathrm{mod}\, 3)$. Then C is a maximal torus of order $(q + 1)^2(q^2 - q + 1)$ and $\pi \cap \pi(G) \subseteq \pi(q + 1)$.

We continue to treat the Hall π-subgroups of ${}^2E_6(q^2)$.

LEMMA 13. Let $G = {}^2E_6(q^2)$. Then $\theta = \theta(G) = \pi \setminus \rho$ and G contains a Hall π-subgroup if and only if π satisfies one of the following:

(1) $\pi \cap \pi(G) \subseteq \pi(q-1)$; here if $3 \in \pi$ then the Hall π-subgroup of G is represented as $S \leftthreetimes T$, where T is a Hall θ-subgroup of some maximal torus of order $(q - 1)^4(q + 1)^2$, $S \in Syl_3(G)$, and if $3 \notin \pi$ then $H = T$;

(2) $\pi \cap \pi(G) \subseteq \pi(q+1)$ and $|\rho \cap \pi| \leqslant 1$; here if $r \in \pi \cap \rho$ then the Hall π-subgroup of G is represented as $S \leftthreetimes T$, where T is a Hall θ-subgroup of some maximal torus of order $(q+1)^6$, $S \in Syl_r(G)$, and if $\pi \cap \rho = \varnothing$ then $H = T$;

(3) $\pi \cap \pi(G)$ is a subset in one of the sets $\pi(q^2+1)$, $\pi(q^2 \pm q+1)$, $\pi(q^4-q^2+1)$, $\pi(q^4+1)$, $\pi(q^4-q^3+q^2-q+1)$, or $\pi(q^6 - q^3 + 1)$ and $\pi \cap \rho = \varnothing$; here, the Hall π-subgroup of G is Abelian and lies in some torus.

The statement of Theorem 1 is satisfied for the Hall π-subgroup of G in all of these three cases.

Proof. As before, if π satisfies one of (1)-(3) then G contains a Hall π-subgroup. We argue for the converse.

Assume, first, that π contains at least one non-singular prime. Then r is non-singular and $R = A \leq Z(C)$. The group C coincides with a maximal torus of order $(q+1)^6$, $(q-1)^4(q+1)^2$, $(q^2+q+1)^2(q^2-q+1)$,

$(q+1)^2(q^2+1)^2$, $(q^2-q+1)^3$, $(q-1)(q+1)(q^4+1)$, $(q-1)(q+1)(q^4-q^3+q^2-q+1)$, $(q^2-q+1)(q^4-q^2+1)$, or q^6-q^3+1.

If $3, 5 \notin \pi$ then the Hall π-subgroup is Abelian and belongs to one of the above-mentioned tori.

Suppose $3 \in \pi$. Let $q \equiv 1 \pmod 3$. Then C is a Cartan subgroup, its order is equal to $(q-1)^4(q+1)^2$, and hence $\pi \cap \pi(G) \subseteq \pi(q-1)$. Let $q \equiv -1 \pmod 3$. Then C is a maximal torus of order $(q+1)^6$ and $\pi \cap \pi(G) \subseteq \pi(q+1)$. By Lemma 6, $N(C)$ contains a Sylow s-subgroup of G for any $s \in \pi(q+1)$. Since the Sylow 3- and 5-subgroups of $N(C)/C$ coincide with those in $W(E_6)$, and G contains a Hall π-subgroup, we have $\pi \cap \rho = \{3\}$. The case where $5 \in \pi$ can be treated similarly.

Now let $\pi \cap \pi(G) = \{3, 5\}$. Then C contains a Sylow 5-subgroup of G. If $q \equiv 1 \pmod 5$ or $q^2 \equiv -1 \pmod 5$ then the Sylow 5-subgroup of G is Abelian and lies in a torus of order $(q-1)^4(q+1)^2$ or $(q+1)^2(q^2+1)^2$, respectively. In the former case $q-1$ is divided by 3 and the group $N(C)$ does not contain a Sylow 3-subgroup of G in the latter case. Therefore G is not an E_π-group.

If $q \equiv -1 \pmod 5$ then the group $O^{p'}(C) = C_0$ coincides with one of the following: ${}^2A_4(q^2)$, ${}^2A_4(q^2) * A_1(q)$, ${}^2A_5(q^2)$, ${}^2D_5(q^2)$, or ${}^2A_5(q^2) * A_1(q)$. By Lemma 3, $C_0 \cap H$ is a Hall $\{3, 5\}$-subgroup of C_0. From [9, Thms. 4.2 and 4.3], it follows that C_0 is not an $E_{\{3,5\}}$-group. Consequently, G is not an E_π-group in the present case, too.

Lemmas 7-13 can be combined to yield the following:

THEOREM 3. Let G be a finite exceptional Lie-type group with a definition field $GF(q)$ of characteristic p which is not a Suzuki or Ree group. Let π be a set of primes such that $2, p \notin \pi$ and $|\pi \cap \pi(G)| \geqslant 2$, ρ be a set of small primes for G, r be the least prime in $\pi \cap \pi(G)$, and $\tau = \pi \setminus \{r\}$. Then G has property E_π if and only if, for some cyclotomic polynomial $\Phi_i(t)$ involved in the decomposition of a p'-part of the order of G, π satisfies one of the following conditions:

(1) $\pi \cap \rho = \varnothing$ and $\pi \cap \pi(G) \subseteq \pi(\Phi_i(q))$; here, the Hall π-subgroup is Abelian;

(2) $\pi \cap \rho = \{r\}$, $\pi \cap \pi(G) \subseteq \pi(\Phi_i(q))$, and $i = e(r, q)$; here there exists a maximal torus H such that H contains a Hall τ-subgroup of G and $N(H)$ contains a Hall π-subgroup of G.

We end to treat the Hall π-subgroups of Suzuki and Ree groups.

