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HALL SUBGROUPS OF ODD ORDER IN FINITE GROUPS

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We complete the description of Hall subgroups of odd order in finite simple groups initiated by F. Gross, and as a consequence, bring to a close the study of odd order Hall subgroups in all finite groups modulo classification of finite simple groups. In addition, it is proved that for every set π of primes, an extension of an arbitrary D_π-group by a D_π-group is again a D_π-group. This result gives a partial answer to Question 3.62 posed by L. A. Shemetkov in the "Kourovka Notebook."

INTRODUCTION

Ph. Hall's definition maintains that a finite group possesses property E_π (in other words, it is an E_π-group) for some set π of primes if it contains a π-subgroup whose index is not divisible by the primes in π. (Such is conventionally called a Hall π-subgroup.) If a group G contains a Hall π-subgroup, and all of its Hall π-subgroups are conjugate, then G is called a C_π-group. Finally, if G is a C_π-group, and its π-subgroups each is contained in some Hall π-subgroup, then G is said to possess property D_π.

It is well known that, for every set π of primes, the class of all E_π-groups is closed under normal subgroups and homomorphic images. Consequently, if a finite group possesses a Hall π-subgroup then its composition factors each also possess one. Traditionally, therefore, the problem of describing Hall subgroups of finite groups which are close to simple ones takes center-stage in their study. For finite Chevalley groups, a similar problem is pointed out, in particular, in [1].

We also know that property C_π is inherited under extensions. The question if property D_π is inherited under extensions remains open. (Shemetkov brought it in [2] as Question 3.62.)

Descriptions of Hall subgroups of the symmetric groups are obtained in Hall [3] and in Thompson [4]. Hall subgroups of the sporadic groups and of the finite Lie-type groups, for the case where the characteristic of a base field belongs to π, are described in Gross [5, 6] and in Revin [7, 8]. Due to Gross is also the description of Hall π-subgroups of the classical groups of arbitrary characteristic p, if $2, p \notin \pi$ (cf. [9]), and of the groups $GL_n(p^s)$ and $Sp_{2n}(p^s)$ if $3, p \notin \pi$ (cf. [6]).

The problem of describing finite groups with property E_π and their Hall π-subgroups for the case where $2 \notin \pi$ is of special interest, for the class of all E_π-groups will be closed under extensions, as follows from the main result of [10], which uses the classification of finite simple groups. In particular, Gross proved that properties E_π and C_π are equivalent if $2 \notin \pi$. Therefore, for an arbitrary finite group to contain a Hall

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