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Maximal Orders of Abelian Subgroups in Finite Chevalley Groups

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Abstract—In the present paper, for any finite group G of Lie type (except for ${}^2F_4(q)$), the order $a(G)$ of its large Abelian subgroup is either found or estimated from above and from below (the latter is done for the groups $F_4(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$, and ${}^2E_6(q^2)$). In the groups for which the number $a(G)$ has been found exactly, any large Abelian subgroup coincides with a large unipotent or a large semisimple Abelian subgroup. For the groups $F_4(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$, and ${}^2E_6(q^2)$, it is shown that if an Abelian subgroup contains a noncentral semisimple element, then its order is less than the order of an Abelian unipotent group. Hence in these groups the large Abelian subgroups are unipotent, and in order to find the value of $a(G)$ for them, it is necessary to find the orders of the large unipotent Abelian subgroups. Thus it is proved that in a finite group of Lie type (except for ${}^2F_4(q)$) any large Abelian subgroup is either a large unipotent or a large semisimple Abelian subgroup.

Key words: *linear algebraic groups, Chevalley groups, Abelian subgroups, unipotent subgroups, semisimple subgroups.*

1. INTRODUCTION

In the present paper we study the structure and the orders of large Abelian subgroups in finite Chevalley groups. Various special Abelian subgroups in finite groups of Lie type were studied by many authors. For example, Carter [1, 2] found the structure of connected centralizers in algebraic groups over an algebraically closed field of characteristic $p > 0$, as well as the structure and the orders of their fixed point subgroups with respect to the Frobenius automorphism. In particular, he found the orders of maximal tori in all classical simply connected Chevalley groups. The investigation of connected centralizers in exceptional Chevalley groups was finished by Deriziotis in [3,4]. In the series of papers [5–8], Barry and Wong found the orders and the structure of the large unipotent Abelian subgroups in finite classical Chevalley groups.

In the present paper, for any finite group G of Lie type (except for ${}^2F_4(q)$), the order $a(G)$ of a large Abelian subgroup is either found or estimated from below and from above (the latter is done for the groups $F_4(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$, and ${}^2E_6(q^2)$). In groups for which the number $a(G)$ has been found exactly, any large Abelian subgroup coincides with a large unipotent or a large semisimple Abelian subgroup. For the groups $F_4(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$, and ${}^2E_6(q^2)$, it is shown that if an Abelian subgroup contains a noncentral semisimple element, then its order is less than the order of an Abelian unipotent group. Hence in these groups the large Abelian subgroups are unipotent, and in order to find the value of $a(G)$ for them, it is necessary to find the orders of the large unipotent Abelian subgroups.

The notation and the definitions used in this paper can be found in [9–11]. If φ is a homomorphism of the group G and g is an element of G, then G^φ and g^φ are the images under φ of G and g, respectively.

The notation related to finite groups of Lie type is used as in [10]. Unless otherwise stated, by a *Chevalley group* we understand either a universal Chevalley group or any of its quotient groups modulo a subgroup of its center. In the study of Chevalley groups, we denote the field of order q by $GF(q)$, its characteristic by p, the multiplicative group of $GF(q)$ by $GF(q)^*$, and the algebraic closure of $GF(q)$ by $K = \overline{GF(q)}$. The Chevalley group over $GF(q)$ corresponding to the root system Φ is denoted by $\Phi(q)$. The Weyl group corresponding to the root system Φ is denoted by $W(\Phi)$. The twisted groups are denoted by

$$^2A_n(q^2), \quad ^2D_n(q^2), \quad ^2E_n(q^2), \quad ^3D_4(q^3), \quad ^2B_2(q), \quad ^2G_2(q), \quad ^2F_4(q);$$

Φ^+ (Φ^-) stands for the set of positive (negative) roots of the root system Φ; $\Delta = \{r_1, \dots, r_k\}$ is the set of the fundamental weights numbered as in [10, Eq. (3.4)]. An element of the Chevalley group $\Phi(q)$ is said to be *semisimple* if its order is relatively prime to p, and *unipotent* if its order is a power of p. Similarly, *semisimple and unipotent subgroups of* $\Phi(q)$ are defined as subgroups whose orders are relatively prime to p (p'-subgroups) and are powers of p, respectively. The *extended Dynkin diagram of a Chevalley group* is the diagram obtained by supplementing the initial Dynkin diagram with the root $-r_0$ (where r_0 is the highest root) and joining it with the other vertices by the usual rule. Extended Dynkin diagrams for all types of root systems are given in [3].

For a finite group G, we denote the maximum of the orders of its Abelian subgroups by $a(G)$. If G is a finite group of Lie type, then by $a_u(G)$ and $a_s(G)$ we denote the maximal orders of its unipotent Abelian and semisimple Abelian subgroups, respectively. The set of Abelian subgroups of G whose orders are equal to $a(G)$, $a_u(G)$, or $a_s(G)$, will be denoted by $A(G)$, $A_u(G)$, or $A_s(G)$, respectively. The elements of the sets $A(G)$, $A_u(G)$, and $A_s(G)$ will be called *large Abelian, large unipotent Abelian*, and *large semisimple Abelian subgroups*, respectively.

2. AUXILIARY RESULTS CONCERNING LINEAR ALGEBRAIC GROUPS

In this section we present the necessary information concerning the structure of linear algebraic groups and obtain some auxiliary results to be used for estimating the orders of Abelian subgroups. The main definitions and results concerning the structure and the properties of linear algebraic groups can be found in [11]. If G is a linear algebraic group, then by G^0 we denote the identity component of G. A linear algebraic group is said to be *semisimple* if its radical is trivial, and *reductive* if its unipotent radical is trivial (in both cases the linear algebraic group is not assumed to be connected). It is well known (see, for example, [11]) that a connected semisimple linear algebraic group is the central product of connected simple linear algebraic groups, and a connected reductive linear algebraic group G is the central product of a torus S and a semisimple group M, where $S = Z(G)^0$, $M = [R, R]$, and the group $S \cap M$ is finite.

Suppose that G is a connected reductive linear algebraic group, T is its maximal torus (by a *torus* we always mean a connected diagonalizable group), B is a Borel subgroup containing the torus T. There exists a Borel subgroup B^- in G such that $B \cap B^- = T$. Let Φ be the root system of the group G with respect to T, let $\varphi\colon N_G(T) \to N_G(T)/T = W$ be the canonical homomorphism onto the Weyl group W of G, and let $X_\alpha (\alpha \in \Phi)$ be the root subgroups with respect to T (i.e., the T-invariant unipotent subgroups of the groups B and B^-). The action of the Weyl group W on the root system Φ is defined as follows [11, 24.1]. Suppose that for each element w in W, its representative n_w in G is fixed. Then the Weyl group acts on the roots of the root system Φ by the rule $\alpha^w(t) = \alpha(t^{n_w})$ for all $\alpha \in \Phi$, $t \in T$. It is well known that $B = TU$, where $U = \langle X_\alpha : \alpha \in \Phi^+ \rangle$ is a maximal unipotent subgroup of G, and $B^- = TU^-$, where U^- denotes the group $\langle X_\alpha : \alpha \in \Phi^- \rangle$. Given an order on the set Φ^+, any element of U can be uniquely expressed as a product of elements of the root subgroups U_α (taken in the given order).