LEMMA 14. Let G be a Suzuki or Ree group, p be the characteristic of a definition field for G, π be a set of primes such that $|\pi \cap \pi(G)| \geqslant 2$, $2, p \notin \pi$, and $\theta = \theta(G) = \pi \setminus \{3\}$. Then G possesses property E_π if and only if one of the following conditions is satisfied:

(1) $G = {}^2B_2(q)$ and $\pi \cap \pi(G)$ is contained in one of the sets $\pi(q-1)$ or $\pi(q \pm \sqrt{2q} + 1)$; here, the Hall π-subgroup of G is Abelian and is contained in some maximal torus of order $q-1$ or $q \pm \sqrt{2q} + 1$, respectively;

(2) $G = {}^2G_2(q)$ and $\pi \cap \pi(G)$ is contained in one of the sets $\pi(q \pm 1)$ or $\pi(q \pm \sqrt{3q} + 1)$; here, the Hall π-subgroup of G is Abelian and is contained in some maximal torus of order $q \pm 1$ or $q \pm \sqrt{3q} + 1$, respectively;

(3) $G = {}^2F_4(q)$ and $\pi \cap \pi(G)$ is contained in one of the sets $\pi(q^2-1)$, $\pi(q \pm \sqrt{2q} + 1)$, $\pi(q^2+1)$, $\pi(q^2 \pm q\sqrt{2q} \mp \sqrt{2q} - 1)$, or $\pi(q^2 \pm q\sqrt{2q} + q \pm \sqrt{2q} + 1)$; here if $3 \in \pi$ then $H = S \leftthreetimes T$, where T is an Abelian Hall θ-subgroup of ${}^2F_4(q)$, $S \in Syl_3({}^2F_4(q))$, and if $3 \notin \pi$ then $H = T$.

The statement of Theorem 1 is satisfied for the Hall π-subgroup of G in all of these three cases.

Proof. Groups ${}^2B_2(q)$ and ${}^2G_2(q)$ were treated in [20] and [21], respectively, and so we refrain from comments in presenting the structure of Hall π-subgroups in these. The Hall π-subgroups of odd order in Tits groups were studied in [5], and so for groups ${}^2F_4(q)$, we put $q > 2$.

If $3 \notin \pi$ then the Hall π-subgroup of ${}^2F_4(q)$ is Abelian and lies in some maximal torus. If $3 \in \pi$ then C

is a maximal torus of ${}^2F_4(q)$ of order $(q^2-1)^2$. By Lemma 2, $N(C)$ contains a Sylow 3-subgroup of ${}^2F_4(q)$. Hence $\pi\cap\pi({}^2F_4(q))\subseteq\pi(q^2-1)$, and ${}^2F_4(q)$ then satisfies the conclusion of Theorem 1.

4. HALL SUBGROUPS OF ODD ORDER IN FINITE GROUPS OF LIE TYPE

In this section we prove the theorem which combines Theorem 3 and the results of [9].

THEOREM 4. Let G be a finite Lie-type group with a definition field $GF(q)$ of characteristic p which is not a Suzuki or Ree group. Let π be a set of primes such that $2,p\notin\pi$ and $|\pi\cap\pi(G)|\geqslant 2$, ρ a set of small primes, r the least prime in $\pi\cap\pi(G)$, and $\tau=\pi\setminus\{r\}$. Then G possesses property E_π if and only if, for some cyclotomic polynomial $\Phi_i(t)$ involved in the decomposition of a p'-part of the order of G, π satisfies one of the following:

(1) $\pi\cap\rho=\varnothing$ and $\pi\cap\pi(G)\subseteq\pi(\Phi_i(q))$; here, the Hall π-subgroup is Abelian and lies in some torus;

(2) $\pi\cap\rho=\{r\}$, $\pi\cap\pi(G)\subseteq\pi(\Phi_i(q))$, and $i=e(r,q)$; here there exists a maximal torus H such that H contains a Hall τ-subgroup of G and $N(H)$ contains a Hall π-subgroup of G;

(3) $\pi\cap\rho=\varnothing$, $\tau\cap\pi(G)\subseteq\pi(\Phi_i(q))$, and $|G|_r=|W|_r=|N(T)/T|_r$, where $T=Z(C_G(A))$ is a maximal torus and A a Hall τ-subgroup of G; here, either the Hall π-subgroup of G is Abelian, or G is not a D_π-group.

Proof. If G is an exceptional group of Lie type then the conclusion of the theorem follows from Theorem 3, proved above.

Let G be a classical finite group. We treat all possible cases for G. By [9, Thm. 4.9], G is not an E_π-group iff G contains a Hall $\{t,s\}$-subgroup for all $t,s\in\pi$. If $2\notin\pi$ then property E_π is inherited under extensions, and so we may assume that G coincides with $GL_n(q)$, $Sp_n(q)$, $GU_n(q)$, or $GO_n^\varepsilon(q)$, where $\varepsilon=\pm$. We assume hereafter that $t,s\in\pi\cap\pi(G)$, $a=e(q,t)$, $b=e(q,s)$, and $t<s$.

Let $G=GL_n(q)$. In view of [9, Thm. 4.1], G is an $E_{\{t,s\}}$-group iff $n<bs$, and one of the following cases is realized:

(1) $a=b$;

(2) $a=t-1$, $b=t$, $(q^{t-1}-1)_t=t$, and $\left[\frac{n}{t-1}\right]=\left[\frac{n}{t}\right]$;

(3) $a=t-1$, $b=t$, $(q^{t-1}-1)_t=t$, $\left[\frac{n}{t-1}\right]=\left[\frac{n}{t}\right]+1$, and $n\equiv t-1\,(\mathrm{mod}\,t)$;

(4) $a=t-1$, $b=1$, $(q^{t-1}-1)_t=t$, and $\left[\frac{n}{t-1}\right]=\left[\frac{n}{t}\right]$.

The condition that $n<bs$ means that s is a large prime for G. Thus $|\pi\cap\rho|\leqslant 1$ and $\tau\cap\rho=\varnothing$. Further, if $t,s\in\tau$ then $a=b$. Consequently, $\tau\cap\pi(G)\subseteq\Phi_b(q)$. We treat the four cases singly.

