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MAXIMAL ORDERS OF ABELIAN SUBGROUPS IN FINITE SIMPLE GROUPS

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We bring out upper bounds for the orders of Abelian subgroups in finite simple groups. (For alternating and classical groups, these estimates are, or are nearly, exact.) Precisely, the following result, Theorem A, is proved. Let G be a non-Abelian finite simple group and $G \not\cong L_2(q)$, where $q = p^t$ for some prime number p. Suppose A is an Abelian subgroup of G. Then $|A|^3 < |G|$. Our proof is based on a classification of finite simple groups. As a consequence we obtain Theorem B, which states that a non-Abelian finite simple group G can be represented as ABA, where A and B are its Abelian subgroups, iff $G \cong L_2(2^t)$ for some $t \geqslant 2$; moreover, $|A| = 2^t+1$, $|B| = 2^t$, and A is cyclic and B an elementary 2-group.

INTRODUCTION

In the present article we work to find upper bounds for the orders of Abelian subgroups in finite simple groups. For alternating and classical groups, these estimates are (or are nearly) exact. In any case the following is valid:

THEOREM A. Let G be a non-Abelian finite simple group and $G \not\cong L_2(q)$, where $q = p^t$ for some prime number p. Suppose A is an Abelian subgroup of G. Then $|A|^3 < |G|$.

In proving the theorem, we make use of the classification of finite simple groups given in [1, Table 2.4].

A consequence is obtaining the answer to Question 4.27 in [2]; the solution to this problem was first announced in [3] where it was underpinned by some other ideas.

THEOREM B. A non-Abelian finite simple group G can be represented as a product ABA of its Abelian subgroups A and B if and only if $G \cong L_2(2^t)$ for some $t \geqslant 2$; moreover, $|A| = 2^t + 1$, $|B| = 2^t$, and A is a cyclic group and B an elementary 2-group.

Abelian subgroups of finite simple groups have been studied extensively. In [4], for instance, an estimate is obtained for the order of a maximal torus in all Chevalley groups. For universal classical Chevalley groups, bounds for the orders of semisimple subgroups of nilpotency class at most 2 are estimated in [5]. In [6-9], the reader can find estimates for the orders of Abelian unipotent subgroups in classical groups, and also descriptions of those subgroups.

For many of the Chevalley groups of exceptional type, exact estimates for the orders of Abelian unipotent subgroups are still not found; the estimates of which we have knowledge will be given in Lemmas 4.1 and 4.5.

Chevalley groups are structured so as to allow us to conjecture that the order of an arbitrary Abelian subgroup does not exceed a maximum of the orders of a greatest Abelian p-subgroup and a greatest Abelian p'-subgroup. In the present article, we confirm this conjecture for projective special linear and symplectic groups.

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The article is divided into some sections, in each of which a specified type of finite simple groups is treated. In Sec. 1, we address the case of symmetric and alternating groups. Classical groups are studied in Secs. 2 and 3, exceptional Lie-type groups — in Sec. 4, and sporadic groups — in Sec. 5. In the final Sec. 6, we prove Theorem B.

For the groups amenable to exact estimates, we specify the structure of greatest Abelian subgroups. By a greatest Abelian subgroup, throughout the article, we mean an Abelian subgroup of maximal order.

The notation and definitions are borrowed from [10, 11]. Denote by $A(G)$ the greatest Abelian subgroup of G, and by $A_p(G)$ and $A_{p'}(G)$ the greatest Abelian p- and p'-subgroups, respectively. Let p be some prime number; then $O_p(G)$ is a maximal normal p-subgroup of G. For a subset M of G, we use $\langle M\rangle$ to denote a group generated by the set M, and write $|M|$ for the cardinality of M. If H is a subgroup of G, then $C_G(H)$ and $N_G(H)$ are, respectively, the centralizer and the normalizer of H in G; $|G:H|$ is the index of H in G. If H is normal in G, written $H \trianglelefteq G$, then G/H is a factor group of G w.r.t. H. By $Z(G)$ we denote the center of G; $A\times B$ is a direct product of groups A and B, and $A * B$ is a central product.

In dealing with Chevalley groups, we denote by F_q a field of order q, by p its characteristic, and by F_q^* a multiplicative group of F_q. For a Chevalley group corresponding to a root system Φ over F_q, write $\Phi(q)$. A set of positive roots in the root system Φ is denoted by Φ^+, and a set of fundamental roots — by $\{r_1,\dots,r_k\}$, where the numbering is chosen in accordance with [11]. A root subgroup corresponding to a root $r\in\Phi$ is denoted by X_r, and an element in that root subgroup — by $x_r(t)$, $t\in F_q$. An element x in the Chevalley group $\Phi(q)$ is called *semisimple*, if its order is coprime to p, and we call it *unipotent* if its order is the power of p. Similarly, a *semisimple subgroup* in $\Phi(q)$ is one whose order is coprime to p (p'*-subgroup*) and its *unipotent subgroup* is one whose order is the power of p.

1. ALTERNATING GROUPS

THEOREM 1.1. A greatest Abelian subgroup in an alternating group A_n is conjugate to one of the following groups:

(1) $\langle(1,2,3),\dots,(3k-2,3k-1,3k)\rangle$ if $n=3k$;

(2) $\langle(1,2)(3,4),(1,3)(2,4),(5,6,7),\dots,(3k-1,3k,3k+1)\rangle$ if $n=3k+1$;

(3) $\langle(1,2)(3,4),(1,3)(2,4),(5,6)(7,8),(5,7)(6,8),(9,10,11),\dots,(3k-1,3k,3k+1)\rangle$ if $n=3k+2$;

(4) $\langle(1,2,3,4,5)\rangle$ if $n=5$.

Also, the orders of greatest Abelian subgroups in alternating (A_n) and symmetric (S_n) groups are given thus:

$|A(A_{3n})|=3^n$; $|A(A_{3n+1})|=4\cdot 3^{n-1}$; $|A(A_{3n+2})|=16\cdot 3^{n-2}$; $|A(A_5)|=5$; $|A(S_{3n})|=3^n$; $|A(S_{3n+1})|=4\cdot 3^{n-1}$; $|A(S_{3n+2})|=2\cdot 3^n$.

For any n, the group $A(A_n)$ is unique up to conjugation.

Remark. That Theorem A is valid for alternating groups follows easily from Theorem 1.1. Indeed, routine computations help us check that $|A(A_n)|^3<|A_n|$ for $n\geqslant 7$. We have $A_5\cong L_2(4)$ and $A_6\cong L_2(9)$, which proves Theorem A for A_n.

Proof of the theorem. We point out the following well-known fact. Let $H\le S_n$ and assume that H is Abelian and acts transitively on a set $\{1,\dots,n\}$. Then $|H|=n$.

In fact, consider a stabilizer $St_H(i)$ of some element $i\in\{1,\dots,n\}$ in the group H. Since H acts transitively, for any $j\in\{1,\dots,n\}$, there exists a $\tau\in H$ for which $i^\tau=j$. For any $\sigma\in St_H(i)$, therefore, we have

$$j^\sigma=i^{\tau\sigma}=i^{\sigma\tau}=i^\tau=j,$$

that is, if $\sigma \in St_H(i)$ then $\sigma \in St_H(j)$ for all $j \in \{1, \dots, n\}$. Hence $\sigma = \varepsilon$ is an identical permutation, that is, $St_H(i) = \{\varepsilon\}$. Furthermore, $|H| = |H : St_H(i)| \cdot |St_H(i)|$, $|H : St_H(i)| = n$, and consequently $|H| = n$.

Finally, the whole set $\{1, \dots, n\}$ splits into disjoint subsets $I_1, \dots, I_k$, on each of which the Abelian subgroup G of S_n acts transitively. Thus $|G| \leqslant \prod\limits_{j=1}^{k} |I_j|$.

Write $P_n = \max\limits_{n_1+\dots+n_k=n} (\prod\limits_{j=1}^{k} n_j)$. By the above, $|A(S_n)| = P_n$. It is not hard to see that P_n satisfies the following recurrent relation:

$$P_n = \max_{0<m\leqslant n} (P_{n-m} \cdot m), \; P_0 = 1.$$

Using this, by induction we obtain the equalities

$$P_{3n} = 3^n; \; P_{3n+1} = 4 \cdot 3^{n-1}; \; P_{3n+2} = 2 \cdot 3^n.$$

The theorem is proved for $A(S_n)$.

Note that $A_n < S_n$, and hence $|A(A_n)| \leqslant |A(S_n)|$. In the group A_{3n}, there exists an Abelian subgroup G generated by permutations $(1,2,3), (4,5,6), \dots, (3n-2, 3n-1, 3n)$, that is, G can be represented as a direct product of cyclic groups of order 3. The order of G is equal to 3^n; therefore, $|A(A_{3n})| = 3^n$. It is worth mentioning that any greatest Abelian subgroup F in A_{3n} is represented as a direct product of cyclic groups of order 3, that is, it is generated by permutations $(k_1, k_2, k_3), (k_4, k_5, k_6), \dots, (k_{3n-2}, k_{3n-1}, k_{3n})$; therefore, $G^\sigma = F$, where σ is a permutation in S_{3n} sending 1 to k_1, 2 to k_2, and so on. If σ is odd, we may take a permutation $(1,2)\sigma = \tau$, which is even. Since $G^{(1,2)} = G$, we have $G^\tau = F$, that is, G and F are conjugate in A_{3n}.