Any element of G can be uniquely represented in the form un_wtv, where $v \in U$, $t \in T$, $u \in U \cap n_w U^- n_w^{-1}$ (see, for example, [11, Theorem 28.3]). This representation of the elements of G is called their *Bruhat decomposition*.

Suppose that G is a connected simple algebraic group, π is its exact rational representation, Γ_π is the lattice generated by the weights of π. Denote the lattice generated by the roots of Φ by Γ_{ad}, and the lattice generated by the fundamental weights by Γ_{sc}. The lattices Γ_{sc}, Γ_π, and Γ_{ad} are independent of the particular representation of G, and the following inclusions hold: $\Gamma_{ad} \le \Gamma_\pi \le \Gamma_{sc}$.

It is known that, for a root system of a given type, there are several distinct simple algebraic groups called *isogenies*. They differ by the structure of Γ_π and by the order of the finite center. In the case when the lattice Γ_π coincides with Γ_{sc}, the group G is said to be *simply connected* and is denoted by G_{sc}. If the lattice Γ_π coincides with Γ_{ad}, then the group G is said to be *of adjoint type* and is denoted by G_{ad}. Any linear algebraic group with root system Φ can be obtained as the quotient of the group G_{sc} modulo a subgroup lying in its center. The center of G_{ad} is trivial, and this group is simple as an abstract group.

Let c_i be the coefficient of the fundamental root r_i in the root r_0. The prime numbers dividing the coefficients c_i are called *bad primes*.

Further, let us recall the following fundamental result concerning the structure of algebraic groups.

Lemma 2.1 [11, Secs. 21.3, 22.2]. *Let G be a connected simple linear algebraic group. Then all Borel subgroups of G are conjugate. Moreover, the maximal tori and the maximal connected unipotent subgroups of G are just the maximal tori and the maximal connected unipotent subgroups of the Borel groups. Besides, all maximal tori and maximal connected unipotent subgroups are conjugate, and any semisimple* (*respectively, unipotent*) *element belongs to a maximal torus* (*respectively, a maximal connected unipotent subgroup*).

Now let us recall the relationship between finite groups of Lie type and simple linear algebraic groups. Let G be a connected simple linear algebraic group defined over an algebraically closed field of characteristic $p > 0$, and let σ be an endomorphism of G such that the set of its fixed points G_σ is finite. In what follows, any endomorphism σ satisfying this property will be called a *Frobenius automorphism*, although it does not necessarily coincide with the classical Frobenius automorphism. Note that σ is an automorphism if G is regarded as an abstract group, and σ is an endomorphism if G is regarded as an algebraic group. In general, the automorphism σ has the form $q\sigma_0$, where $q = p^\alpha$ is raising to the qth power, and σ_0 is a graph automorphism of order 1, 2, or 3. Then $O^{p'}(G_\sigma)$ is a group of Lie type over a finite field of characteristic p, and any group of Lie type (normal or twisted) can be obtained in this way. In what follows, by σ we shall always denote a Frobenius automorphism.

Let T be a maximal σ-invariant torus of a connected simple algebraic group G. In what follows, by a *maximal torus* of G_σ (respectively, of $O^{p'}(G_\sigma)$) we shall mean a group of the form T_σ (respectively, $T_\sigma \cap O^{p'}(G_\sigma)$). Note (see, for example, [10, Chap. 7]) that for any σ-invariant torus T we have $G_\sigma = T_\sigma O^{p'}(G_\sigma)$. (In [10] the group G_σ is denoted by $\widehat{G}$.)

We shall need the following lemma.

Lemma 2.2. *Let $R = S * M$ be a σ-invariant reductive subgroup of maximal rank in a connected simple group G, where S is the central torus and M is a semisimple subgroup of G. Suppose that the simple components of M_σ are $G_1, \dots, G_k$. Let $z_{i,j} = |Z(G_i) \cap Z(G_j)|$, $z_i = |Z(G_i) \cap S_\sigma|$. Then*

$$O^{p'}(R_\sigma) = G_1 * \dots * G_k * S_\sigma \qquad \text{and} \qquad |R_\sigma : O^{p'}(R_\sigma)| \le \prod_{i \ne j} z_{i,j} \prod_i z_i.$$

Proof. The equality $O^{p'}(R_\sigma) = G_1 * \cdots * G_k * S_\sigma$ is well known; therefore, we shall only prove the second inequality. In [3, Proposition 2.4.2] it was proved that

$$|R_\sigma| = |M_\sigma| \cdot |S_\sigma| = |G_1| \cdots |G_k| \cdot |S_\sigma|.$$

Since $G_i \cap G_j = Z(G_i) \cap Z(G_j)$ for $i \neq j$, we have

$$|G_1 * \cdots * G_k * S_\sigma| \leq |R_\sigma| \Big/ \left(\prod_{i \neq j} z_{i,j} \prod_i z_i \right),$$

whence the required inequality follows. □

We shall now prove an auxiliary result that will be used in the study of Abelian subgroups of finite Chevalley groups.

Lemma 2.3. *Suppose that G is a connected reductive linear algebraic group over an algebraically closed field of characteristic p, R is its reductive (not necessarily connected) subgroup of maximal rank, $(|R : R^0|, p) = 1$, $s \in R^0$ is a semisimple element, and T is an arbitrary maximal torus in R^0 containing s. Then the group $C_R(s)$ is reductive (although it is not necessarily connected). It is generated by the torus T together with the root subgroups U_α such that $\alpha(s) = 1$, and together with those representatives of elements of the Weyl group $n_w \in N_R(T)$ that commute with s. The identity component $C_R(s)^0$ is generated by the torus T and the subgroups U_α such that $\alpha(s) = 1$. In particular, the group $C_R(s)/C_R(s)^0$ is isomorphic to a section of the Weyl group of G. Moreover, all unipotent elements of $C_R(s)$ belong to $C_R(s)^0$.*

Proof. In the group R^0, let us fix a Borel subgroup B containing T. It is clear that all generators mentioned in the lemma belong to $C_R(s)$. Let us prove that $C_R(x)$ is generated by the elements mentioned in the lemma. First, we shall show that the group R (which is not necessarily connected) admits the Bruhat decomposition. Let x be an arbitrary element of R. Then B^x is a Borel subgroup of the group R^0. By Lemma 2.1, there exists an element $s \in R^0$ such that $B^x = B^s$. Then the element xs^{-1} normalizes the subgroup B. The torus $T^{xs^{-1}}$ is a maximal torus of the group B. Since all maximal tori in B are conjugate (Lemma 2.1), there is an element g in B such that $T^{xs^{-1}} = T^g$. Hence it can be assumed that xs^{-1} normalizes the torus T. Then $xs^{-1} = n_w t$ for some $n_w \in N_R(T)$, $t \in T$. Since t normalizes B and xs^{-1} normalizes B, the element n_w also normalizes B; hence it normalizes the maximal (connected) unipotent subgroup U of the group B. Since s lies in R^0, it has a Bruhat decomposition, i.e., it can be represented as

$$u_1 n_{w_1} t_1 v_1, \qquad \text{where} \quad u_1 \in U \cap n_{w_1} U^- n_{w_1}^{-1}, \quad n_{w_1} \in N_{R^0}(T), \quad t \in T,\ v_1 \in U.$$

Hence the element x can be represented as $x = n_w t u_1 n_{w_1} t_1 v_1$. Since the elements t and n_w normalize U, we obtain the representation of the element x as

$$x = u_2 n_{w_2} t_2 v_2, \qquad \text{where} \quad u_2 \in U \cap n_{w_2} U^- n_{w_2}^{-1}, \quad n_{w_2} \in N_R(T), \quad t_2 \in T,\ v_2 \in U.$$

Since this Bruhat decomposition coincides with the Bruhat decomposition of the element x in the group G, this decomposition is unique.