Let $a=b$. If the Sylow r-subgroup of G is Abelian then the Hall π-subgroup of G lies in some torus whose order is divided by $\Phi_a(q)$, so $\pi\cap\rho=\varnothing$, and hence condition (1) of the theorem is met. Assume that the Sylow r-subgroup of G is non-Abelian. Consider a subgroup $G_1=GL_{\left[\frac{n}{a}\right]}(q^a)$. For any $t\in\pi\cap\pi(G)$, G_1 contains a Sylow t-subgroup of G. Indeed, $e(q,t)=a$, and hence

$$|G|_t=\prod_{m<n,\ a|m}(q^a-1)_t\left(\frac{m}{a}\right)_t=\prod_{1\leqslant k\leqslant\left[\frac{n}{a}\right]}(q^a-1)_t\,(ka)_t=|G_1|_t\,.$$

Thus $\pi(|G:G_1|)\cap\pi=\varnothing$. Consequently, the Hall π-subgroup of G_1 coincides with one in G. Let H and N be, respectively, a subgroup of diagonal matrices and a subgroup of monomial matrices of G_1. Then N contains a Hall π-subgroup of the groups G_1 and G. Therefore the maximal torus T of G such that

$H \leq T$ and $N \leq N(T)$ contains a Hall τ-subgroup of G and $N(T)$ contains a Hall π-subgroup of G. Thus G satisfies condition (2) of the theorem.

Theorem 4.1 in [9] implies that (2)-(4) can be realized only if $t = r$. Assume that $a = r-1$, $b = r$, $(q^{r-1}-1)_r = r$, and $\left[\frac{n}{r-1}\right] = \left[\frac{n}{r}\right]$, or $\left[\frac{n}{r-1}\right] = \left[\frac{n}{r}\right]+1$ and $n \equiv r-1 \pmod{r}$. Then the condition that $ar > n$ holds in view of [9, Cor. 4.2]; so, $r \notin \rho$. Thus $\pi \cap \rho = \varnothing$. Consider a subgroup $G_1 = GL_{\left[\frac{n}{r}\right]}(q^r) \times GL_{r-1}(q)$ of G. We know that $n = (\left[\frac{n}{r}\right]+1)(r-1) + \left[\frac{n}{r}\right]$ (cf. proof of Thm. 4.1 in [9]). Consequently, the desired group G_1 does indeed exist in G. Clearly, G_1 contains a Hall τ-subgroup of G. Since $ar > n$ and $(q^{r-1}-1)_r = r$, we have $|G|_r = r$. Obviously, $|G_1|_r = r$. Let H be a subgroup of diagonal matrices in the group $GL_{\left[\frac{n}{r}\right]}(q^r)$. Then H contains a Hall τ-subgroup of G. Moreover, there exists a maximal torus T of G such that $H \leq T$, and $N(T)$ contains a Hall π-subgroup of G. Thus G satisfies condition (3) of the theorem. Note also that the Hall π-subgroup of G is Abelian in this instance.

Finally, let $a = r-1$, $b = 1$, $(q^{r-1}-1)_r = r$, and $\left[\frac{n}{r-1}\right] = \left[\frac{n}{r}\right]$. We comment on the structure of a Hall π-subgroup.

For some natural m and b, $n = mr+b = m(r-1)+(b+m)$. Since $b+m < r-1$, $m < r-1$. Furthermore, the Hall τ-subgroup of G lies in the subgroup H of diagonal matrices of G, and $|G|_r = |\mathrm{Sym}_n|_r$, where Sym_n is a symmetric group of degree n, which coincides with the Weyl group for G. Consequently, the subgroup

$$G_1 = \underbrace{GL_1(q) \times \ldots \times GL_1(q)}_{b \text{ times}} \times \underbrace{GL_r(q) \times \ldots \times GL_r(q)}_{m \text{ times}}$$

of G contains a Hall π-subgroup of G. Moreover, the Hall τ-subgroup of G_1 lies in a product of the diagonal matrices of direct factors of G_1 and the Hall π-subgroup lies in a product of the monomial subgroups of direct factors of G_1.

Let $s \in \tau$ and $(q-1)_s = s^\alpha \neq 1$. Denote the group $\mathbb{Z}_{s^\alpha}$ by S. Clearly, the Hall π-subgroup of G is a direct product of Hall π-subgroups in the groups $GL_1(q) = GF(q)^*$ and $GL_r(q)$. The structure of a Hall π-subgroup in $GL_1(q)$ is obvious, and so we need only show how such is structured in $GL_r(q)$. Its Hall τ-subgroup T lies in a group of diagonal matrices, and the Sylow r-subgroup R acts on T by permuting diagonal elements of the matrices. Therefore the Hall π-subgroup H of $GL_r(q)$ has the form $R \leftthreetimes T$, and if $h \in H$ is some r-element of H then the group $C_H(h)$ consists of scalar matrices. On the other hand, $C_{GL_r(q)}(h) \geq \mathbb{Z}_r \times \mathbb{Z}_{q-1} \times \mathbb{Z}_{q-1} \geq \mathbb{Z}_r \times S \times S$. By reason of this fact, the Hall π-subgroup of G_1 (and consequently of G) contains no subgroup isomorphic to the group

$$G_2 = \underbrace{S \times \ldots \times S}_{b+m(r-1)+2 \text{ times}} \times \mathbb{Z}_r.$$

Alternatively, G_1 does contain a subgroup isomorphic to G_2. Therefore G is not a D_π-group. And G then satisfies condition (3) of the theorem.

Let $G = GU_n(q^2)$. In view of [9, Thm. 4.3], G is an $E_{\{t,s\}}$-group iff one of the following cases is realized:

(1) $a = b \equiv 0 \pmod{4}$ and $n < bs$;

(2) $a = b \equiv 2 \pmod{4}$ and $2n < bs$;

(3) $a = b \equiv 0 \pmod{4}$ and $n < bs$;

(4) $t \equiv 1 (\mathrm{mod}\ 4)$, $a = t-1$, $b = 2t$, $(q^n-1)_t = t$, and $\left[\frac{n}{t-1}\right] = \left[\frac{n}{t}\right]$;

(5) $t \equiv 3 \pmod{4}$, $a = \frac{t-1}{2}$, $b = 2t$, $(q^n-1)_t = t$, and $\left[\frac{n}{t-1}\right] = \left[\frac{n}{t}\right]$;

(6) $t \equiv 1 \pmod{4}$, $a = t-1$, $b = 2t$, $(q^n-1)_t = t$, and $\left[\frac{n}{t-1}\right] = \left[\frac{n}{t}\right] + 1$ and $n \equiv t-1 \pmod{t}$;

(7) $t \equiv 3 \pmod 4$, $a = \frac{t-1}{2}$, $b = 2t$, $(q^n-1)_t = t$, and $\left[\frac{n}{t-1}\right] = \left[\frac{n}{t}\right] + 1$ and $n \equiv t-1 \pmod t$;

(8) $t \equiv 1 \pmod 4$, $a = t-1$, $b = 2$, $(q^n-1)_t = t$, $n < 2s$, and $\left[\frac{n}{t-1}\right] = \left[\frac{n}{t}\right]$;

(9) $t \equiv 3 \pmod 4$, $a = \frac{t-1}{2}$, $b = 2$, $(q^n-1)_t = t$, $n < 2s$, and $\left[\frac{n}{t-1}\right] = \left[\frac{n}{t}\right]$.