In the group A_{3n+1}, there is an Abelian subgroup G generated by permutations $(1,2)(3,4)$, $(1,3)(2,4)$, $(5,6,7)$, ..., $(3n-1, 3n, 3n+1)$; its order is equal to $4 \cdot 3^{n-1}$, and so $|A(A_{3n})| = 4 \cdot 3^{n-1}$. A proof that any greatest Abelian group and G are conjugate in A_{3n+1} goes along the same line as in the A_{3n} case.

Lastly, if G is an Abelian subgroup of A_{3n+2}, then either $|G| = 3n+2$, or G is represented as $G = G_1 \times G_2$, where $G_1 < A_{k_1}$, $G_2 < A_{k_2}$, and $k_1 + k_2 = 3n+2$. If $n \geqslant 2$, for the indices k_1 and k_2, we have the following cases.

1. Let $k_1 = 3n_1 + 1$ and $k_2 = 3n_2 + 1$. Then $|G| = |G_1 \times G_2| = |G_1| \cdot |G_2| \leqslant |A(A_{3n_1+1})| \cdot |A(A_{3n_2+1})| = 16 \cdot 3^{n-2}$.

2. Let $k_1 = 3n_1$ and $k_2 = 3n_2+2$. Then $|G| = |G_1 \times G_2| = |G_1| \cdot |G_2| \leqslant |A(A_{3n_1})| \cdot |A(A_{3n_2+2})| = 16 \cdot 3^{n-2}$.

Using induction on n yields the estimate $|A(A_{3n+2})| \leqslant 16 \cdot 3^{n-2}$. On the other hand, for $n \geqslant 2$, A_{3n+2} contains a subgroup G generated by permutations $(1,2)(3,4)$, $(1,3)(2,4)$, $(5,6)(7,8)$, $(5,7)(6,8)$, $(9,10,11)$, ..., $(3n, 3n+1, 3n+2)$; its order is equal to $16 \cdot 3^{n-2}$, and so $|A(A_{3n})| = 16 \cdot 3^{n-2}$. As above, we can prove that any greatest Abelian subgroup and G are conjugate in A_{3n+2}. The theorem is proved.

2. RESULTS REVISITED

Theorems 11.3.2, 14.5.1, and 14.5.2 in [11] imply

LEMMA 2.1. The following isomorphisms hold:
(1) $A_{n-1}(q) \cong L_n(q) \cong PSL_n(q)$, $n \geqslant 2$;
(2) $B_n(q) \cong P\Omega_{2n+1}(q)$, $n \geqslant 3$;
(3) $C_n(q) \cong PSp_{2n}(q)$, $n \geqslant 2$;
(4) $D_n(q) \cong P\Omega_{2n}^{+}(q)$, $n \geqslant 4$;

(5) ${}^2D_n(q^2) \cong P\Omega_{2n}^-(q)$, $n \geqslant 4$;
(6) ${}^2A_n(q^2) \cong PSU_{n+1}(q^2)$, $n \geqslant 2$;
(7) $B_2(3) \cong {}^2A_4(2^2)$, $B_n(2^\alpha) \cong C_n(2^\alpha)$, $B_2(q) \cong C_2(q)$.

The next lemma collects the results obtained in [6-9]. The cases of orthogonal groups of small dimensions of which no mention is made in the lemma can be found in [8].

LEMMA 2.2. Let $q = p^\alpha$. Then:
(1) $|A_p(GL_n(q))| = q^{[\frac{n^2}{4}]}$;
(2) $|A_p(Sp_{2n}(q))| = q^{\frac{n(n+1)}{2}}$;
(3) $|A_p(O_{2n+1}(q))| = q^{\frac{n(n-1)}{2}+1}$ for $n \geqslant 3$ and $p \neq 2$;
(4) $|A_p(O_{2n}^+(q))| = q^{\frac{n(n-1)}{2}}$ for $n \geqslant 4$;
(5) $|A_p(O_{2n}^-(q))| = q^{\frac{(n-2)(n-1)}{2}+2}$ for $n \geqslant 4$, q is odd;
(6) $|A_p(O_{2n}^-(q))| = q^{\frac{n(n-1)}{2}}$ for $n \geqslant 4$, q is even;
(7) $|A_p(U_n(q^2))| = q^{[\frac{n^2}{4}]}$.

LEMMA 2.3 [5]. Let V be an n-dimensional vector space over a finite field F_q, assume that $G \leq GL(V)$, $H \lhd G$, $(|G/H|, q) = 1$, and suppose that the nilpotency class of G/H is at most 2. Then:
(1) $|G/H| < q^n$.
(2) if G preserves the nondegenerate bilinear form f on V then $|G/H| \leqslant 2^{\varepsilon(n)}\delta^{[\frac{n}{2}]}$, where

$$\varepsilon(n) = \begin{cases} 0 & \text{if } q \text{ or } n \text{ is even,} \\ 1 & \text{if } q \text{ and } n \text{ are odd;} \end{cases} \qquad \delta = \begin{cases} 8 & \text{if } q = 3 \text{ or } 5, \\ 1+q & \text{otherwise.} \end{cases}$$

LEMMA 2.4 [4, Thm. 2.4]. Let G be an adjoint Chevalley group, which is not a Suzuki or Ree group. Suppose that U is a maximal unipotent subgroup of G, and $q = |Z(U)|$. Then $\frac{1}{d}(q-1)^r \leqslant |A_{p'}(G)| \leqslant \frac{1}{d}(q+1)^r$, where r is the Lie rank of G and d is the order of the center of a universal Chevalley group. (In [11], that order is determined for all Chevalley groups.) If G is a Suzuki or Ree group, again we let U be a maximal unipotent subgroup of G, $q = |Z(U)|$, and $q_1 = \sqrt{q}$. Then $(q_1 - 1)^r \leqslant |A_{p'}(G)| \leqslant (q_1+1)^r$, where r is the Lie rank of G.

LEMMA 2.5 [5, Lemma 1.1]. Let V be a finite-dimensional vector space over a field F_q. Suppose A is a subgroup of $GL(V)$, and $(|A|, q) = 1$. Then V is decomposed into a direct sum of proper irreducible A-submodules.

LEMMA 2.6 [5, Lemma 1.2]. Let V be a finite-dimensional vector space over F_q and f be the nondegenerate bilinear form on V. If A is a subgroup of $GL(V)$, A preserves f, and $(|A|, q) = 1$, then $V = C_V(A) \oplus^\perp [V, A]$ is an orthogonal direct sum of A-submodules $C_V(A) = \{v \in V \mid va = v \text{ for all } a \in A\}$ and $[V, A] = \{va - v \mid v \in V,\ a \in A\}$.

3. CLASSICAL SIMPLE GROUPS

A classical simple group, we recall, is an adjoint classical Chevalley group. Up to isomorphism, this is one of the following:
(1) $A_n(q) = L_{n+1}(q) = PSL_{n+1}(q)$;
(2) $B_n(q) = O_{2n+1}(q) = P\Omega_{2n+1}(q)$;
(3) $C_n(q) = S_{2n}(q) = PSp_{2n}(q)$;
(4) $D_n(q) = O_{2n}^+(q) = P\Omega_{2n}^+(q)$;

(5) ${}^2A_n(q^2) = U_{n+1}(q) = PSU_{n+1}(q^2)$;

(6) ${}^2D_n(q^2) = O_{2n}^-(q) = P\Omega_{2n}^-(q)$.

THEOREM 3.1. Greatest Abelian subgroups of classical simple groups are estimated thus:

$|A(L_2(q))| = q + 1$ if q is even;

$|A(L_2(q))| = q$ if q is odd;

$|A(L_3(q))| = q^2 + q + 1$ if $(3, q-1) = 1$;

$|A(L_3(q))| = q^2$ if $(3, q-1) = 3$;

$|A(L_n(q))| = q^{[n^2/4]}$ if $n \geqslant 4$;

$|A(U_4(2))| = |A(S_4(3))| = 27$;

$q^3 \leqslant |A(S_4(q))| \leqslant (q+1)q^2$ if $q \geqslant 4$;

$q^4 \leqslant |A(O_7(q))| \leqslant 2(q+1)^2q^3$ if q is odd;

$q^{\frac{n(n-1)}{2}+1} \leqslant |A(O_{2n+1}(q))| \leqslant 2\delta q^{\frac{n(n-1)}{2}}$, where q is odd, δ is as in Lemma 2.3, and $n \geqslant 4$;

$|A(S_{2n}(q))| = q^{\frac{n(n+1)}{2}}$ for $n \geqslant 3$;

$q^{\frac{n(n-1)}{2}} \leqslant |A(O_{2n}^+(q))| \leqslant \delta q^{\frac{n(n-1)}{2}}$;

$q^{\frac{(n-2)(n-1)}{2}+2} \leqslant |A(O_{2n}^-(q))| \leqslant \delta^2 q^{\frac{(n-1)(n-2)}{2}+2}$, where q is odd;

$q^{\frac{n(n-1)}{2}} \leqslant |A(O_{2n}^-(q))| \leqslant \delta q^{\frac{n(n-1)}{2}}$, where q is even;

$q^2 \leqslant |A(U_3(q))| \leqslant (q+1)^2$;

$|A(U_4(q))| = q^4$ if q is odd;

$q^4 \leqslant |A(U_4(q))| \leqslant (q^2+1)q^2$ if $q > 3$ and is even;

$q^{[n^2/4]} \leqslant |A(U_n(q))| \leqslant (q^2+1)q^{[n^2/4]}$ for $n \geqslant 5$.