If $x \in C_R(s)$, then, by using the Bruhat decomposition, we can write $x = u n_w t v$, where $v \in U$, $t \in T$, and $u \in U \cap n_w U^- n_w^{-1}$. Since s normalizes U, $N(T)$, U^- and commutes with x, the uniqueness of the decomposition implies that each of the elements u, n_w, and v commutes with s. Moreover, since s normalizes each root subgroup U_α, the uniqueness of the decomposition of U into the product of root subgroup U_α $(\alpha > 0)$ implies that $\alpha(s) = 1$ whenever u or v contains a nontrivial factor from U_α. Thus x belongs to the group generated by the torus T and by those U_α and n_w that commute with s.

Since T and all U_α with $\alpha(s)=1$ are connected, the subgroup H generated by them is closed, connected, and normal in $C_R(s)$. Since the Weyl group is finite, we have $|G_R(s):H|<\infty$, and hence $H=C_R(s)^0$.

Since the roots of the group $C_R(s)$ with respect to the torus T appear in pairs (i.e., if $\alpha(s)=1$, then $-\alpha(s)=1$ as well), the group $C_R(s)$ is reductive. Indeed, if $C_R(s)$ has a nontrivial unipotent radical V, then it is normalized by T, and hence it contains a root subgroup U_α. The subgroup V is normalized by the root group $U_{-\alpha}$, which provides us with an element in V which is not unipotent; we obtain a contradiction.

Since $(|R:R^0|,p)=1$, all unipotent elements of R lie in R^0; hence all unipotent elements of $C_R(s)$ lie in $C_{R^0}(s)$. The fact that in a connected reductive group R^0 any unipotent element of $C_{R^0}(s)$ lies in $C_{R^0}(s)^0$ is well known (see, for example, [12, Sec. 2.2]). □

Let x be a semisimple element in a connected reductive linear algebraic group G. Then, by Lemma 2.3, $C_G(x)^0$ is a connected reductive subgroup of maximal rank, and $[C_G(x)^0, C_G(x)^0]$ is a semisimple group whose root system is an additively closed subsystem of the root system of G. In what follows, such subgroups will be referred to as *subsystem* subgroups. Since only finite groups are studied in the present paper, the elements of prime order $r\neq p$ are of particular interest. It turns out that the following lemma holds.

Lemma 2.4 [13, 14.1]. *Suppose that G is a simple connected linear algebraic group over an algebraically closed field of characteristic $p>0$, and an element $x\in G$ has prime order $r\neq p$. Let*

$$C' = [C_G(x)^0, C_G(x)^0]$$

be a subsystem subgroup. If Δ is the Dynkin diagram of the root system of the group C', then one of the following assertions holds:

1) *the diagram Δ is obtained from the Dynkin diagram of G by removing some vertices;*
2) *the diagram Δ is obtained from the extended Dynkin diagram of G by removing a vertex r_i, where $r=c_i$ is the coefficient of r_i in r_0.*

In particular, if r is not a bad number for the group G, then $\dim(Z(C_G(x)^0))\geq 1$.

Concluding this section, let us recall the algorithm of Borel and de Siebenthal for finding all subsystems of a root system Φ; see [14]. Consider the extended Dynkin diagram of the system Φ. The diagrams of all possible subsystems of Φ are obtained by removing several vertices from the extended Dynkin diagram of Φ.

3. GENERAL STRUCTURE OF ABELIAN SUBGROUPS IN CONNECTED REDUCTIVE LINEAR ALGEBRAIC GROUPS, IN FINITE CHEVALLEY GROUPS, AND IN WEYL GROUPS

Lemma 3.1. *Suppose that G is a connected reductive linear algebraic group, and A is its closed Abelian subgroup. Then the following assertions hold:*

1) *the group A can be represented as $A_s\times A_u$, i.e., as the direct product of its semisimple part and its unipotent part, respectively* (*see* [10, 15.5]);
2) *the group G contains a reductive subgroup R of maximal rank such that*

$$A\leq R,\qquad A_u\leq R^0,\qquad A_s\cap R^0 = A_{s0}\leq Z(R^0);$$

3) *if $W_R=N_R(T)/T$ and $W_{R0}=N_{R^0}(T)/T$ for some maximal torus T of the group R, then the group A_s/A_{s0} is isomorphically embedded into W_R/W_{R0}.*

If A is a finite group consisting of elements that are fixed by a Frobenius automorphism σ of G, then R is σ-invariant.

Proof. Suppose that s is a semisimple element of A_s. Let $R = C_G(s)$. It is clear that $A \leq R$. By Lemma 2.3, we have $A_u \leq R^0$, and R^0 is a connected reductive subgroup of maximal rank in G. The subgroup R^0 is normal in R; therefore, any element of R normalizes $Z(R^0)$, and hence normalizes $Z(R^0)^0$. By Lemma 2.3, we have $(|R : R^0|, p) = 1$. If there exists a semisimple element $s_1 \in A_s$ such that $s_1 \in R^0$ but $s_1 \notin Z(R^0)$, then let us consider $C_R(s_1)$. It is clear that $A \leq C_R(s_1)$. By Lemma 2.3, $C_R(s_1)^0$ is a connected reductive subgroup of maximal rank in G. As above, we have $A_u \leq C_R(s_1)^0$, and any element of $C_R(s_1)$ normalizes $Z(C_R(s_1)^0)^0$. Replacing the group R by $C_R(s_1)$, we obtain a reductive subgroup of maximal rank in G containing A, of smaller dimension. Indeed, the dimension decreases, since the dimension of the identity component decreases. The process described above is finite, since at each step the dimension decreases, and the dimension of G is finite. Note that if A is an Abelian subgroup in a finite group of Lie type, then A consists of fixed points of a Frobenius automorphism σ, and hence the groups obtained at each step of the process described above are σ-invariant. Therefore, if $A \leq G_\sigma$, then the group R is σ-invariant.

Further, we have

$$A_s/A_{s0} \cong A_s R^0/R^0 \leq R/R^0.$$

By Lemma 2.3, any element of R can be represented as $n_w x$, where $x \in R^0$; hence the group R/R^0 is isomorphic to the group

$$N_R(T)/N_{R^0}(T) \cong W_R/W_{R0}$$

for any maximal torus T in R^0. $\square$

As a simple corollary of Lemma 3.1, note that if $\Phi_R = \Phi$, then $A_s = A_{s0} \leq Z(G)$. Indeed, the reductive group R which is mentioned in the lemma coincides in this case with G (and coincides with R^0), but we have $A_s \cap R^0 = A_s \leq Z(R^0)$.

In Lemma 3.1 sections of the Weyl group appear; hence it is necessary to find the orders of large Abelian subgroups of the Weyl groups for all simple algebraic groups. Besides, in what follows we shall often encounter the situation in which a semisimple Abelian subgroup is the set of fixed points for a Frobenius automorphism of a torus T of dimension n. To estimate the orders of such subgroups, we shall need the following lemma.