It is not hard to verify that if $t, s \in \tau$ then $a = b$, that is, $\tau \cap \pi(G) \subseteq \pi(\Phi_a(q))$. Furthermore, in items (1)-(3), (8), and (9) s is a large prime for G. As indicated below, both of the numbers t and s are large in (4) and (7). Consequently, $\tau \cap \rho = \varnothing$. Now we consider all of the nine cases singly.

Let $a = b \equiv 0 \pmod 4$ and $n < bs$. If the Sylow r-subgroup of G is Abelian then the whole π-subgroup is contained in some maximal torus, and so G satisfies condition (1) of the theorem. Suppose that G contains a non-Abelian Sylow r-subgroup. Then the subgroup $G_1 = GL_{\left[\frac{n}{a}\right]}(q^a)$ of G contains a Hall π-subgroup of G. In this case the Hall π-subgroup of G_1 is contained in its monomial matrix group. Consequently, there exists a maximal torus T such that $N(T)$ contains a Hall π-subgroup of G. Hence G satisfies condition (2) of the theorem. Cases (2) and (3) can be treated similarly.

If $a \neq b$ then $t = r$. Assume that one of (4)-(7) holds. Then $ar > n$, which was stated in the proof of Theorem 4.3 in [9]. Consequently, $|G|_r = r$ and G contains an Abelian Sylow r-subgroup. In addition, $sb > sr > n$, and so s is a large prime. Hence the Hall π-subgroup of G is Abelian, and so G satisfies condition (3) of the theorem.

Cases (8) and (9) can be treated in the same way as was (4) for $GL_n(q)$. In each of these cases, G satisfies condition (3) of the theorem, and it is not a D_π-group in this instance.

At the moment assume that $G = GO_n^\varepsilon(q)$, where $\varepsilon = \pm$. In view of [9, Thm. 4.4], G contains a Hall $\{t, s\}$-subgroup iff one of the following cases holds:

(1) $\varepsilon = +$, $a = b \equiv 0 \pmod 2$, and $n \leqslant bs$;

(2) $\varepsilon = +$, $a = b \equiv 1 \pmod 2$, and $n \leqslant 2bs$;

(3) $\varepsilon = -$, $a = b \equiv 0 \pmod 2$, and $n \leqslant bs$;

(4) $\varepsilon = -$, $a = b \equiv 1 \pmod 2$, and $n \leqslant 2bs$;

(5) $\varepsilon = -$, $a \equiv 1 \pmod 2$, $b = 2a$, and $n = 4a$;

(6) $\varepsilon = -$, $b \equiv 1 \pmod 2$, $a = 2b$, and $n = 4b$.

As we did for the groups $GL_n(q)$ and $GU_n(q^2)$, we can prove that if (1)-(4) hold then G satisfies conditions (1) or (2) of the theorem. In items (5) and (6), the Hall π-subgroup of G is cyclic, which was mentioned in the proof of Theorem 4.4 in [9]. Consequently, G satisfies condition (3) of the theorem, and its Hall π-subgroup is Abelian.

Finally assume that $G = Sp_{2n}(q)$. G contains a Hall $\{t, s\}$-subgroup iff one of the following cases is realized (cf. [9, Thm. 4.5]):

(1) $a = b \equiv 0 \pmod 2$ and $2n < bs$;

(2) $a = b \equiv 1 \pmod 2$ and $n < bs$.

As above, it is not hard to verify that if either one of (1), (2) is realized then G satisfies conditions (1) or (2) of the theorem.

5. INHERITING D_π IN FINITE GROUPS

We prove that, for the case where $2 \notin \pi$, an extension of a D_π-group by a D_π-group is again a D_π-group. We say that G satisfies condition $(*)$ if it satisfies conditions (1) and (2) of Theorem 4 or one of (1)-(3) of Lemma 14 (with the condition that $|\pi \cap \pi(G)| \geqslant 2$ dropped).

From Theorems 1 and 4 and Lemma 14, it follows that if a finite Lie-type group G satisfies $(*)$ then its Hall π-subgroup M is representable as $M = R \leftthreetimes T$, where R is a Sylow r-subgroup of G, r is a minimal number in $\pi \cap \pi(G)$, $\tau = \pi \setminus \{r\}$, and T is a Hall τ-subgroup of G. Furthermore, there exists a maximal torus H such that $T \leq H$ and $M \leq N(H)$.

Below we also need an additional definition bearing on algebraic groups. Let $G = O^{p'}(\overline{G}_\sigma)$ be a finite group of Lie type and $\overline{G}$ its corresponding simple algebraic group. Let $\overline{T}$ be some maximal torus of $\overline{G}$ and $\overline{D}$ a $\overline{T}$-invariant connected semisimple subgroup of $\overline{G}$ such that $\overline{D} \cap \overline{T}$ is a maximal torus in $\overline{D}$. Then $\overline{D}$ is called a *subsystem* subgroup of $\overline{G}$. Obviously, $\overline{T} \cdot \overline{D}$ is a connected reductive subgroup of maximal rank in $\overline{G}$. If $\overline{R} = \overline{T} \cdot \overline{D}$ is a σ-invariant group then $\overline{D} = \left[\overline{R}, \overline{R}\right]$ is also one. The *subsystem* subgroup of G is $D = \overline{D}_\sigma \cap G = \overline{D} \cap G$. Clearly, every non-trivial subsystem subgroup of G either coincides with G or has the form $G_1 * \ldots * G_k$, where all G_i are Lie-type groups of lesser rank than is G, but possibly with a larger definition field.

LEMMA 15. Let $G = O^{p'}(\overline{G}_\sigma)$ be a finite group of Lie type satisfying condition $(*)$, R be a σ-invariant semisimple connected subgroup (of not necessarily maximal rank) in $\overline{G}$, and $O^{p'}(R_\sigma) = G_1 * \ldots * G_k$, where all G_i are finite groups of Lie type. Then all groups $G_1, \ldots, G_k$ satisfy condition $(*)$. Moreover, if r is the least prime in $\pi \cap \pi(G)$, $e = e(q, r)$, and $|G_i|_r > 1$, then $|G_i|$ is divided by $\Phi_e(q)$.