Remark 1. For classical groups, Theorem A is verified by direct computations via the estimates given in Theorem 3.1.

Remark 2. The groups for which exact estimates are obtained admit obtaining a description of greatest Abelian subgroups up to conjugation. For $A_p(A_n(q))$ and $A_p(C_n(q))$, this was done in [6]. Up to conjugation, a greatest Abelian subgroup is one of the following:

$A_p(A_{2n+1}(q)) = \langle X_r \mid r \geqslant r_{n+1}\rangle$;

$A_p(A_{2n}(q)) = \langle X_r \mid r \geqslant r_n\rangle$ or $A_p(A_{2n}(q)) = \langle X_r \mid r \geqslant r_{n+1}\rangle$;

$A_p(A_2(q)) = \langle x_{r_1}(a)x_{r_2}(a)x_{r_1+r_2}(b) \mid a \in F_q,\ b \in F_q\rangle$;

$A_p(C_n(q)) = \langle X_r \mid r \geqslant r_n\rangle$.

For the other cases, a description is immediately obtained thus.

Let $L_2(q) \cong SL_2(q)$ for q even. Choose a matrix A in $GL_2(q)$ such that $|A| = q^2 - 1$, that is, the element A generates a group $F_{q^2}^*$. It follows that $A(L_2(q)) = \langle A^{q-1}\rangle$, that is, the result is a cyclic group generated by A^{q-1}.

Let $L_3(q) \cong SL_3(q)$ for $(3, q-1) = 1$. Choose a matrix A in $GL_3(q)$ so that $|A| = q^3 - 1$, that is, the element A generates $F_{q^3}^*$. Then $A(L_3(q)) = \langle A^{q-1}\rangle$.

The proof of the theorem will proceed through a number of lemmas.

LEMMA 3.1. Let

$f_1(n,q) = \max(q^n - 1, (q-1)q^{[\frac{n^2}{4}]})$;

$f_2(n,q) = (2, q-1)q^{\frac{n(n+1)}{2}}$ for $n \geqslant 3$, $f_2(2,q) = (2,q-1)(q+1)q^2$, and $f_2(1,q) = \max(\delta, (2,q-1)q)$;

$f_3(n,q) = (2, n-1)\cdot 2\cdot\delta\cdot q^{[\frac{n}{2}]([\frac{n}{2}]-1)/2}$ if $n \geqslant 8$ and q is odd; $f_3(2k+1, q) = q^{\frac{k(k+1)}{2}}$ if $k \geqslant 2$ and q is even; $f_3(2k,q) = \delta q^{\frac{k(k-1)}{2}}$ if $k \geqslant 4$ and q is even; $f_3(1,q) = 2$, $f_3(2,q) = 2q$, $f_3(3,q) = 4q$, $f_3(4,q) = 2\cdot(q+1)\cdot q^2$, $f_3(5,q) = 4\cdot(q+1)\cdot q^2$, $f_3(6,q) = 2\cdot(q+1)^2\cdot q^3$, $f_3(7,q) = 4\cdot(q+1)^2\cdot q^3$ if q is odd; $f_3(1,q) = 1$, $f_3(2,q) = q+1$, $f_3(3,q) = q+1$, $f_3(4,q) = (q+1)q^2$, $f_3(6,q) = (q+1)^2q^3$ if q is even;

$f_4(n,q) = (2, n-1) \cdot 2 \cdot \delta^{(2,n)} \cdot q^{[\frac{n-1}{2}]([\frac{n-1}{2}]-1)/2+(2,n)}$ if $n \geqslant 8$ and q is odd; $f_4(n,q) = f_3(n,q)$ if $n \leqslant 7$ or q is even;

$f_5(n,q) = (q+1)(q^2+1)q^{[\frac{n^2}{4}]}$ if $n \geqslant 5$; $f_5(1,q) = q+1$, $f_5(2,q) = (q+1)^2$, $f_5(3,q) = (q+1)^3$, $f_5(4,q) = (q+1) \cdot q^4$ if $q \geqslant 3$; $f_5(4,2) = 81$.

Then, for any $j = 1,2,3,4,5$,

$$f_j(n+m,q) \geqslant f_j(n,q) f_j(m,q). \tag{1}$$

In proving the present lemma, we will use the following:

Proposition 3.1. Let $f(x)$ and $g(x)$ be polynomials of degrees n_1 and n_2, respectively. Suppose that $f(x)$ and $g(x)$, together with their derivatives, are not less than 0 for $x > 0$. Also, assume that $n_1 < n_2$ and $g(x_0) \geqslant f(x_0)$, $g'(x_0) \geqslant f'(x_0)$, ..., $g^{(n_1)}(x_0) \geqslant f^{(n_1)}(x_0)$ at some point $x_0 > 0$. Then $g(x) \geqslant f(x)$ for all $x \geqslant x_0$.

Proof. By assumption, $g^{(n_1+1)}(x) \geqslant 0$ for all $x > 0$, and $f^{(n_1+1)}(x) = 0$; therefore, $g^{(n_1+1)}(x) \geqslant f^{(n_1+1)}(x)$ for all $x \geqslant x_0$. Since $g^{(n_1)}(x_0) \geqslant f^{(n_1)}(x_0)$, we have $g^{(n_1)}(x) \geqslant f^{(n_1)}(x)$ for all $x \geqslant x_0$. And the required result will follow by repeating this argument for derivatives of lesser orders.

Proof of Lemma 3.1. We prove inequality (1) for the case $f_3(n,q)$, q is odd. For n, $m \geqslant 8$,

$$f_3(n,q) \cdot f_3(m,q) = 2(2,n-1)\delta q^{[\frac{n}{2}]([\frac{n}{2}]-1)/2} 2(2,m-1)\delta q^{[\frac{m}{2}]([\frac{m}{2}]-1)/2} \leqslant$$

$$2(2,n+m-1)\delta q^{[\frac{n+m}{2}]([\frac{n+m}{2}]-1)/2} = f_3(n+m,q);$$

therefore, it suffices to consider the case where $m \leqslant 7$. We claim, for instance, that $f_3(6,q) f_3(6,q) = 2(q+1)^2 q^3 \cdot 2(q+1)^2 q^3 \leqslant 2\delta q^{15} = f_3(12,q)$. In fact, by the definition of δ, we need only prove that

$$2(q+1)^3 \leqslant q^9. \tag{2}$$

It is not hard to see that the polynomials in q on the right- and left-hand sides of the inequality satisfy the conditions of the proposition at point $q = 2$; hence, inequality (2) is true for all $q \geqslant 2$.

Arguing similarly for other i, m, n, and q proves the lemma.

LEMMA 3.2. Let $G \leq GL_n(q)$, with G satisfying the following conditions:

(1) $G = H_s \times H_u$, where H_s and H_u are, respectively, semisimple and unipotent components of G;

(2) the nilpotency class of H_s does not exceed 2;

(3) H_u is Abelian.

Then $|G| \leqslant f_1(n,q)$.

Proof. Assume that the statement of the lemma is untrue, and G is its counterexample of minimal order.

By Lemma 2.5, the group H_s is decomposed into irreducible blocks. We can therefore assume that

$$H_s = \begin{pmatrix} H_{s_1} & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & H_{s_k} \end{pmatrix},$$

where all blocks of H_{s_j} are irreducible and numbered in order of increasing dimension. The so structured group H_s indicates that all matrices in H_s are in blockwise-diagonal form, and each block of H_{s_j} can be conceived of as a semisimple irreducible subgroup of the group $GL_{n_j}(q)$; any H_{s_j} has its nilpotency class at most 2. In addition, $n_1 + \dots + n_k = n$. Note that now $H_s \leq H_{s_1} \times \dots \times H_{s_k}$.