Lemma 3.2. *Suppose that S is a σ-invariant torus of dimension n in a connected simple algebraic groups G, where $\sigma = q\sigma_0$ is a Frobenius automorphism. Let S^g be one of its conjugate σ-invariant tori in G. By $X(S)$ denote the group of rational characters of S. Then in the Weyl group W of G there exists an element w such that $X(S)^w \subseteq X(S)$ and the group $(S^g)_\sigma$ is isomorphic to the group $X(S)/(\sigma w - 1)X(S)$. In particular, since the element $\sigma_0 w$ is of finite order, we have*

$$|(S^g)_\sigma| \leq (q+1)^n.$$

Proof. Since the torus S is σ-invariant, its centralizer C in G is a connected σ-invariant reductive subgroup of maximal rank. Then the group C^g is also σ-invariant. The groups C and C^g contain some σ-invariant maximal tori T and T_1 [15, Sec. 10.10]; without loss of generality, we may assume that they are also conjugated by g. Let W_1 be the Weyl group of C. In [1], the corollary of Proposition 2 asserts that in this case the element $g^\sigma g^{-1}$ lies in $N_G(T)$, and its image under the canonical homomorphism of $N_G(T)$ onto the Weyl group belongs to $N_W(W_1)$. Proposition 8 from [1] asserts that in this case we have

$$(T^g)_\sigma \cong X(T)/(\sigma w - 1)X(T).$$

Since $S \leq Z(C)$, the torus S lies in T. By the bijective correspondence between closed subgroups of the torus T and subgroups of the group $X(T)$ of its characters (see [16, Chap. III]), we have

$$(S^g)_\sigma \cong X(S)/(\sigma w - 1)X(S). \quad \square$$

In Table 1 below we list the orders of the large Abelian subgroups in the Weyl groups of the classical simple groups. The group $W(A_n)$ is isomorphic to S_{n+1}; the orders and the structure of the large Abelian subgroups of this group are found in [17], and we cite them in Table 1 without proof.

The groups $W(B_n)$ and $W(C_n)$ are isomorphic; hence we shall only consider the group $W(B_n)$. If Φ is a root system of type B_n and $e_1, \dots, e_n$ is an orthonormal basis of the Euclidean space in which the system B_n lies, then Φ can be written as

$$\{\pm e_i \pm e_j,\ i \neq j,\ \pm e_i;\ \ i, j = 1, \dots, n\}.$$

The Weyl group $W(B_n)$ acts on the set $\{\pm e_1, \dots, \pm e_n\}$ of $2n$ roots. Let A be an Abelian subgroup of $W(B_n)$. Then $I_1, \dots, I_k$ are all A-orbits in $\{\pm e_1, \dots, \pm e_n\}$. Consider the group G of all transformations of the Euclidean space spanned by $e_1, \dots, e_n$ under which the set $\{\pm e_1, \dots, \pm e_n\}$ is invariant. It is clear that $W(B_n) \leq G$. Moreover, these groups are isomorphic, but we shall not use this fact. Let us find $a(G)$ and prove that $a(G) = a(W(B_n))$. The proof of Theorem 1.1 from [17] implies that

$$|A| \leq |I_1| \times \cdots \times |I_k|.$$

Let $f(2n)$ be the order of a large Abelian subgroup in G.

Assume that among the sets $I_1, \dots, I_k$ there are sets of odd order. Without loss of generality, we may assume that one of these sets is the set I_1 and that the basis vector e_1 belongs to I_1. Then the vector $-e_1$ does not lie in I_1. Indeed, assume the converse. The group A acts transitively on I_1; hence there is an element σ in A that takes e_1 to $-e_1$. But in this case the order of the element σ is even, and it can be assumed that σ is a 2-element. Besides, σ does not belong the stabilizer $\mathrm{St}_A(I_1)$ of the orbit I_1 in A. Hence its image under the natural homomorphism $\varphi\colon A \to A/\mathrm{St}_A(I_1)$ is also of even order. But the order $|A/\mathrm{St}_A(I_1)| = |I_1|$ is odd; we obtain a contradiction. Thus the element $-e_1$ belongs to another set; without loss of generality we may assume that this set is I_2. Since G is a group of linear transformations of Euclidean space, the relation $\sigma(-e_1) = -\sigma(e_1)$ holds for any $\sigma \in G$. Since the group A acts transitively on the sets $I_1, \dots, I_k$, it follows that if $\sigma \in \mathrm{St}_A(I_1)$, then $\sigma \in \mathrm{St}_A(I_1) \cap \mathrm{St}_A(I_2)$, and for any $v \in I_1$, the vector $-v$ belongs to I_2. Let $m = |I_1| = |I_2|$. Then, by the above arguments, we have

$$|A| \leq m \cdot f(2n - 2m).$$

Consider the group A_1 such that its action on the orbits $I_3, \dots, I_k$ is the same as for the group A, and the set $I_1 \cup I_2$ splits into two-element orbits $\{\pm v\}$ under the action of A_1. By construction, $A_1 \leq G$ and the group A_1 is Abelian. Besides, we have

$$|A| = m \cdot |\mathrm{St}_A(I_1)| < 2^m \cdot |\mathrm{St}_A(I_1)| = |A_1|.$$

Therefore, the set $\{\pm e_1, \dots, \pm e_n\}$ splits into orbits of even order under the action of the group $A \in A(G)$. By induction on n, let us show that all orbits are of order 2 or 4 and, therefore, $f(2n) \leq 2^n$. Indeed, if an orbit of A is of order greater than or equal to 6, then, as in the odd case, we can construct an Abelian subgroup of order $2^3 \cdot 2^{n-3}$, whence $|A| \leq 6 \cdot 2^{n-3} < 2^3 \cdot 2^{n-3}$. Therefore,

$$f(2n) \leq 2^n.$$

On the other hand, there is an Abelian subgroup of $W(B_n)$, and hence of G that takes e_i to $\pm e_i$ for all i. Its order is equal to 2^n, and hence $a(W(B_n)) = 2^n$.

Note that $W(D_n)$ can be embedded isomorphically into $W(B_n)$ as a subgroup of index 2; hence

$$a(W(D_n)) \leq a(W(B_n)).$$

Table 1

Type of G	Order $a(W)$
A_n	3^k if $n=3k-1$ $4\cdot 3^{k-1}$ if $n=3k$ $2\cdot 3^k$ if $n=3k+1$
B_n, C_n, D_n	$\leq 2^n$

For even n, this estimate is attained, and for odd n, it is not. In what follows, we shall only need this upper estimate, although it is not hard to find the exact estimate as well.

Now we shall list the known estimates for the orders of large unipotent Abelian subgroups.

Lemma 3.3 [5–8]**.** *Let $q=p^\alpha$. Then*

1) $a_u(A_n(q)) = q^{[(n+1)^2/4]}$ *(see* [5]*)*;
2) $a_u(C_n(q)) = q^{n(n+1)/2}$ *(see* [5, 6, 8]*)*;
3) $a_u(B_n(q)) = q^{n(n-1)/2+1}$ *for* $n \geq 4$, $p \neq 2$ *(see* [5, 7]*)*;
4) $a_u(B_3(q)) = q^5$ *for* $p\neq 2$ *(see* [5, 7]*)*;
5) $a_u(D_n(q)) = q^{n(n-1)/2}$ *for* $n\geq 4$ *(see* [5, 7]*)*;
6) $a_u({}^2D_n(q^2)) = q^{(n-2)(n-1)/2+2}$ *for* $n\geq 5$ *(see* [7]*)*;
7) $a_u({}^2D_4(q^2)) = q^6$ *(see* [7]*)*;
8) $a_u({}^2A_n(q^2)) = q^{[(n+1)^2/4]}$ *(see* [8]*)*.