Proof. Let $D = O^{p'}(R_\sigma)$. Since $\Phi_{n\alpha}(t)$ divides $\Phi_n(t^\alpha)$ whenever some polynomial $\Phi_n(q^\alpha)$ divides $|D|$, $\Phi_{n\alpha}(q)$ divides $|D|$ and hence also $|G|$. Therefore if s is a large prime for G then it is also one for D. In addition, if r is a minimal prime in $\pi \cap \pi(G)$, $e = e(q, r)$, and r divides $|G_i|$, then $\Phi_{ke}(q)$ divides $|G_i|$ for some natural k; consequently, either $q^{ke} - 1$ divides $|G_i|$, or $q^{ke/2} + 1$ divides $|G_i|$. Since $\Phi_e(q)$ divides $q^{ke} - 1$ or $q^{ke/2} + 1$, $\Phi_e(q)$ divides $|G_i|$, whence the result.

LEMMA 16. Let $G = O^{p'}(\overline{G}_\sigma)$ be a finite simple group of Lie type and X a subgroup of $\operatorname{Aut} G$. Then X satisfies one of the following:

(1) X normalizes some proper connected σ-invariant subgroup $H < \overline{G}$;

(2) there exists a finite simple group H such that $H \leq X \leq \operatorname{Aut} H$;

(3) $X \cap G$ lies in the normalizer of some Jordan subgroup of $\overline{G}$;

(4) $G = E_8(q)$, char $GF(q) > 5$, and $Y < G_0 \cap X < N_G(Y)$, where $Y \cong A \times B$, $A \cong Alt_5$ and $B \cong Alt_6$ are alternating groups of degrees 5 and 6, respectively, $N_G(Y)/Y$ is a quaternion group, $C_{\overline{G}}(B) = A$, $C_{\overline{G}}(A) \cong \mathrm{Sym}_6$ is a symmetric group of degree 6, $B = [C_{\overline{G}}(A), C_{\overline{G}}(A)]$, and Y is defined uniquely up to conjugation in G.

Proof. Theorem 1 in [22] was proved for the case where σ is some classical Frobenius automorphism. That proof, note, can be carried over verbatim to the case of an arbitrary automorphism whose fixed-point set is finite.

THEOREM 5. Let $G = O^{p'}(\overline{G}_\sigma)$ be a finite simple Lie-type group satisfying condition $(*)$. Then the following conditions are equivalent:

(1) every π-subgroup of G contains a normal Hall τ-subgroup;

(2) $\operatorname{Aut} G$ is a D_π-group;

(3) every extension of G by an arbitrary D_π-group is a D_π-group;

(4) G is a D_π-group.

Proof. The implication (2) ⇒ (3) follows from the corollary to Theorem 1 in [23]. (3) ⇒ (4) is trivial. (4) ⇒ (1) follows from Theorem 1.

We embark on (1) ⇒ (2). Assume that this implication is false, and that $G_0 = O^{p'}(\overline{G}_\sigma)$ is a counterexample of minimal order. Put $G_1 = \operatorname{Aut} G_0$. Let M_1 be a maximal π-subgroup of G_1 which is not Hall. Put

$G = \hat{G}_0 = \overline{G}_\sigma$. Condition (1) of the theorem is equivalent for G_0 and $G = \hat{G}_0$ since $N(T_0)\overline{T}_\sigma = N(T)$ for every maximal torus $T_0 = \overline{T}_\sigma \cap G_0$. We handle a group $M = M_1 \cap G$. There are three cases to consider.

Case (a). M contains a non-trivial τ-subgroup T. In view of condition (1), M also contains a normal Hall τ-subgroup. We may assume that T is a normal Hall τ-subgroup of M. Consider a group $C_G(T)$. Since T is contained in some maximal torus H of G which contains a Hall τ-subgroup of G, $C_{\overline{G}}(T)$ is a reductive (not necessarily connected) σ-invariant subgroup of maximal rank in $\overline{G}$. Furthermore, $C_{\overline{G}}(T)^0 = C = DR$ is a characteristic σ-invariant subgroup of $C_{\overline{G}}(T)$; here, $D = [C, C]$ and $R = Z(C)^0$. It follows that $C_\sigma = D_\sigma R_\sigma$ and $O^{p'}(D_\sigma) = G_1 * \ldots * G_k$. By Lemma 15, $G_1, \ldots, G_k$ satisfy $(*)$. Since G is a counterexample of minimal order, $\operatorname{Aut} G_1/Z(G_1), \ldots, \operatorname{Aut} G_k/Z(G_k)$ are D_π-groups. By [23, cor. to Thm. 1], it then follows that the group C_σ and its every extension by a D_π-group are D_π-groups. Since M_1 normalizes C_σ and is a D_π-group, M_1C_σ is also a D_π-group. Note that C_σ contains a Hall τ-subgroup of G.

The group M_1 is a maximal π-subgroup; hence, it is a Hall π-subgroup of M_1C_σ. And T is then a Hall τ-subgroup of G. The group $N_G(T)$ is, therefore, normalized by M_1 and contains a Hall π-subgroup of G. The Hall π-subgroup of every non-Abelian composition factor in $N_G(T)$ coincides with a Sylow r-subgroup of that factor and, consequently, is nilpotent. By [3, Thm. D5; 23, cor. to Thm. 1], every extension of $N_G(T)$ by a D_π-group, and the group $M_1N_G(T)$ in particular, will enjoy property D_π. By maximality, M_1 is a Hall π-subgroup of $M_1N_G(T)$. Therefore $M = M_1 \cap G$ is a Hall π-subgroup of G.

Consider a group $N_{G_1}(M)$. By [10], G_1 is a C_π-group. In particular, $N_{G_1}(M)$ contains a Hall π-subgroup of G_1. Since the group G_1/G is solvable, $N_{G_1}(M)/M \cong N_{G_1}(M)G/G$ is also solvable and is a D_π-group. Consequently, M_1/M is contained in the Hall π-subgroup N_1/M of $N_{G_1}(M)/M$. And M_1, then, is contained in the Hall π-subgroup N_1 of G_1, a contradiction with the choice of M_1.