Further we may assume that all H_{s_j} coincide and hence $H_s \cong H_{s_1}$. Indeed, $H_u \neq \{1\}$, since otherwise $G = H_s$, and by Lemma 2.3, we would have $|G| \leqslant q^n - 1 \leqslant f_1(n,q)$, which contradicts the assumption that G is a counterexample. By assumption, $AB = BA$ for any matrices $A \in H_s$ and $B \in H_u$. Write B as follows:

$$B = \begin{pmatrix} B_{1_1} & B_{1_2} & \dots & B_{1_k} \\ B_{2_1} & B_{2_2} & \dots & B_{2_k} \\ & \dots & & \\ B_{k_1} & B_{k_2} & \dots & B_{k_k} \end{pmatrix},$$

where the dimension of a block coincides with that of a corresponding block in the representation of the group H_s, that is, B_{i_j} has n_j columns and n_i rows. By commutativity, $H_{s_i}B_{i_j} = B_{i_j}H_{s_j}$ and $H_{s_j}B_{j_i} = B_{j_i}H_{s_i}$. By the Schur lemma, the sets $B_{(j,i)} = \{B_{j_i} \mid B \in H_u\}$ and $B_{(i,j)} = \{B_{i_j} \mid B \in H_u\}$ form a division ring. Consequently, if dimensions of the blocks corresponding to groups H_{s_i} and H_{s_j} are different, or these groups are not conjugate, then $B_{(i,j)} = B_{(j,i)} = \{0\}$. In this way, if dimensions of the blocks do not coincide, or some blocks (groups) are not conjugate, then the group H_u, in the same basis as H_s, takes up the following form:

$$H_u = \begin{pmatrix} H_{u_1} & 0 \\ 0 & H_{u_2} \end{pmatrix}.$$

It follows that $G \leq G_1 \times G_2$, where G_1 and G_2 are subgroups in $GL_{n_1}(q)$ and $GL_{n_2}(q)$ satisfying the assumption of the lemma. Since G is a minimal counterexample, Lemma 3.1 implies

$$|G| \leqslant |G_1| \cdot |G_2| \leqslant f_1(n_1,q) \cdot f_1(n_2,q) \leqslant f_1(n_1+n_2,q) = f_1(n,q).$$

We have arrived at a contradiction with G being a minimal counterexample.

Thus, not only are dimensions equal for all blocks H_{s_j}, but also all subgroups of H_{s_j} are conjugate. Up to conjugation, therefore, the group H_s has the following form:

$$H_s = \underbrace{\begin{pmatrix} H_{s_1} & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & H_{s_1} \end{pmatrix}}_{k \text{ times}}. \tag{3}$$

Consequently, $H_s \cong H_{s_1}$. Assume that the dimension of H_{s_1} equals $\frac{n}{k}$. By Lemma 2.3, $|H_s| = |H_{s_1}| \leqslant (q^{\frac{n}{k}} - 1)$. Since all blocks B_{i_j} of the matrices in H_u form a division ring, and any finite division ring is a field, and, since they lie in $M_{\frac{n}{k}}(q)$, we may assume that these blocks form a subfield in $F_{q^{\frac{n}{k}}}$. The group H_s has the form (3); therefore, we can think that $H_u \leq GL_k(q^{\frac{n}{k}})$. From (1) of Lemma 2.2, it follows that $|H_u| \leqslant q^{\frac{n}{k}[\frac{k^2}{4}]}$. This implies that

$$|G| \leqslant |H_s| \cdot |H_u| \leqslant (q^{\frac{n}{k}} - 1) \cdot q^{\frac{n}{k}[\frac{k^2}{4}]} \leqslant f_1(n,q).$$

Contradiction. Hence such G does not exist, proving the lemma.

LEMMA 3.3. Suppose $G \leq Sp_{2n}(q)$ and G satisfies the following conditions:

(1) $G = H_s \times H_u$, where H_s and H_u are, respectively, semisimple and unipotent components of G;

(2) the nilpotency class of H_s is at most 2;

(3) H_u is Abelian.

Then $|G| \leqslant f_2(n,q)$.

Proof. Assume that the statement of the lemma is untrue, and G is its minimal counterexample. If the group H_s has no proper submodules then it is a unique irreducible block.

Among all proper H_s-submodules, if any, choose an H_s-submodule U of minimal dimension. Consider $D = C_{H_s}(U)$. If $D = \{1\}$ then H_s acts faithfully on U. By Lemma 2.3, $|H_s| < q^{\dim(U)} = q^k$. As in the proof of the previous lemma, we can obtain $|G| \leqslant |H_s| \cdot |H_u| \leqslant (q^k - 1)q^{k[\frac{n^2}{k^2}]} \leqslant f_2(n,q)$. (The latter inequality holds for $n \geqslant 3$ and $k \geqslant 2$.) For $n = 1$, the situation that obtains is the known case where $Sp_2(q) \cong SL_2(q)$. For the case where $n = 2$ and $k = 2$, note that $H_s * Z(GL_4(q))$ satisfies the estimate $|H_s * Z(GL_4(q))| \leqslant (q^2 - 1)$. Since $|H_s \cap Z(GL_4(q))| \leqslant (2, q-1)$, we have $|H_s| \leqslant (2,q-1)(q+1)$, that is, the inequality $|G| \leqslant f_2(n,q)$ holds in this case, too. If $k = 1$ then $|H_s| = (2, q-1)$. By (2) of Lemma 2.2, $|H_u| \leqslant q^{\frac{n(n+1)}{2}}$, and so $|G| \leqslant (2,q-1)q^{\frac{n(n+1)}{2}} \leqslant f_2(n,q)$.

We can thus assume $D > \{1\}$. Then $C_V(D)$ and $[V, D]$ are proper nontrivial H_s-submodules. By Lemma 2.6, $V = C_V(D) \oplus^{\perp} [V, D]$; hence, H_s can be represented as a subgroup in the group $H_{s_1} \times H_{s_2}$, where H_{s_1} and H_{s_2} are semisimple class 2 nilpotent subgroups in the groups $Sp_{2n_1}(q)$ and $Sp_{2n_2}(q)$, respectively. Repeating the above argument for H_{s_1} and H_{s_2} yields $H_s \leq H_{s_1} \times \ldots \times H_{s_k}$, where all H_{s_j} are irreducible, semisimple, nilpotent subgroups of class 2 in $Sp_{2n_j}(q)$.

We make use of the estimate obtained in Lemma 3.2 and the fact that H_u can be treated as a subgroup of $GL_k(q^{\frac{2n}{k}})$, that is, $|H_u| \leqslant q^{\frac{2n}{k}[\frac{k^2}{4}]}$. As indicated above, H_s has the form

$$H_s = \underbrace{\begin{pmatrix} H_{s_1} & & 0 \\ & \ddots & \\ 0 & & H_{s_1} \end{pmatrix}}_{k \text{ times}}$$

that is, $H_s \cong H_{s_1}$, and hence $|H_s| \leqslant \delta^{\frac{n}{k}}$ by Lemma 2.3. It follows that $|G| \leqslant \delta^{\frac{n}{k}} \cdot q^{\frac{2n}{k}[\frac{k^2}{4}]} \leqslant q^{\frac{n^2}{2} + \frac{n}{2}} \leqslant f_2(n,q)$ for $n \geqslant 3$. For $n = 2$ and $k = 2$, $|G| \leqslant \delta \cdot q^2 \leqslant f_2(n,q)$. The lemma is proved.

LEMMA 3.4. Let G be a subgroup of the orthogonal group $O_n(q)$ satisfying the following conditions:

(a) $G = H_s \times H_u$, where H_s and H_u are, respectively, semisimple and unipotent components of G;

(b) the nilpotency class of H_s is at most 2;

(c) H_u is Abelian.

Then the following properties hold:

(1) if $G \leq O^+_{2n}(q)$ then $|G| \leqslant f_3(2n,q)$;

(2) if $G \leq O^-_{2n}(q)$ then $|G| \leqslant f_4(2n,q)$;

(3) if $G \leq O_{2n+1}(q)$ then $|G| \leqslant f_3(2n+1,q)$.

Proof. Assume that the conclusion of the lemma is untrue, and G is a counterexample of minimal order.

If H_s lacks proper submodules then H_s is a unique irreducible block itself.

Among all proper H_s-submodules, if any, choose an H_s-submodule U of minimal dimension. Consider $D = C_{H_s}(U)$. If $D = \{1\}$ then H_s acts faithfully on U. By Lemma 2.3, $|H_s| < q^{\dim(U)} = q^k$. Let $G \leq O^+_{2n}(q)$. As in the proof of Lemma 3.2, we can obtain $|G| \leqslant |H_s| \cdot |H_u| \leqslant (q^k - 1)\min(q^{k[\frac{n^2}{k^2}]}, q^{\frac{n(n-1)}{2}}) \leqslant f_i(n,q)$. (The latter inequality holds for $n \geqslant 8$ and $k \geqslant 2$.) The cases where $G \leq O^-_{2n}(q)$ and $G \leq O_{2n+1}(q)$ admit a similar treatment. Small-dimensional cases will be treated separately, a bit later. If $k = 1$ then $|H_s| = (2, q-1)$. And, taking the value of the order of the greatest Abelian unipotent subgroup from Lemma 2.2, we arrive at $|G| \leqslant (2,q-1)|H_u| \leqslant f_i(n,q)$.

We can thus assume that $D > \{1\}$. Then $C_V(D)$ and $[V, D]$ are proper nontrivial H_s-submodules. By Lemma 2.6, $V = C_V(D) \oplus^{\perp} [V, D]$; hence, H_s is represented as a subgroup in $H_{s_1} \times H_{s_2}$, where H_{s_1} and H_{s_2} are subgroups in $O_{n_1}(q)$ and $O_{n_2}(q)$, respectively. Repeating the above argument for H_{s_1} and H_{s_2} yields $H_s \leq H_{s_1} \times \ldots \times H_{s_k}$, where all H_{s_j} are subgroups in $O_{n_j}(q)$ and are irreducible.