Here in parentheses we indicate the papers in which the Abelian unipotent subgroups of the corresponding groups were studied. Besides, in these papers the structure of large unipotent Abelian subgroups is described.

Lemma 3.4 [17, Lemma 4.5]**.** *Suppose that q is a power of an odd prime p. Then the following estimates hold*:

1) $q^9 \leq a_u(F_4(q)) \leq q^{14}$;
2) $q^{16} \leq a_u(E_6(q)) \leq q^{20}$;
3) $q^{27} \leq a_u(E_7(q)) \leq q^{32}$;
4) $q^{36} \leq a_u(E_8(q)) \leq q^{61}$;
5) $q^{12} \leq a_u({}^2E_6(q^2)) \leq q^{20}$.

The proof of Lemma 4.5 in [17] for the groups $E_6(q)$, $E_7(q)$, and $E_8(q)$ does not employ the fact that q is odd; hence the estimates given in the lemma for these groups hold in any characteristic. The estimates for the rest of exceptional groups (except for ${}^2B_2(q)$, ${}^2G_2(q)$, and ${}^2F_4(q)$) are listed in the following lemma.

Lemma 3.5. *The following assertions hold*:

1) $a_u({}^3D_4(q^3)) = q^5$;
2) $q^{12} \leq a_u({}^2E_6(q^2)) \leq q^{20}$;
3) $a_u(G_2(q)) = q^3$ *if q is not divisible by* 3;
4) $a_u(G_2(q)) = q^4$ *if q is divisible by* 3;
5) $q^{11} \leq a_u(F_4(q)) \leq q^{17}$ *if q is even.*

Proof. It is well known (see, for example, [18]) that the group $G_2(q)$ contains a unipotent Abelian subgroup of order q^3 if q is not divisible by 3, and an Abelian subgroup of order q^4 if q is divisible by 3. In the group ${}^3D_4(q^3)$ there is an Abelian unipotent subgroup of order q^5, in the group ${}^2E_6(q^2)$ there is an Abelian subgroup of order q^{12}, in the group $F_4(q)$ for even q there is an Abelian subgroup of order q^{12}; hence it only remains to prove the upper estimates for all groups.

First, we consider the group $G_2(q)$. Let P be the parabolic subgroup with Levi factor generated by the fundamental short root r_1. Then we have $P = HA_1(q)U_{r_2}$, where $U_{r_2} = \langle X_r : r \geq r_2\rangle$.

Here the partial order is given by

$$r = \alpha_1 r_1 + \alpha_2 r_2 \geq 0 \iff \alpha_1 \geq 0,\ \alpha_2 \geq 0.$$

Note that $[U_{r_2}, U_{r_2}] = X_{r_0}$; hence the group $U_{r_2}/X_{r_0} = \overline{U_{r_2}}$ is Abelian. It can be endowed with the skew-symmetric bilinear form $< \cdot\,, \cdot >$ defined by the rule

$$[u, v] = x_{r_0}(< \bar{u}, \bar{v} >) \qquad \text{for } u, v \in U_{r_2}.$$

The table of constants that occur in the Chevalley commutator relations for the group $G_2(q)$ (see [10, p. 211]) implies that this form is nondegenerate whenever q is not divisible by 3. The dimension of the vector space $\overline{U_{r_2}}$ is equal to 4; the vectors

$$\overline{x_{r_2}(1)} = x_1, \quad \overline{x_{r_1+r_2}(1)} = x_2, \quad \overline{x_{2r_1+r_2}(1)} = x_3, \quad \overline{x_{3r_1+r_2}(1)} = x_4$$

form a basis of this space. Thus the entire unipotent subgroup U of $G_2(q)$ can be represented as $X_{r_1}U_{r_2}$. Let A be a large Abelian subgroup of U. It is clear that $X_{r_0} = Z(U) \leq A$. Suppose that the image of A under the natural homomorphism $\varphi\colon A \to A/(A \cap U_{r_2})$ is nonidentity. Let x be an element of A whose image x^φ is nonidentity. Then we can define the action of x^φ on $\overline{U_{r_2}}$ by conjugation; besides, there exists a unique element $x_{r_1}(t)$ such that $x^\varphi = x^\varphi_{r_1}(t)$. Since the group A is Abelian, the vector space $A \cap U_{r_2}/X_{r_0}$ is contained in the space of eigenvectors of x^φ. For any nontrivial element x^φ, its matrix in the basis x_1, x_2, x_3, x_4 has the following form (the structure constants are chosen as in [10]):

$$\begin{pmatrix} 1 & t & t^2 & -t^3 \\ 0 & 1 & 2t & 3t^2 \\ 0 & 0 & 1 & -3t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If $p > 3$, then the space of eigenvectors of x^φ has dimension 1. Thus we have

$$|A| = |A/(A \cap U_{r_2})| \cdot |(A \cap U_{r_2})/X_{r0}| \cdot |X_{r0}| \leq q^3.$$

If $A^\varphi = \{1\}$, then $A \leq U_{r_2}$. In this case, the group A/X_{r_0} is an isotropic subspace of $\overline{U_{r_2}}$; therefore, its dimension is at most 2. Hence

$$|A| = |A/X_{r_0}| \cdot |X_{r_0}| \leq q^3.$$

If q is divisible by 3, then the skew-symmetric form defined on $\overline{U_{r_2}}$ is degenerate. Let $L(x_2, x_3)$ be the linear span of the vectors x_2 and x_3. The quotient space $\overline{U_{r_2}}/L(x_2, x_3)$ possesses the skew-symmetric form induced by the form $< \cdot\,, \cdot >$. Let A be a large Abelian subgroup of U. If A has an element x such that $x^\varphi \neq 1$, then the group $(A \cap U_{r_2})/X_{r_0}$ is contained in the space of fixed points of the transformation x^φ. The form of the matrix of x^φ obtained above implies that the space of the vectors fixed by x^φ is of dimension at most 2. Thus we have

$$|A| = |A/(A \cap U_{r_2})| \cdot |(A \cap U_{r_2})/X_{r_0}| \cdot |X_{r_0}| \leq q^4.$$

If $A \leq U_{r_2}$, then the image of A under the homomorphism

$$A/X_{r_0} \to (A/X_{r_0})/(A/X_{r_0} \cap L(x_2, x_3))$$

is an isotropic space; hence its dimension is 1. Therefore,

$$|A| = |(A/X_{r_0})/(A/X_{r_0} \cap L(x_2, x_3))| \cdot |L(x_2, x_3)| \cdot |X_{r_0}| \leq q^4.$$

Finally, let $p=2$. In this case, the dimension of the space of eigenvectors for the action of any nontrivial element x of $A\setminus(A\cap U_{r_2})$ is equal to 2. The skew-symmetric bilinear form defined above is nondegenerate on this space; hence the image of the Abelian group $A\cap U_{r_2}$ under the natural homomorphism

$$\varphi\colon A\cap U_{r_2}\to (A\cap U_{r_2})/X_{r_0}$$

is an isotropic space of dimension 1. Thus we have

$$|A|=|A/(A\cap U_{r_2})|\cdot|(A\cap U_{r_2})/X_{r_0}|\cdot|X_{r_0}|\le q^3.$$

This completes the examination of the group $G_2(q)$.