Case (b). M is a non-trivial r-group. In view of [24, Thm. 30], every automorphism of $G = \overline{G}_\sigma$ extends to an automorphism of $\overline{G}$ which commutes with σ. Thus we may assume that $M_1 \leq \operatorname{Aut} \overline{G}$. Since M is non-trivial, the same is true for its center $Z(M)$ and, hence, for the group $Z = \Omega_1(Z(M))$. Consider a group $C = C_{\overline{G}}(Z)$. Since Z is closed and M_1-invariant, C is likewise. Furthermore, C^0 is a characteristic subgroup of C; consequently, it is M_1-invariant. Let $U = R_u(C^0)$ be a unipotent radical of C^0. Since U is a characteristic subgroup of C^0, it is also M_1-invariant.

First assume that U is non-trivial. Put $N_1 = N_{\overline{G}}(U)$, $U_1 = UR_u(N_1)$, $N_i = N_{\overline{G}}(U_{i-1})$, and $U_i = U_{i-1}R_u(N_i)$. Since $\overline{G}$ is finite-dimensional, the chain of embeddings $N_0 \leq N_1 \leq \ldots$ will stabilize at some step k. The group N_0 is M_1-invariant, and so are all N_i therefore. In addition, Z consists of σ-fixed elements; consequently, the groups C, C^0, and U are σ-invariant, and so therefore are all N_i, U_i. By [13, Prop. 30.3], $P = N_k$ is a σ- and M_1-invariant parabolic subgroup of $\overline{G}$. Since $N_0 \neq 1$, we have $P \neq \overline{G}$. Thus P_σ is a proper parabolic subgroup of G and P_σ is representable as $L_\sigma V_\sigma$, where $V_\sigma = R_u(P)_\sigma$, $L_\sigma = D_\sigma * S_\sigma$ is the Levi factor of P_σ, $D_\sigma = [L, L]_\sigma$ is a subsystem subgroup of G, and $S_\sigma = (Z(L)^0)_\sigma$ is some torus of order $(q-1)^k$, $k = \dim(Z(L))$. By Lemma 15, all non-Abelian composition factors of the group $P_\sigma = P_1$ satisfy $(*)$. Thus P_1 and M_1P_1 are D_π-groups in view of corollary to Theorem 1 in [23]. By maximality, M_1 is a Hall π-subgroup of M_1P_1.

Since r divides $|P_\sigma|$ and is coprime with p, r divides $|L_\sigma|$. It follows that either r divides $|D_\sigma|$, or r divides $|S_\sigma|$. In the former case the order of D_σ is divisible by $\Phi_e(q)$, where $e = e(r, q)$, by Lemma 15. Hence the group P_σ contains non-trivial τ-elements. Consequently, M contains a non-trivial τ-subgroup, a contradiction with the choice of M_1. In the latter case r divides $q-1$. It follows that $e(r, q) = 1$, and the torus S_σ contains non-trivial τ-elements, which is a contradiction with the choice of M_1 again.

TABLE 2

$\overline{G}$	J	$N_{\overline{G}}(J)$
A_{r^n-1}, $n \geqslant 1$	r^{2n}	$r^{2n} \cdot Sp_{2n}(r)$
E_6	3^3	$3^3 \cdot 3^3 \cdot SL_3(3)$
E_8	5^3	$5^3 \cdot SL_3(5)$
F_4	3^3	$3^3 \cdot SL_3(3)$

Next suppose that U is trivial. Then C is a reductive subgroup of $\overline{G}$. And we are faced with the following options:

1. Let $\dim(C) = 0$. Since Z is finite, it is a d-group. $N_{\overline{G}}(Z)$ is a finite group in view of the fact that d-groups are rigid (cf. [13, cor. to Prop. 16.3]). By Lemma 16, Z satisfies one of (2)-(4) of that lemma. Since Z is a non-trivial normal Abelian subgroup of $N_{\overline{G}}(Z)$, Z cannot satisfy (2) or (4). Consequently, Z contains a Jordan subgroup of $\overline{G}$. The Jordan subgroup is a maximal elementary Abelian subgroup of $N_{\overline{G}}(Z)$ (cf. condition 2 in [25]); therefore, Z is a Jordan subgroup of $\overline{G}$. Since Z lies in G, $N_{\overline{G}}(Z)$ also lies in G. In Table 2, we list all possible Jordan subgroups J of odd order (cf. [25]).

If $\overline{G} = A_{r^n-1}(K)$ then r divides $q-1$. Consequently, $N_{\overline{G}}(Z)$ is a τ'-group. Therefore $M_1 N_{\overline{G}}(Z)$ is a D_π-group, and hence M is a Sylow r-subgroup of $N_{\overline{G}}(Z)$. The latter is impossible since $N_{\overline{G}}(Z)$ acts regularly on Z and is generated by r-elements, which contradicts the condition that $Z \leq Z(M)$. In all other cases (cf. Table 2), either $N_{\overline{G}}(Z)$ is a τ'-group, which is again a contradiction with the choice of M_1, or G contains a π-subgroup of the form $L = J \leftthreetimes T$, where J is a normal Sylow r-subgroup of L and T is a Hall τ-subgroup of L which is not normal in it. This leads us to a contradiction with condition (1) of the theorem.

2. Assume that $Z(C^0) = 1$, $\dim(C) > 0$, and that C is of lesser rank than is $\overline{G}$. Then C^0 is a σ- and M_1-connected invariant semisimple group, and by Lemma 15, the composition factors of $M_1(C^0)_\sigma$ all satisfy condition $(*)$; hence, C^0 is a D_π-group. Therefore M is a Hall π-subgroup of $M_1(C^0)_\sigma$. If r divides $|(C^0)_\sigma|$, then $(C^0)_\sigma = G_1 * \ldots * G_k$, since $Z(C^0)$ is trivial; so, $(C^0)_\sigma = G_1 * \ldots * G_k$ contains non-trivial τ-elements by Lemma 15, which contradicts the choice of M_1. For the other case we obtain $Z \cap C^0 = 1$.