Since G is minimal, again we may assume that H_s has the form

$$H_s = \underbrace{\begin{pmatrix} H_{s_1} & & 0 \\ & \ddots & \\ 0 & & H_{s_1} \end{pmatrix}}_{k \text{ times}}$$

that is, $H_s \cong H_{s_1}$, and by Lemma 2.3, $|H_s| \leqslant \delta^{[\frac{n}{2k}]}$. It follows that $|G| \leqslant \delta^{[\frac{n}{2k}]} \min(q^{\frac{n}{k}[\frac{k^2}{4}]}, |H_u|) \leqslant f_3(n, q)$. (The latter inequality holds for $n \geqslant 8$.)

For groups $O_n^-(q)$, we note, all blocks of the group H_s, for $k = \frac{n}{2}$, cannot be conjugate: the dimension of a maximal isotropic subspace is equal to $\frac{n}{2} - 1$, and so $G \leq G_1 \times G_2$, where $G_1 \leq O_{n_1}^+$ and $G_2 \leq O_{n_2}^-$, whence $|G| \leqslant f_3(n_1, q) \cdot f_4(n_2, q) \leqslant f_4(n, q)$. For $k > \frac{n}{2}$, we have $|G| \leqslant \delta^{[n/2k]} \cdot q^{\frac{n}{k}[\frac{k^2}{4}]} \leqslant f_4(n, q)$. (The latter inequality holds for $n \geqslant 8$).

We proceed to small-dimensional cases. Obtaining estimates specified in the lemma does not depend on which of the groups $O_n^+(q)$ or $O_n^-(q)$ is chosen, and so below we do not discriminate between these. Unless otherwise stated, we adhere to the notation used in proving the first part of the lemma. For instance, U denotes a proper H_s-submodule of minimal dimension. The function $f_i(n, q)$ was constructed in a way that it is not less than a maximum of the order of a semisimple group of nilpotency class at most 2 and the order of an Abelian unipotent subgroup, multiplied by an order of the center $Z(O_n(q))$. Therefore, we need only handle the case where $H_s > Z(O_n(q))$, $H_u \neq \{1\}$. In addition, Lemma 3.1 allows us to treat the cases with $G \leq G_1 \times G_2$, where G_1 and G_2 are subgroups of orthogonal groups in a smaller dimension.

For groups $O_1(q)$, $O_2(q)$, and $O_3(q)$, nontrivial cases are an impossibility. For $O_5(q)$ and $O_7(q)$, every subgroup fitting in the nontrivial case is decomposable, since 5 and 7 are primes, and so we may well omit them.

Consider a group $O_4(q)$. A nontrivial case appears when $\dim(U) = k = 2$. Let $C_{H_s}(U) = \{1\}$; then $H_s \leq GL_2(q)$ and $|H_s * Z(GL_2(q))| \leqslant q^2 - 1$. Since $|H_s \cap Z(GL_2(q))| \leqslant (2, q-1)$, we have $|H_s| \leqslant (2, q-1)(q+1)$. In [8], it was proved that $|H_u| \leqslant q^2$, yielding the required estimate. Now let $C_{H_s}(U) \neq \{1\}$. Then the group H_s takes up the blockwise-diagonal form, and blocks along the diagonal coincide. Hence $|H_s| \leqslant \delta$ and $|H_u| \leqslant q^2$, yielding the required estimate again.

Finally, consider $O_6(q)$. Nontrivial cases are realizable with $\dim(U) = k = 2$ and $\dim(U) = k = 3$. In the former case, arguing in a way we did for $O_4(q)$ yields $|H_s| \leqslant (2, q-1)(q+1)$. In [8], it was proved that $|H_u| \leqslant q^4$, yielding the required value. Let $k = 3$. If $C_{H_s}(U) \neq \{1\}$ then H_s splits into equal blocks; hence, $|H_s| \leqslant (2, q-1)\delta$. We can assume $H_u \leq GL_2(q^3)$; therefore, $|H_u| \leqslant q^3$. Consequently, the statement of the lemma is true. Let $C_{H_s}(U) = \{1\}$. Then $H_s \leq GL_3(q)$, $|H_s * Z(GL_3(q))| \leqslant q^3 - 1$, $|H_s \cap Z(GL_3(q))| \leqslant (2, q-1)$; hence, $|H_s| \leqslant (2, q-1)\frac{q^3-1}{q-1} \leqslant (2, q-1)(q+1)^2$. Moreover, again $|H_u| \leqslant q^3$, yielding the required value. The lemma is proved.

For unitary groups — denoted by $U_n(q^2)$ throughout the proof of Lemma 3.5 — a statement similar to Lemma 2.6 holds; see [9]. Again, the bound for the order of a maximal Abelian semisimple subgroup in a unitary group, which we are able to derive from Lemma 2.3, is equal to just $q^{2n} - 1$. This is too big for groups in a small dimension, and so we need a somewhat refined estimate for the order of a semisimple group.

First note that if H_s satisfies the conditions

(1) $(q, |H_s|) = 1$, and

(2) $H_s/(H_s \cap Z(U_n(q^2)))$ is an Abelian group,

then its order does not exceed $(q+1)^n$. Indeed, by Lemma 2.4, the order of a greatest Abelian semisimple subgroup in ${}^2A_{n-1}(q^2)$ is at most $\frac{(q+1)^{n-1}}{|Z(SU_n(q^2))|}$, and $|U_n(q^2) : SU_n(q^2)| = q+1$, whence the result.

LEMMA 3.5. Let $G \leq U_n(q^2)$ and G satisfy the following conditions:

(1) $G = H_s \times H_u$, where H_s and H_u are, respectively, semisimple and unipotent components of G;

(2) $H_s/(H_s \cap Z(U_n(q^2)))$ is Abelian;

(3) H_u is Abelian.

Then $|G| \leqslant f_5(n, q)$.

Proof. As above, assume that the statement of the lemma is untrue, and G is its counterexample of minimal order.

Let U be a proper H_s-submodule of minimal dimension. If $\dim(U) = n$ then $H_u = \{1\}$, and the statement of the lemma is true for G. We therefore let $\dim(U) = k < n$. If $k = 1$ then $H_s \leq Z(GL_n(q^2))$, that is, H_s is a scalar matrix group, since otherwise the G would split into

$$\begin{pmatrix} G_1 & 0 & & 0 \\ 0 & G_2 & & 0 \\ & & \ddots & \\ 0 & 0 & & G_m \end{pmatrix}.$$

Each block corresponds to a subspace of eigenvectors for some matrix in H_s, in which case different blocks correspond to different eigenvalues, and the vectors corresponding to the various eigenvalues are orthogonal. Therefore, we may assume that the blocks G_i are subgroups in unitary groups of smaller dimensions, whence $G \leq G_1 \times \ldots \times G_m$, and hence $|G| \leqslant |G_1| \cdot \ldots \cdot |G_m| \leqslant f_5(n_1, q) \cdot \ldots \cdot f_5(n_m, q) \leqslant$ (by Lemma 4.1) $f_5(n, q)$, which contradicts the choice of G for $m > 1$. If $m = 1$ then $|H_s \cap Z(GL_n(q^2))| \leqslant q+1$, by Lemma 2.2, $|H_u| \leqslant q^{[\frac{n^2}{4}]}$, and the statement of the lemma is thus true.

If $k > 1$, we repeat the argument of Lemma 3.2 to see that

$$|G| \leqslant \frac{q^{2k}-1}{q-1} q^{2k[\frac{n^2}{4k^2}]},$$

which proves the lemma for $n \geqslant 5$. Here, just as was done for orthogonal groups, small-dimensional cases will be treated separately.

In fact, we need only consider the case where G is indecomposable, $H_s \neq Z(U_n(q^2))$, and $H_u \neq \{1\}$. The sole unitary group of small dimension satisfying the above conditions is $U_4(q^2)$.

A nontrivial case is realized with $\dim(U) = 2$. If $C_{H_s}(U) \neq \{1\}$ then H_s splits into equal blocks, which are subgroups in $U_2(q^2)$. Therefore, $|H_s| \leqslant (q+1)^2$. The group H_u can be treated as a subgroup of $GL_2(q^4)$. In some basis for a 4-dimensional vector space over F_{q^2}, the group in question takes up the form

$$\begin{pmatrix} E & B \\ 0 & E \end{pmatrix},$$

where the first two vectors of the basis are taken from the module U, and the next two — from its orthogonal complement. Hence the matrix of unitary form in that basis is shaped thus:

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

Direct computations show that $B = \{0\}$, hence $H_u = \{1\}$, and the statement of the lemma is thus true.