Consider the group ${}^3D_4(q^3)$. Let P be the parabolic subgroup of $D_4(q^3)$ such that the Dynkin diagram of its Levi factor is obtained by removing the vertex r_2 from the Dynkin diagram of $D_4(q^3)$. Then P is τ-invariant (here τ is the twisting automorphism of order 3). Its fixed point subgroup has the form LU_1, where L is the Levi factor isomorphic to $A_1(q^3)H$. The group U_1 possesses the property $[U_1,U_1]=(X_{r_0})_\sigma=X$. Hence the group $U_1/X=\overline{U_1}$ is Abelian. It can be expressed in the form $\overline{U_2}\oplus\overline{U_3}$, where $\overline{U_2}$ is a vector space of dimension 2 over $GF(q)$, and $\overline{U_3}$ is a vector space of dimension 2 over $GF(q^3)$. As in the case of the group $G_2(q)$, each of these spaces can be endowed with the nondegenerate skew-symmetric bilinear form $<\cdot,\cdot>$ defined by the rule

$$[u,v]=x_{r_0}(<\bar u,\bar v>).$$

This form is nondegenerate, since all roots in a root system of type D_4 are of equal length, and hence all constants in the Chevalley commutator relations are of modulus 1. As in the case of $G_2(q)$, we define an action of L on $\overline{U_1}$. Then, for any nontrivial unipotent element of L, its fixed point subgroup in $\overline{U_1}$ is of order at most q. Thus the inequalities similar to those written for the case of $G_2(q)$ imply the assertion of the lemma for the group ${}^3D_4(q^3)$.

Let us pass to the group ${}^2E_6(q^2)$. Consider the parabolic subgroup in $E_6(q^2)$ whose Levi factor is isomorphic to $A_5(q^2)H$. This parabolic subgroup is τ-invariant, where τ is the twisting automorphism of order 2. Its fixed point subgroup can be expressed as LU_1, where L is the Levi factor isomorphic to ${}^2A_5(q^2)H$, and U_1 is the unipotent radical possessing the property

$$[U_1,U_1]=(X_{r_0})_\tau=X=Z(U_1).$$

Again, the group $U_1/X=\overline{U_1}$ can be represented as $\overline{U_2}\oplus\overline{U_3}$, where $\overline{U_2}$ is a vector space of dimension 12 over $GF(q)$, and $\overline{U_3}$ is a vector space of dimension 4 over $GF(q^2)$. Each of these spaces can be endowed with the nondegenerate bilinear skew-symmetric form $<\cdot,\cdot>$ defined by the rule

$$[u,v]=x_{r_0}(<\bar u,\bar v>).$$

Then any unipotent Abelian group A can be represented in the form A_1A_2, where $A_2=A\cap U_1$, $A_1\le {}^2A_5(q^2)$, and the image of A_2 under the natural homomorphism $A_2\to A_2/X$ is a direct sum of isotropic subspaces. Thus we have

$$|A|\le|A_1|\cdot|A_2|\le q^9\cdot q^6\cdot q^4\cdot q\le q^{20}.$$

Finally, consider the group $F_4(q)$, where q is even. It contains a parabolic subgroup $P=LU_{r_1}$ such that the Dynkin diagram of its Levi factor is obtained by removing the long root r_1. Then the Levi factor L of this group is isomorphic to $C_3(q)H$. The group U_{r_1} has the property $[U_{r_1},U_{r_1}]=X_{r_0}$. Besides, we have $|Z(U_{r_1}|=q^7$. The quotient group $U_{r_1}/Z(U_{r_1})=\overline{U_{r_1}}$ is elementary Abelian, and it can be regarded as a vector space of dimension 8 over $GF(q)$. Again, the vector space $\overline{U_{r_1}}$ can be endowed with a skew-symmetric bilinear form. Further, if A is a unipotent Abelian subgroup of $F_4(q)$, then A can be represented as A_1A_2, where $A_1\le C_3(q)$

and $A_2 \leq U_{r_1}$. The image of the group A_2 under the natural homomorphism $A \to A/(A \cap Z(U_{r_2}))$ is an isotropic subspace; hence we have

$$|A| \leq |A_1| \cdot |A_2| \leq q^6 \cdot q^4 \cdot q^7 \leq q^{17}. \quad \square$$

4. LARGE ABELIAN SUBGROUPS IN FINITE GROUPS OF LIE TYPE

In this section the above results are applied to the study of finite groups of Lie type. It is proved that a large Abelian subgroup of a finite group of Lie type, except for groups of small rank, coincides with a large unipotent Abelian subgroup. In nonsimple groups of Lie type, a large Abelian subgroup is the product of a large Abelian unipotent subgroup and the center of the corresponding group. Note that large Abelian subgroups of the groups $A_n(q)$ and $C_l(q)$, $l \geq 3$, were found in [17]. The groups ${}^2B_2(2^{2n+1})$ and ${}^2G_2(3^{2n+1})$ were studied in [19,20]. The group ${}^2F_4(q)$ is not studied in the present paper.

We now recall the structure of σ-fixed points of reductive σ-invariant subgroups of maximal rank in simple linear algebraic groups (see [1, Propositions 1, 2, 6 and 8]).

Lemma 4.1. *Suppose that G is a simple connected linear algebraic group; σ is its Frobenius automorphism; $G_1 = M * S$ is a σ-invariant connected reductive subgroup of maximal rank, where M is semisimple and S is the central torus; G_1^g is a σ-invariant subgroup conjugate to G_1. Let Δ_1 be the Dynkin diagram of G_1, and let W_1 be the Weyl group of G_1. Then*

$$g^\sigma g^{-1} \in N_G(G_1) \cap N_G(T) = N$$

(for some maximal torus T in G_1), and there is a bijection $\pi\colon N \to N_W(W_1)/W_1$.

Let $\pi(g^\sigma g^{-1}) = w$; by τ denote the image of the element w under the natural homomorphism $\varphi\colon N_W(W_1) \to \mathrm{Aut}_W(\Delta_1)$ and the graph automorphism of M corresponding to the symmetry τ. Then $(M^g)_\sigma \cong M_{\sigma\tau}$.

Let $\overline{P_1}$ be the sublattice of $X(T)$ generated by all rational linear combinations of the roots belonging to Δ_1. Then

$$(S^g)_\sigma \cong (X(T)/\overline{P_1})/(\sigma w - 1)(X(T)/\overline{P_1}).$$

Besides, $|(G_1^g)_\sigma| = |M_\sigma^g| \cdot |S_\sigma^g|$.