Let T be some maximal torus of C^0. Since C^0 is connected, $C_{C^0}(T) = T$. From [13, Thm. 22.3], it follows that $C_1 = C_{\overline{G}}(T)$ is a connected reductive subgroup of maximal rank in $\overline{G}$. Clearly, $Z \leq C_1$. Moreover, $Z \nleq Z(C_1)$, for otherwise Z would lie in some maximal torus, and so $C_{\overline{G}}(Z) = C$ would be a subgroup of maximal rank in $\overline{G}$, which is a contradiction with the conditions on C. Thus $Z_1 = Z \cap [C_1, C_1] \neq 1$; in particular, $[C_1, C_1] \neq 1$. We have $C_{C^0}(T) = T$, and so $[C_1, C_1] \cap C^0 \leq Z(C_1)$. Since $C_{\overline{G}}(Z) = C$, $C_{[C_1,C_1]}(Z_1)$ is a finite group. Thus Z_1 is a Jordan subgroup of $[C_1, C_1]$. This, in view of the above, leads us to a contradiction either with $Z \leq Z(M)$ or with condition (1) of the theorem.

3. Let $Z(C^0) \neq 1$ or C be a subgroup of maximal rank in $\overline{G}$. If C is not a subgroup of maximal rank, then $Z(C^0) \neq 1$ is contained in some maximal torus of $\overline{G}$, since it is contained in all maximal tori of the connected group C^0. Consequently, $C_{\overline{G}}(Z(C^0))$ is a proper σ- and M_1-invariant subgroup of maximal rank in $\overline{G}$, and C can be replaced by the group $C_{\overline{G}}(Z(C^0))$. Further, we assume that C is a reductive subgroup of maximal rank in $\overline{G}$.

Put $C_1 = C^0$ and $F = (C_1)_\sigma = D_\sigma * S_\sigma$, where $D_\sigma = [C_1, C_1]_\sigma$ is a subsystem subgroup of G and $S_\sigma = (Z(C_1)^0)_\sigma$ is some torus. Consider a group $M_1 F$. Since G is a counterexample of minimal order,

TABLE 3

G	r	$\lvert(C_{\overline{G}}(S))_\sigma\rvert$	$e(q,r)$
$A_{kr-1}(q)$		$(q^{kr}-1)/(q-1)$	1
${}^2A_{kr-1}(q^2)$, k is odd		$(q^{kr}+1)/(q+1)$	2
$G_2(q)$	3	q^2+q+1	1
$G_2(q)$	3	q^2-q+1	1
$F_4(q)$	3	$(q^2+q+1)^2$	1
$F_4(q)$	3	$(q^2-q+1)^2$	1
$E_6(q)$	3	$(q^2+q+1)^3$	1
$E_6(q)$	3	q^6+q^3+1	1
$E_6(q)$	3	$(q^2-q+1)^2(q^2+q+1)$	2
$E_8(q)$	3	$(q^2+q+1)^4$	1
$E_8(q)$	3	$(q^2-q+1)^4$	2
$E_8(q)$	3	$(q^2-q+1)(q^6-q^3+1)$	2
$E_8(q)$	5	$(q^4+q^3+q^2+q+1)^2$	1
$E_8(q)$	5	$(q^4-q^3+q^2-q+1)^2$	2
${}^3D_4(q^3)$	3	$(q^2+q+1)^2$	1
${}^3D_4(q^3)$	3	$(q^2-q+1)^2$	2
${}^2E_6(q^2)$	3	$(q^2+q+1)^2(q^2-q+1)$	1
${}^2E_6(q^2)$	3	$(q^2-q+1)^3$	2
${}^2E_6(q^2)$	3	q^6-q^3+1	2

and all non-Abelian composition factors of F satisfy $(*)$, M_1F is a D_π-group. By maximality, M_1 is a Hall π-subgroup of M_1F.

Further, either r divides $|D_\sigma|$, or r divides $|N(F)|$. In the former case, F contains non-trivial τ-elements by Lemma 15. Hence M contains a non-trivial τ-subgroup, which contradicts the choice of M_1. Suppose that the second case is realized and F is a τ'-group. Denote $e(r,q)$ by e. Then S_σ is some torus such that the order of its group $N(S_\sigma)$ is divided by $\Phi_{ke}(q)$, for some $k\in\mathbb{Z}$, and the order of $(C_{\overline{G}}(S))_\sigma$ is not divided by $\Phi_e(q)$. In Table 3 we list all groups G and all primes r for which such a situation obtains (cf. [17-19]).

Our next goal is to treat all of these cases singly. Note, from the outset, that $\pi\cap\pi(N(S_\sigma))=\{r\}$ and $N(S_\sigma)$ is an M_1-invariant group, and so $M_1N(S_\sigma)$ enjoys D_π. This, in view of the fact that M_1 is maximal, implies that M is a Sylow r-subgroup of $N(S_\sigma)$.

First assume that $G=A_{kr-1}(q)$. Then $S_\sigma=(C_{\overline{G}}(S))_\sigma$ is a cyclic group. Consequently, S_σ contains a unique subgroup $A=\langle a\rangle$ of order r, which lies in the center of M. Indeed, A is a characteristic subgroup of S_σ, hence $A\cap(M)\neq 1$, and so $A\cap Z(M)=A$. Since $S_\sigma\trianglelefteq M_1N(S_\sigma)$, A is normal in M_1. If $C_{\overline{G}}(A)\not\leq N_{\overline{G}}(S)$, or, which is the same, $C_G(A)\not\leq N(S_\sigma)$, then $C_{\overline{G}}(A)^0$ is a σ- and M_1-invariant subgroup of maximal rank in $\overline{G}$, distinct from S. Hence $(C_{\overline{G}}(A))_\sigma$ contains non-trivial τ-elements, which is a contradiction with the choice of M_1 again. Therefore we may assume that $C_G(A)\leq N(S_\sigma)$, that is, a is a regular element.

Let R_1 be a Sylow r-subgroup in G_1 containing a Sylow r-subgroup of M_1; then $R=R_1\cap G$ is a Sylow r-subgroup of G containing M. Since $R\trianglelefteq R_1$, R contains an element b of order r in $Z(R_1)$. Clearly, $b\in Z(R)$, and so $b\in C_G(a)\leq N(S_\sigma)$. Direct computations show that $Z(R)$ is contained in the group of diagonal matrices of G; hence, $C_G(Z(R)) \leq C_G(b)$ and the group $C_G(Z(R))$ contains non-trivial τ-elements. Consequently, $b \notin A$. From [15, Prop. 30], it follows that $N(S_\sigma)/S_\sigma$ is a cyclic group. Therefore $\langle a \rangle \times \langle b \rangle = \Omega_1(M)$ is a characteristic subgroup of M.