We let $C_{H_s}(U) = \{1\}$. The group H_s takes up the form

$$\begin{pmatrix} H_{s_1} & 0 \\ 0 & H_{s_1} \end{pmatrix}.$$

In the same basis, H_u has a presentation by matrices of the form

$$\begin{pmatrix} E & B \\ 0 & E \end{pmatrix}.$$

We can write $V = U \oplus U_1$, where U and U_1 are proper H_s-submodules corresponding to irreducible blocks. We introduce some notation. Denote by $\overline{B}$ the matrix

$$\begin{pmatrix} E & B \\ 0 & E \end{pmatrix};$$

u (without indices) is an element of U, u_1 is one of U_1, and $\overline{u_1} = u\overline{B} - u$ is an image of u in U_1 under the action of some matrix $\overline{B}$ in H_u. Note that U_1 is a fixed-point space for the group H_u. Consider an Hermitian product (u_1, w), where u_1 and w are arbitrary vectors in U_1 and U, respectively. We then have $(u_1, w) = (u_1\overline{E}, w\overline{E}) = (u_1, w) + (u_1, \overline{w_1})$; consequently $(u_1, \overline{w_1}) = 0$. Since u_1 and w are arbitrary, the space U_1 is isotropic. Consider (u, u). For any matrix $\overline{B}$ in H_u and for every vector u in U, we have $(u, u) = (u\overline{B}, u\overline{B}) = (u + \overline{u_1}, u + \overline{u_1}) = (u, u) + (u, \overline{u_1}) + (\overline{u_1}, u)$; consequently, $(u, \overline{u_1}) + (\overline{u_1}, u) = 0$. Since $(u, \overline{u_1}) = (\overline{u_1}, u)^q$, it follows that $(u, \overline{u_1})[1 + (u, \overline{u_1})^{q-1}] = 0$. This implies either $(u, \overline{u_1}) = 0$ or $(u, \overline{u_1})^{q-1} = -1$. In the former case the unitary form should be degenerate, which it is not.

We handle the second case. Let $(u, \overline{u_1}) = x$. Then $1 + x^{q-1} = 0$, $1 = x^{q^2-1}$, hence $x^{q-1}(x^{q+1} + 1) = 0$, whence $x^2 = 1$. Since $x^{q-1} = -1$, for q odd, we arrive at a contradiction. If q is even, we obtain $x = 1$, and then

$$(u, \overline{u_1}) = 1. \tag{4}$$

If $(u, \overline{u_1}) = 1$ and $(u, \overline{v_1}) = 1$, then $(u, \overline{u_1} - \overline{v_1}) = 0$. Hence $\overline{u_1} - \overline{v_1}$ lies in the subspace orthogonal to u. Also, U_1 is an isotropic subspace; therefore, the dimension of the orthogonal complement of u in U_1 is at most 1. Hence the number of vectors in U_1 satisfying (4) is at most q^2. The number of such vectors is not less than $|H_u|$. In fact, if $\overline{u_1} = \overline{u_1}'$ for some matrices $\overline{B_1}$ and $\overline{B_2}$ in H_u, then $\overline{B_1 + B_2} \in H_u$, since $\{B\}$ is a field. This implies that $u\overline{B_1 + B_2} = u$, and hence $B_1 = B_2$. Thus, different elements in H_u send the vector $u \in U$ to different elements $\overline{u_1} \in U_1$; hence, $|H_u| \leqslant |\text{the orthogonal complement of } u \text{ in } U_1| = q^2$. The lemma is proved.

Proof of Theorem 3.1. Let A be an Abelian subgroup of a classical group G. Then $A = H_s \times H_u$, where H_s and H_u are, respectively, semisimple and unitary subgroups in A. Let $\overline{A}$ be a preimage of A in the universal central extension (cf. [12] for definition) under the natural homomorphism. Then $\overline{A}$ satisfies the conditions of Lemmas 3.2-3.5, and the estimate for $|A|$ is obtained from one in the appropriate lemma, by dividing by the order of the center of a suitable group. The theorem is proved.

4. EXCEPTIONAL CHEVALLEY GROUPS

We bring out known bounds for the orders of $A_p(G)$, where G is a Chevalley group of exceptional type; see [12].

LEMMA 4.1. Let $G = L(q)$ be a Chevalley group of exceptional type over a field F_q and p be the characteristic of F_q. Then the following estimates are correct:

(1) if $p \neq 3$ then $|A_p(G_2(q))| = q^3$;
(2) if $p = 3$ then $|A_3(G_2(q))| = q^4$;
(3) $|A_2(^2B_2(2^{2n+1}))| = 2^{3n+1}$;
(4) $|A_3(^2G_2(q))| = q^2$, $q = 3^{2n+1}$;
(5) $|A_2(^2F_4(q))| = q^5$, $q = 2^{2n+1}$;
(6) $|A_p(^3D_4(q^3))| = q^5$;
(7) if $p = 2$ then $q^{12} \leqslant |A_2(F_4(q))| \leqslant q^{16}$;
(8) if $p = 2$ then $q^{12} \leqslant |A_2(^2E_6(q^2))| \leqslant q^{20}$;
(9) if $p = 2$ then $|A_2(E_6(q))| = q^{16}$;
(10) if $p = 2$ then $|A_2(E_7(q))| = q^{27}$;
(11) if $p = 2$ then $|A_2(E_8(q))| = q^{36}$.

Let G be a Chevalley group of normal type over a field F_q and k be its Lie rank. Recall that any element h in the Cartan subgroup H in G (cf. [11] for definition) has the form $h(\chi)$, where χ is defined as follows. Let P be a set of all linear combinations of fundamental roots $\{r_1, \dots, r_k\}$ with integral coefficients and let Q be a set of rational combinations of the same roots. Then $\{\chi\}$ is precisely the set of homomorphisms of P into F_q^*, which are extended to homomorphisms of Q into F_q^*. For any $r \in \Phi$ and $t \in F_q$, the equality $h(\chi)x_r(t)h^{-1}(\chi) = x_r(\chi(r)t)$ holds.

LEMMA 4.2. Let $h(\chi) \neq 1$ be some element in a Cartan subgroup of the Chevalley group $L(q)$ and U be a maximal unipotent subgroup in $L(q)$. Then the centralizer $C_U(h(\chi))$ of $h(\chi)$ in U is equal to U_{Φ_χ}, where $\Phi_\chi = \{r \mid \chi(r) = 1\} \subset \Phi$ and $U_{\Phi_\chi} = \langle X_r \mid r \in \Phi_\chi \rangle$. Also, Φ_χ is a subsystem of Φ.

Proof. By the above, it suffices to prove that Φ_χ is a subsystem of Φ. Recall that the subset Ψ of a finite-dimensional Euclidean space over $\Re$ is called a root system if the following conditions are satisfied:

(1) Ψ is finite, generates the vector space as a whole, and is freed of zero vectors;
(2) if $r \in \Psi$ then the vectors multiple to r in Ψ are just $\pm r$;
(3) if $r \in \Psi$ then the reflection w_r leaves the set Ψ invariant, where $w_r(s) = s - \frac{2(r,s)}{(r,r)}r$;
(4) if $r, s \in \Psi$ then $\frac{2(r,s)}{(r,r)}$ is an integer.

Conditions (1), (2), and (4) are satisfied since Φ_χ is a subset of Φ. We check (3). For r and s, we have $\chi(r) = \chi(s) = 1$. Then

$$\chi(w_r(s)) = \chi\left(s - \frac{2(r,s)}{(r,r)}r\right) = \chi(s) \cdot \chi^{-1}\left(\frac{2(r,s)}{(r,r)}r\right) = [\underbrace{\chi(r) \cdot \dots \cdot \chi(r)}_{\frac{2(r,s)}{(r,r)} \text{ times}}]^{-1} = 1.$$

The lemma is proved.

LEMMA 4.3 [12, Sec. 2.4]. Let Q be a nontrivial p-subgroup of a Chevalley group G. Then G contains a proper parabolic subgroup P such that $Q \leq O_p(P)$ and $N_G(Q) \leq P$. Here, $O_p(P)$ is a maximal normal p-subgroup of P.

LEMMA 4.4 [12, Sec. 2.2]. Let P_I be a parabolic subgroup of a Chevalley group G of normal type. Then it has the form $P_I = L_I U_I H$, where L_I is the Levi factor, which is a central product of Chevalley groups of which each is obtained from a connected component in the Dynkin diagram for G, by banishing vertices which enter the set I, $U_I = \langle X_r \mid r = \sum m_i r_i,\ m_i > 0$, for some $i \in I\rangle$. Again, for any Chevalley group G, we have $P_I = L_I U_I H$, where $H \leq N_G(L_I)$, $U_I = O_p(P_I)$ and $U_I \cap L_I H = 1$, $N_G(U_I) = P_I$, where L_I is a central product of Chevalley groups.

In what follows, we write r_0 to denote a longest root in the root system Φ for a Chevalley group in question. Note that $Z(U) = X_{r_0}$. We describe how the normalizer of X_{r_0} is structured; see [12].

Let G be a Chevalley group of normal type. Then $N_G(x_{r_0}(t)) = P_{r_0} = N_G(X_{r_0})$ for $t \neq 0$. Here, the subgroup P_{r_0} is obtained as follows: first, an extended Dynkin diagram is built up by adding to the ordinary diagram a vertex $-r_0$ bridged to the other vertices in the usual way, and then P_{r_0} is found by discarding the vertices bridged to $-r_0$.