Note that Lemma 4.1 deals with the group G_σ. In the case when the group G is not simply connected, the finite group $O^{p'}(G_\sigma)$ does not coincide with G_σ; hence the order of $(G_1^g)_\sigma \cap O^{p'}(G_\sigma)$ is less than the number given in the lemma. In this case, we have $G_\sigma = \widehat{H} O^{p'}(G_\sigma)$, where $|\widehat{H} : H| = d_1/d$. Here $\widehat{H}$ is a maximal torus of G_σ, H is a maximal torus of $O^{p'}(G_\sigma)$, d_1 is the order of the center of $O^{p'}(G_\sigma)$, and d is the order of the center of $(G_{sc})_\sigma$. Thus to find the order of the subgroup $(G_1^g)_\sigma$ in the group $O^{p'}(G_\sigma)$, the number given in the lemma should be multiplied by d_1/d. Indeed, a connected reductive subgroup of maximal rank contains a maximal torus of a connected simple linear algebraic group, and hence

$$(G_1^g)_\sigma = \widehat{H}((G_1^g)_\sigma \cap O^{p'}(G_\sigma)).$$

Therefore,

$$|(G_1^g)_\sigma : ((G_1^g)_\sigma \cap O^{p'}(G_\sigma))| = \frac{d_1}{d};$$

thus,

$$|(G_1^g)_\sigma \cap O^{p'}(G_\sigma)| = \frac{d_1}{d} |(G_1^g)_\sigma|.$$

We now consider various particular types of finite Chevalley groups. In what follows, we assume the assumption and notation of Lemma 4.1. Suppose that G is a simple connected linear algebraic group with root system Φ, σ is a Frobenius automorphism of this group ($\sigma = q\sigma_0$, $q = p^\alpha$), and $G_0 = O^{p'}(G_\sigma)$. Let $A = A_s \times A_u$ be an Abelian subgroup of G_0. It can be assumed that $A_s \nleq Z(G_0)$. By Lemma 3.1, the group G has a connected reductive σ-invariant subgroup R of maximal rank (R^0 in the notation of Lemma 3.1) containing the group

$$A_0 = A_u \times A_{s0} \qquad (A_{s0} = A_s \cap R),$$

where the group A/A_0 is isomorphic to a section of the Weyl group $W(\Phi)$, $A_{s0} \leq Z(R)$. Let $S = Z(R)^0$.

Type ^2A_n

Lemma 4.2 [2, Proposition 8]**.** *In the notation of Lemma* 4.1*, let G be a group of type A_n, and let G_σ be the twisted form of G. Suppose that G_1 is a σ-invariant reductive subgroup of maximal rank in G corresponding to a partition λ of the integer $n+1$. Let G_1^g be the σ-invariant reductive subgroup of G obtained by twisting G_1 by the element $w \in W$ defined by the rule $\pi(g^\sigma g^{-1}) = w$. Let τ be the image of w under the homomorphism $N_W(W_1) \to \mathrm{Aut}_W(\Delta_1)$. Let n_i be the number of parts of λ that are equal to i, so that $\mathrm{Aut}_W(\Delta_1) \cong S_{n_2} \times S_{n_3} \times \cdots$. Suppose that $\sigma_0\tau$ ($|\sigma_0| = 2$) gives rise to partitions $\mu^{(2)}, \mu^{(3)}, \ldots$ of the integers $n_2, n_3, \ldots$, respectively. Then the simple components of the semisimple group $(M^g)_\sigma$ are of type $A_{i-1}(q^{\mu_j^{(i)}})$ for even $\mu_j^{(i)}$ and of type $^2A_{i-1}(q^{2\mu_j^{(i)}})$ for odd $\mu_j^{(i)}$.*

The order of the semisimple part $(S^g)_\sigma$ of the group $(G_1^g)_\sigma$ is given by

$$(q+1)|(S^g)_\sigma| = \prod_{i,j\,:\,\mu_j^{(i)} \text{ is even}} (q^{\mu_j^{(i)}} - 1) \prod_{i,j\,:\,\mu_j^{(i)} \text{ is odd}} (q^{\mu_j^{(i)}} + 1).$$

Since the group A/A_0 is isomorphic to a section of the Weyl group $W(A_n) \cong S_{n+1}$, it follows from Table 1 that $|A/A_0| \leq 3^{(n+1)/3}$. Therefore, by using Lemmas 3.3 and 4.2, we obtain the following estimate:

$$(q+1)|A| \leq \frac{d_1}{d} 3^{(n+1)/3} \prod_{i,j\,:\,\mu_j^{(i)} \text{ is even}} (q^{\mu_j^{(i)}} - 1) \prod_{i,j\,:\,\mu_j^{(i)} \text{ is odd}} (q^{\mu_j^{(i)}} + 1) \prod_{i,j} (i, q^{\mu_j^{(i)}} - 1) q^{\mu_j^{(i)}[i^2/4]}. \quad (1)$$

Let $S = Z(R)^0$. Since there are no bad numbers for a root system of type A_n, it follow from Lemma 2.4 that $\dim(S) \geq 1$. First, we consider the case in which $\dim(S) = 1$. Then the group R has only two simple components, since otherwise the dimension of the central torus would be more than 1. Therefore,

$$R = A_{m_1-1}(K) * A_{m_2-1}(K) * S,$$

where $m_1 + m_2 = n+1$. Let us show that in this case $R = C_{A_n(K)}(S)$. Indeed, the centralizer of a torus in a connected reductive group is connected; hence $R \leq C_{A_n(K)}(S)$. On the other hand, $A_n(K)$ is the unique subsystem connected subgroup containing $A_{n_1-1}(K) * A_{n_2-1}(K)$. Since its center has dimension 0, it does not coincide with S. Thus any element of $A_n(K)$ that centralizes S lies in R.

Assume that $A_s \neq A_{s0}$. Since the group A_s normalizes S and $A_{s0} \in R$, we obtain an action of the group A_s/A_{s0} by automorphisms of the character group $X(S)$ of S. This action is defined as follows. If $x \in A_s/A_{s0}$ and $\chi \in X(S)$, then for any $s \in S$ we have $\chi^x(s) = \chi(s^x)$. Here by s^x

we mean the conjugation by any representative of x in the group A_s. Clearly, this conjugation does not depend on the choice of the representative, since A_{s0} centralizes S. The action of x is an automorphism of the group $X(S)$. Indeed, the fact that the operation is preserved is obvious, since x is a homomorphism of the group $X(S)$. Since any element of a group has an inverse, the action of x is bijective, and hence is an automorphism of the group $X(S)$. Since the dimension of the torus S is 1, the group $X(S)$ is isomorphic to $\mathbb{Z}$. The only nontrivial automorphism of $\mathbb{Z}$ is an automorphism of order 2. Thus if the group A_s/A_{s0} is nontrivial, then its order is equal to 2.

Besides, in this case we have $|A_{s0}\cap S|\le 2$. Indeed, A_s is an Abelian group; hence each element of A_{s0} is fixed by the action of x. By the definition of the action of x on $X(S)$, it follows that for the generating character χ in $X(S)$ and an arbitrary element $t\in A_{s0}\cap S$, the following relation holds:

$$\chi(t)=\chi^{-1}(t^x)=\chi^{-1}(t)=\chi(t^{-1}),$$

i.e., $t^{-1}=t$ and, in particular, $|A_{s0}\cap S|\le 2$. Since $d_1/d\le 1$, the factor d_1/d can be replaced by 1. The factor $3^{(n+1)/3}$ appeared in estimating the order of the group A/A_0. Since in our case $|A/A_0|=2$, the factor $3^{(n+1)/3}$ is replaced by 2. Besides, since $|S\cap A_0|\le 2$, instead of $|S|$ in (1) we can also put 2. Thus, by (1), we have either

$$|A|\le 4(m_1,q+1)q^{[m_1^2/4]}\cdot(m_2,q+1)q^{[m_2^2/4]}$$

or

$$|A|\le 4\biggl(\frac{n+1}{2}-1,q^2-1\biggr)q^{2[(n+1)/16]}$$

(the last possibility occurs if $m_1=m_2$, i.e., $n+1$ is even). Let us show that for $n\ne 2$ and $q\ne 3$ in both cases we have $|A|\le q^{[(n+1)^2/4]}$.