Thus $\langle a \rangle$ and $\langle a \rangle \times \langle b \rangle$ are M_1-invariant groups. Note also that r is the least number in π. In fact, r is least in $\pi \cap \pi(G)$ by assumption. If there exists an $s < r$, $s \in \pi$, then $q^{s-1} - 1$ divides $|G|$ and is divided by s, which contradicts the choice of r. Since $|a| = r$ and $\pi(r-1) \cap \pi = \varnothing$, M_1 centralizes $\langle a \rangle$. In view of $2 \notin \pi$, we have $\pi \cap \pi(r+1) = \varnothing$, and so $\pi \cap \pi(r^2 - 1) = \varnothing$. Consequently, every r'-element of M_1 centralizes $\langle a \rangle \times \langle b \rangle$. Further, the group $C_{M_1}(\langle a \rangle \times \langle b \rangle)$ is normal in M_1 and contains a Sylow r-subgroup of M_1. Consequently, every Sylow r-subgroup of M_1 lies in $C_{M_1}(\langle a \rangle \times \langle b \rangle)$. The group M_1 centralizes b and hence normalizes $C_G(b)$. The group $C_G(Z(R))$, and so $C_G(b)$, should contain non-trivial τ-elements, which they do not by the choice of M_1. Similarly we argue for the group ${}^2A_{kr-1}(q^2)$.

For exceptional groups, the argument is uniform based on the groups $A_{kr-1}(q)$ and ${}^2A_{kr-1}(q^2)$ treated above. We dwell, for instance, on the case where $G = E_8(q)$, $r = 5$, and $|S_\sigma| = (q^4 - q^3 + q^2 - q + 1)^2$. By [15, Lemmas 26 and 27], the group $S_\sigma = (C_{\overline{G}}(S))_\sigma$ is unique up to conjugation in $E_8(q)$. From [19, Table 2], it follows that $E_8(q)$ contains a subgroup isomorphic to ${}^2\hat{A}_4(q^2) * {}^2\hat{A}_4(q^2)$. Since this subgroup contains a torus of order $(q^4 - q^3 + q^2 - q + 1)^2$, we may assume that $S_\sigma \leq {}^2\hat{A}_4(q^2) * {}^2\hat{A}_4(q^2)$. Then $N(S_\sigma)$ contains a normal subgroup of index 2, representable as a central product $N_1 * N_2$ of isomorphic groups, which lies in the group ${}^2\hat{A}_4(q^2) * {}^2\hat{A}_4(q^2)$. By the above, M_1 centralizes some element $a \in {}^2\hat{A}_4(q^2) * {}^2\hat{A}_4(q^2)$, whose centralizer contains non-trivial τ-elements. We are led to a contradiction with the choice of M_1.

Case (c). The group $M_1 \cap G$ is trivial. Every element of G_1 is uniquely represented as a product of inner-diagonal, field, and graph automorphisms. First assume that M_1 lies in a group generated by the inner-diagonal and field automorphisms of G_2. The group G_2/G is cyclic. Since $M_1 \cap G = \{1\}$, M_1 is also cyclic and $M_1 = \langle h \rangle$. By [26, 7-2], h may be conceived of as an automorphism of G_0.

If the Hall π-subgroup of G is Abelian then G_2 possesses property D_π by [3, Thm. D5]. If it is non-Abelian then $\pi(W(G)) \cap \pi \neq \varnothing$. In view of [24, Thm. 30], h may be treated as an automorphism of $\overline{G}$. Consequently, $K = O^{p'}(\overline{G}_h)$ is a finite group of Lie type, which is a subgroup of $\overline{G}_\sigma = G$. Moreover, there exists an h-invariant maximal torus T of $\overline{G}$ such that $T_h \cap K$ is a Cartan subgroup of K. In particular, $N(T_h \cap K)$ consists of h-fixed elements. Since $\pi(N(T_h \cap K)/(T_h \cap K)) = \pi(W(K)) = \pi(W(G_0))$, the group $N(T_h \cap K)$ contains a non-trivial π-element l. Therefore $\langle h \rangle \leq \langle h, l \rangle$, $\langle h, l \rangle \cap G \neq \{1\}$, and $\langle h, l \rangle$ is a π-group, which contradicts the choice of M_1.

Lastly assume that M_1 does not belong to a group generated by inner-diagonal and field automorphisms. This is possible only if $G_0 = D_4(q^3)$ and $3 \in \pi$. The group $M_1 \cap G_2$ is then a cyclic normal subgroup of index 3 in M_1; hence, $\langle h \rangle = M_1 \cap G_2$. By the above, either the Hall π-subgroup of G is Abelian, or $N_{G_1}(h) \cap G$ contains non-trivial 3-elements. In the former case the result follows from [3, Thm. D5]. In the latter case we may take a Sylow 3-subgroup S in $N_{G_1}(h) \cap G$ which contains a Sylow 3-subgroup of M_1. Then $S \cdot \langle h \rangle$ is a π-subgroup, contains M_1 as a group, and has a non-trivial intersection with G. Therefore $S \cdot \langle h \rangle \neq M_1$, which contradicts the choice of M_1.

Proof of Theorem 2. By [23, Thm. 1], it suffices to prove that an automorphism group of every non-Abelian simple D_π-group G is a D_π-group.

If a simple group is sporadic or alternating then the order of its inner automorphism group is the degree of 2, and so our claim is trivial.

If a simple group is of Lie type and the characteristic of its definition field lies in π, we need only appeal to [8].

Finally, if G is a simple group of Lie type and the characteristic of its definition field does not lie in π, one of the following statements holds:

(1) a Hall π-subgroup is Abelian, and every extension of a D_π-group with an Abelian Hall subgroup by a D_π-group is again a D_π-group (cf. [3, Thm. D5]);

(2) G is a π'-group, as desired;

(3) G satisfies condition $(*)$, and the conclusion follows immediately from Theorem 5.

Remark. The proof of Theorem 5 shows that there exist groups for which $(*)$ does not imply condition (1) of Theorem 5. In this connection, it might be useful to make up a list of all finite groups of Lie type which satisfy $(*)$ and for which (1) of Theorem 5 is satisfied. Such a list would allow us to exhaust the description of groups with property D_π for the case where $2 \notin \pi$.

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