We proceed to obtain estimates of $|A_p(G)|$ for the cases unmentioned in Lemma 4.1.

LEMMA 4.5. Let q be odd and $q = p^\alpha$. Then the following estimates are correct:

(1) $q^9 \leqslant |A_p(F_4(q))| \leqslant q^{14}$;
(2) $q^{16} \leqslant |A_p(E_6(q))| \leqslant q^{20}$;
(3) $q^{27} \leqslant |A_p(E_7(q))| \leqslant q^{32}$;
(4) $q^{36} \leqslant |A_p(E_8(q))| \leqslant q^{61}$;
(5) $q^{12} \leqslant |A_p({}^2E_6(q^2))| \leqslant q^{20}$.

Proof. For all groups of normal type, the estimates are obtained along similar lines, which we sketch below.

Consider a parabolic subgroup $P_{r_0} = U_{r_0} L_{r_0} H$, where L_{r_0} is a Chevalley group for which the order of an Abelian p-subgroup is a known value. The group U_{r_0} possesses the property $[U_{r_0}, U_{r_0}] = Z(U_{r_0}) = X_{r_0}$; therefore, U_{r_0}/X_{r_0} can be treated as a vector space over F_q, with basis $\{\overline{x_r}\}$ and summation given by the rule $\alpha\overline{x_r} + \beta\overline{x_s} = \overline{x_r(\alpha)x_s(\beta)}$. Again, on that space, the nondegenerate antisymmetric bilinear form can be defined by setting $x_{r_0}((\bar{u}, \bar{v})) = [u, v]$.

An Abelian subgroup can then be thought of as an isotropic subspace of known dimension. In this way we obtain estimates for the orders of Abelian p-subgroups in L_{r_0} and in U_{r_0}. To obtain the ultimate value, we need only take a product of these two.

The case of a group ${}^2E_6(q^2)$ is somewhat more complicated. The root system has type F_4, but it consists, not of roots, but of equivalence classes. The equivalence classes are of two types:

$R = \{r\}$: if $r = r^\sigma$, where σ is an automorphism of a root system generated by the symmetry of a Dynkin diagram of type E_6, then R is a class of normal type;

$R = \{r, \bar{r}\}$: if $r \neq r^\sigma = \bar{r}$, then R is a class of special type.

The unipotent group U is generated by root subgroups X_R. Also, the basis classes R_1, R_2, R_3, and R_4 corresponding to fundamental roots in a root system of type F_4 are chosen thus:

$$R_1 = \{r_4\},\ R_2 = \{r_3\},\ R_3 = \{r_2, r_5\},\ R_4 = \{r_1, r_6\}.$$

If the class R is in form $\{r, \bar{r}\}$, then, for definiteness, we assume that $r < \bar{r}$, where the order is given thus: $r > 0 \Leftrightarrow r = \sum \alpha_i r_i$ and the first α_i distinct from 0 is greater than 0.

Proposition 4.1. Let

$$[x_R(t), x_S(u)] = \prod_{iR+jS\in F_4^+} x_{iR+jS}(C_{jiSR} f_{i_j}(t,u)),$$

where C_{jiSR} and $f_{i_j}(t,u)$ are defined as follows:

(1) if R and S are classes of normal type then $f_{1_1}(t,u) = -ut$ and $C_{11SR} = C_{11sr}$;

(2) if R is a class of special type and S a class of normal type then $f_{1_1}(t,u) = -ut$, $f_{2_1}(t,u) = ut\bar{t}$ and $C_{11SR} = C_{11sr}$, $C_{12SR} = C_{11\bar{r}r+s}C_{11sr}$;

(3) if R is normal and S special then $f_{1_1}(t,u) = -ut$, $f_{1_2}(t,u) = tu\bar{u}$ and $C_{11SR} = C_{11sr}$, $C_{21SR} = C_{11\bar{s}r+s}C_{11sr}$;

(4) if R, S, and $R+S$ are classes of special type then $f_{1_1}(t,u) = -ut$ and $C_{11SR} = C_{11s\bar{r}}$;

(5) if R and S are special and $R+S$ normal then $f_{1_1}(t,u) = -(u\bar{t} + \bar{u}t)$ and $C_{11SR} = C_{11s\bar{r}}$.

The proof is by an exhaustive search of all cases, combined with applying properties of structural constants in a root system of type E_6 and the commutator Chevalley formula; see [11, 5.2.2]. Note only that $|C_{ijSR}| = 1$, that is, they do not vanish for any characteristic of F_q.

Write $U_{R_1} = \langle X_R \,|\, R \geq R_1\rangle$. Then $U_{{}^2E_6(q^2)} = U_{{}^2A_5(q^2)}U_{R_1}$, and $U_{R_1} \lhd U_{{}^2E_6(q^2)}$. It follows that $A_p({}^2E_6(q^2)) = A_p({}^2A_5(q^2))A$, where A is an Abelian subgroup in U_{R_1}. Let $W = \langle X_R \,|\, R$ is a class of special type$\rangle$. Then $\overline{U_{R_1}} = U_{R_1}/W$ is Abelian, and on it we can define the antisymmetric bilinear product, as we did for the groups of normal type; an image of the Abelian subgroup, in this event, is an isotropic subspace of known dimension.

Finally, consider $\overline{W} = W/X_{R_0}$. This is again a vector space (but now over F_{q^2}), on the basis vectors of which the scalar product first is defined by setting $(\overline{x_R}, \overline{x_S}) = C_{11SR}$ and then is extended, by linearity, to the whole space. Since W satisfies the identities

$$[u_1u_2, v] = [u_1, v][u_2, v], \ [u, v_1v_2] = [u, v_1][u, v_2],$$

we can prove that for odd q, $(\bar{u}, \bar{v}) = 0$ iff $[u, v] = 1$. Summing up all the above facts yields the estimate specified in the lemma.

Proof of Theorem A. Consider a group $G_2(q)$. Since our theorem is true for the greatest Abelian semisimple and unipotent groups, we may assume that the Abelian subgroup A has "mixed" order, that is, it has both semisimple and unipotent elements. By Lemma 4.3, A lies in some proper parabolic subgroup of $G_2(q)$. The maximal subsystem in G_2 is A_2. Therefore, if $h(\chi) \in A$ for some $h(\chi) \neq 1$, then, by Lemmas 2.2 and 4.2, the order of an Abelian p-subgroup in A is at most q^2, that is, $|A| \leqslant (q^2-1)q^2$, which satisfies the statement of Theorem A. We now assume that such $h(\chi)$ is missing. Note that $N_G(X_{r_0}) = A_1(q)U_{r_2}H$. If A is an Abelian subgroup containing $x_{r_0}(t)$, $t \neq 0$, then $A \leq N_G(X_{r_0})$. It follows that $A \leq H_s \times H_u$, where $H_s \leq SL_2(q)$, $H_u \leq U_{r_2} = \langle X_r \,|\, r \leq r_2\rangle$, and $A \cap H = 1$. It is easy to check that $|H_u| \leqslant q^3$, $|H_s| \leqslant q+1$, whence $|A| \leqslant (q+1)q^3$. Hence $|A|^3 < |G|$. Let $A \cap X_{r_0} = \{1\}$. Writing $A = H_s \times H_u$, with $|H_s| \leqslant q+1$, $|H_u| \leqslant q^3$, again we obtain $|A|^3 < |G|$. (Indeed, if q is not the power of 3 then $|H_u| \leqslant q^2$.) Theorem A is thus proved for $G_2(q)$.

We proceed to groups $F_4(q)$, where $q = p^\alpha$ and is odd. We may assume that the Abelian subgroup A has mixed order, that is, $(|A|, p) = p$ and A is not a p-group. By Lemma 4.3, A lies in some proper parabolic subgroup. If $h(\chi) \neq 1 \in A$ exists, by Lemma 4.2, H_u is then a unipotent subgroup of A and lies in the unipotent subgroup corresponding to some root subsystem of F_4. The maximal subsystem in F_4 is B_4, hence $|H_u| \leqslant q^6$ by Lemmas 2.2 and 4.2, and so $|A| \leqslant q^6(q+1)^3(q-1)$, $|A|^3 < |F_4(q)|$. If such $h(\chi)$ does not exist, then $A \leq U_IL_I$, and hence $|A| \leqslant q^{14}(q+1)^3$ and $|A|^3 < |F_4(q)|$ by Lemmas 2.4 and 4.5.

We turn to groups $F_4(2^t)$. The orders of greatest Abelian 2- and 2′-subgroups, when raised to the third power, do not exceed the order of a whole group. If there exists an Abelian subgroup A for which the inequality $|A|^3 \geqslant |G|$ holds, then the order of A is of mixed type, that is, the order of A is even, but A

is not a 2-group. Since A is Abelian, it contains an element of mixed order. The orders of centralizers of elements in $F_4(2^t)$ are given in [13]. The orders of centralizers of elements of mixed type do not exceed $q^9(q+1)(q^2-1)(q^4-1)$, and the cube of the latter is less than the order of the whole group.

Consider $E_6(q)$, where q is odd. If there exists $h(\chi) \neq 1 \in A$ then, again, H_u is a subgroup in some unipotent group corresponding to a proper root subsystem. The maximal root subsystem in E_6 is D_5. By Lemmas 2.2 and 4.2, we then have $|H_u| \leqslant q^{10}$, $|A| \leqslant q^{10}(q+1)^5(q-1)$, and $|A|^3 < |E_6(q)|$. If such $h(\chi)$ does not exist, then $A \leq U_I L_I$. If $A \geq X_{r_0}$ then $A \leq U_{r_4} L_{r_4}$. And, if we address the proof of Lemma 4.5, we see that either $|A| \leqslant q^{11}(q+1)^5$ or $|A| \leqslant q^{20}(q+1)^4$. In either case A satisfies the statement of Theorem A. Finally, if $A \cap X_{r_0} = \{1\}$ then $|A| \leqslant q^{19}(1+q)^5$, and the statement of Theorem A is satisfied again.

For the other exceptional groups of normal type, the desired estimate is obtained by taking a product of known values of the orders of greatest Abelian p- and p'-groups.

The groups ${}^2B_2(2^{2n+1})$ and ${}^2G_2(q)$, treated in detail in [14, 15], satisfy the statement of Theorem A, $|A({}^2B_2(2^{2n+1}))| = 2^{3n+1}$, and $|A({}^2G_2(q))| = q^2$. For ${}^2F_4(q)$, the value is found by taking a product of the orders of greatest Abelian p- and p'-subgroups.

For the other exceptional Chevalley groups of twisted type, again we make use of parabolic subgroups. For the latter, however, there is no such accurate description as have we do for the case of groups of normal type. Still we can exploit the fact that they lie in parabolic subgroups of the initial groups of normal type.

Consider a group ${}^2E_6(q^2)$. If $h(\chi) \neq 1 \in A$ exists then H_u is contained in some unipotent subgroup of $E_6(q^2)$ corresponding to a proper root subsystem. Searching all the subsystems and taking into account the condition that H_u is twisted, we obtain $|H_u| \leqslant q^9$. Then $|A| \leqslant q^9(q+1)^6$, and Theorem A is true for the present case. If such $h(\chi)$ is missing then $|A| \leqslant q^{20}(q+1)^5$, and the desired inequality obtains for $q \geqslant 4$. For $q = 3$, if $A \geq X_{R_0}$ (for definition of X_{R_0}, see proof of Lemma 4.5), then $A \leq U_{R_1} L_{R_1}$. There are two options: $|A| \leqslant 3^{11} \cdot 4^5$ or $|A| \leqslant 3^{20} \cdot 4^4$. The statement of Theorem A is true in both of these cases. If $A \cap X_{R_0} = \{1\}$ then $|A| \leqslant 3^{19} \cdot 4^5$, which satisfies the statement of Theorem A. The case of a group ${}^2E_6(2^2)$ will be treated together with sporadic groups.

We finish our study of exceptional Chevalley groups by treating ${}^3D_4(q^3)$. A greatest subsystem in D_4, which is sent to itself under the action of a graph automorphism, is $A_1 \times A_1 \times A_1$. Therefore, if $h(\chi) \neq 1 \in A$ exists then $|A| \leqslant q^3(q+1)^2(q-1)$, and the desired inequality holds. If not, then $A \leq U_I L_I$, $|A| \leqslant q^5(q+1)^2$, which confirms the statement of Theorem A for all q except 2. The case of a group ${}^3D_4(2^3)$ will be treated together with sporadic groups.

5. SPORADIC GROUPS

Orders of sporadic groups will be estimated by using results of [6]. Estimates for all groups are obtained following essentially the same line of argument, and so below we concentrate on just two: Co_2 and Fi_{23}. (On them, we demonstrate all tricks through which we obtain the desired estimates.)

1. Co_2. We have $|Co_2| = 2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$. First we argue that the theorem is true for Abelian p-subgroups. Let Q_p be a Sylow p-subgroup. Note, from the outset, that $|Q_p|^3 < |Co_2|$ for $p \geqslant 3$; hence, if there exists an Abelian p-subgroup which fails to satisfy Theorem A, then that subgroup is $A_2(Co_2)$. Furthermore, $|C_{Co_2}(2B)| = 2^{17} \cdot 315$, where $2B$ is some involution of Co_2, and so $|A_2(Co_2)| \leqslant 2^{17}$. Again, Co_2 contains an element of order 8. Consequently, if $|A_2(Co_2)| = 2^{17}$ then $A_2(Co_2)$ has an element of order 4. The maximal power of 2 in the orders of centralizers of such elements is at most 16; therefore, $|A_2(Co_2)| \leqslant 2^{16}$. We have $(2^{16})^3 < |Co_2|$, and so the statement of Theorem A is true also for $A_2(Co_2)$.

If we consider an Abelian subgroup of mixed order, that is, a subgroup which is not a p-group, then it will contain an element of mixed order. Once we have looked through the orders of centralizers of such elements we arrive at the desired estimate.

2. Fi_{23}. We have $|Fi_{23}| = 2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$. For Abelian p-subgroups, obtaining an estimate is still simpler than in the previous case. However, the mixed case gives rise to problems.

We have $|C_{Fi_{23}}(6B)| = 2^7 \cdot 3^6$, where $6B$ is some element of order 6. Thus, if the Abelian subgroup A contains an element of order 6, we can maintain only that $|A| \leqslant 2^7 \cdot 3^6$. The A lies in some maximal subgroup G of Fi_{23}. Since $|A|$ divides $|G|$, we have the following options for G:

(1) if $G = O_7(3)$ then $|A| \leqslant 32 \cdot 27$ and $|A|^3 < |Fi_{23}|$;

(2) if $G = S_3 \times U_4(3) : 2$ then $|A| \leqslant 2^7 \cdot 3^5$ and $|A|^3 < |Fi_{23}|$;

(3) if $G = 3^{1+6} : 2^{3+4} : 3^2 : 2$ then $|A| \leqslant 2^6 \cdot 3^6$ and $|A|^3 < |Fi_{23}|$.

Theorem A is proved.

6. PROOF OF THEOREM B

Clearly, if Abelian subgroups of G satisfy $|A|^3 < |G|$, then G fails to be represented as ABA. We thus need to consider only groups $L_2(q)$. In so doing, we distinguish between the cases with q even and q odd.

Let q be odd, $q \geqslant 7$. Then the orders of maximal Abelian subgroups are equal to q, $\frac{q+1}{2}$, $\frac{q-1}{2}$. If $L_2(q) = ABA$, for the orders of A and B we face the following two options: $|A| = |B| = q$ or $|A| = q$, $|B| = \frac{q+1}{2}$. By routine computations, using the canonical form of elements in $L_2(q)$, we conclude that if u is a nonidentity unipotent element then $Z_{L_2(q)}(u) = U$, where U is a unipotent subgroup of $L_2(q)$. For every Sylow p-subgroup P and for any element x in $L_2(q)$, therefore, $P \cap P^x$ is equal either to 1 or to P.

We proceed to consider both options for A and B. Seek an order of the set AbA, where b is some element in B. We have $|AbA| = |AbAb^{-1}| = \frac{|A|^2}{|A \cap A^{b^{-1}}|} = \begin{cases} |A| \\ |A|^2 \end{cases}$.

If $L_2(q) = ABA$, we arrive at the following two systems of equations (for the first and second versions, respectively):

$$\begin{cases} qx + y = \frac{q^2-1}{2}, \\ x + y = q \end{cases}$$

and

$$\begin{cases} qx + y = \frac{q^2-1}{2}, \\ x + y = \frac{q+1}{2}, \end{cases}$$

where x and y are integers. It is not hard to see that neither system has an integer-valued solution, for any q. Therefore, $L_2(q)$ is not represented as ABA if q is odd.

Let q be even. Then, for the orders of A and B, there are four options for which $|G| \leqslant |A| \cdot |B| \cdot |A|$. These are the following pairs: $(q, q+1)$, (q, q), $(q+1, q-1)$, and $(q+1, q)$. The first two are treated the same way as for q odd, and we have $G \neq ABA$ for them. For the other two, we make use of the fact that $L_2(2^t) \cong SL_2(2^t)$. If the order of A is $q+1$, then A is conjugate in $SL_2(2^{2t})$ to a subgroup of matrices of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^q \end{pmatrix}.$$

Thus $Z_{SL_2(q)}(a) = A$ for a nonidentity element a in A; hence, $A \cap A^x$ is equal either to 1 or to A. Also, $|N_{SL_2(q)}(A)| = 2(q+1)$. Using the same argument as for q odd, we arrive at the following two systems (for the third and fourth versions, respectively):

$$\begin{cases} qx + y = q(q-1), \\ x + y = q - 1 \end{cases}$$

and

$$\begin{cases} qx + y = q(q-1), \\ x + y = q. \end{cases}$$

The system has no integer-valued solution in the third case, but has in the fourth — $x = q - 2$, $y = 2$. We have $|N_{SL_2(q)}(A)| = 2(q+1)$, and so $SL_2(q) = ABA$, which proves Theorem B.

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