For $n\ge 14$ we can write

$$\begin{aligned}4(m_1,q+1)q^{[m_1^2/4]}\cdot(m_2,q+1)q^{[m_2^2/4]}&\le 4(q+1)^2\cdot q^{m_1^2/4+m_2^2/4}\le q^6\cdot q^{m_1^2/4+m_2^2/4}\\&\le q^{m_1^2/4+m_2^2/4+7-1}\le q^{m_1^2/4+m_2^2/4+n/2-1}\\&\le q^{m_1^2/4+m_2^2/4+m_1m_2/2-1}\le q^{[(n+1)^2/4]}.\end{aligned}$$

If $8\le n\le 13$ and $m_1,m_2>1$, then

$$\begin{aligned}4(m_1,q+1)q^{[m_1^2/4]}\cdot(m_2,q+1)q^{[m_2^2/4]}&\le 4(q+1)^2\cdot q^{m_1^2/4+m_2^2/4}\le q^6\cdot q^{m_1^2/4+m_2^2/4}\\&\le q^{m_1^2/4+m_2^2/4+8-2}\le q^{m_1^2/4+m_2^2/4+n-2}\\&\le q^{m_1^2/4+m_2^2/4+m_1m_2/2-1}\le q^{[(n+1)^2/4]}.\end{aligned}$$

If $6\le n\le 7$, $q>2$, and $m_1,m_2>1$, then

$$\begin{aligned}4(m_1,q+1)q^{[m_1^2/4]}\cdot(m_2,q+1)q^{[m_2^2/4]}&\le 4(q+1)^2\cdot q^{m_1^2/4+m_2^2/4}\le q^4\cdot q^{m_1^2/4+m_2^2/4}\\&\le q^{m_1^2/4+m_2^2/4+6-2}\le q^{m_1^2/4+m_2^2/4+n-2}\\&\le q^{m_1^2/4+m_2^2/4+m_1m_2/2-1}\le q^{[(n+1)^2/4]}.\end{aligned}$$

If $8\le n\le 13$, $q>2$, and $m_1=1$, then

$$4(n,q+1)q^{[n^2/4]}\le 4(q+1)\cdot q^{n^2/4}\le q^3\cdot q^{n^2/4}\le q^{n^2/4+4-1}\le q^{[(n+1)^2/4]}.$$

If $n=7$, $q>2$, and $m_1=1$, then

$$4(7,q+1)q^{12}<q^{16}.$$

If $n=6$, $q>2$, and $m_1=1$, then

$$4(6,q+1)q^9 < q^{12}.$$

If $n=5$, $q>2$, and $m_1=1$, then

$$4(5,q+1)q^6 < q^9.$$

If $n=5$, $q>2$, $m_1=2$, $m_2=4$, then

$$4(2,q+1)q(4,q+1)q^4 < q^9.$$

If $n=5$, $q>2$, $m_1=3$, $m_2=3$, then $4(3,q+1)q^2(3,q+1)q^2 < q^9$. The remaining cases are dealt with similarly. Subsequently, in such cases we omit computations and only write out the result.

Let us consider the group ${}^2A_2(3^2)$ separately; in this case $m_1=1$, $m_2=2$. Its large Abelian subgroup is of order 16 (it is a maximal torus which is isomorphic to $\mathbb{Z}_4\times\mathbb{Z}_4$ and which is unique up to conjugation). It is contained in the group $A_2(K)$, where K is an algebraically closed field of characteristic 3, and $A_2(K)$ is simply connected, i.e., $\Gamma_\pi=\Gamma_{sc}$. The group R can be expressed as $R=M*S$, where M is a semisimple group of type A_1, S is a torus of dimension 1, and $M\cap S$ is finite. If M is a group of adjoint type, then its center is trivial, and we have $|A|\le |S|\cdot|a(A_1(3))|\le 4\cdot 3<16$. If M is simply connected, then its center is nontrivial and is of order 2. Let us show that in this case the intersection $M\cap S$ coincides with the center of M. Indeed, $M=\langle X_r, X_{-r}\rangle$, r is a root of a root system of type A_2. Let P be the lattice generated by r, and let $\overline{P}$ be the sublattice of Γ_{sc} generated by all possible rational combinations of r. Then $S=\overline{P}^{\perp}$. In an algebraic group of type A_2, we have $|\Gamma_{sc}:\Gamma_{ad}|=3$. The group $\overline{P}$ is cyclic; let χ be its generator. Then either $\chi\in P$ or $\chi^3\in P$. Let x be a nontrivial element of the center of M. Since $|Z(M)|=2$, we have $x^2=1$. Then $x^r=1$, and hence

$$1=x^{\chi^3}=x^{\chi^2}x^{\chi}=(x^2)^{\chi}x^{\chi}=x^{\chi}=1;$$

therefore, $x\in S$. Hence in this case we also have $|A|\le 4\cdot 3<16$.

Now let $A_s=A_{s0}$. By Lemma 3.1, the group A_{s0} is contained in the center of R; hence the order of A is estimated as follows:

$$|A|\le (q+1)(m_1,q+1)q^{[m_1^2/4]}(m_2,q+1)q^{[m_2^2/4]},$$

or

$$|A|\le (q-1)\left(\frac{n+1}{2},q^2-1\right)q^{2[(n+1)^2/16]}.$$

For $n\ge 5$, we have $|A|\le q^{[(n+1)^2/4]}$. The cases $n=2$, $n=3$ must be studied separately; for $n=4$, the following exceptions exist: $m_1=1$, $m_2=4$, $q=3$ and $m_1=2$, $m_2=3$, $q=2$.

Consider the case $n=2$. If $(3,q+1)=3$, then either the center of ${}^2A_2(q^2)$ is nontrivial and

$$|A|\le (q+1)(m_1,q+1)q^{[m_1^2/4]}(n_2,q+1)q^{[m_2^2/4]}\le 3q^2,$$

or

$$|A|\le \frac{1}{3}(q+1)(m_1,q+1)q^{[m_1^2/4]}(m_2,q+1)q^{[m_2^2/4]}\le q^2.$$

If $(3,q+1)=1$, then the group ${}^2A_2(q^2)$ is simply connected, but its center is trivial. A large Abelian subgroup in this case is a maximal torus of order $(q+1)^2$ which is isomorphic to $\mathbb{Z}_{q+1}\times\mathbb{Z}_{q+1}$

and which is unique up to conjugation. Again, we must examine the case in which $m_1 = 1$, $m_2 = 2$. As for the group ${}^2A_2(3^2)$, the intersection $M \cap S$ coincides with the center of M; hence

$$|A| \leq (q+1)q < (q+1)^2,$$

where $(q+1)^2$ is the order of the large Abelian subgroup.

Now let $n = 3$. If $n_1 = n_2 = 2$, then

$$|A| \leq (q+1)(2, q+1)(2, q+1)q^2 \leq q^4$$

for $q \neq 3$. In the group ${}^2A_3(3^2)$, there are three possibilities for the center. The center can be trivial and then ${}^2A_3(3^2)$ is of adjoint type. Hence

$$|A| \leq \frac{1}{4} \cdot 4 \cdot 2 \cdot 2 \cdot 3^2 \leq 3^4.$$

The center can be of order 2 and then

$$|A| \leq \frac{1}{2} \cdot 4 \cdot 2 \cdot 2 \cdot 3^2 \leq 2 \cdot 3^4.$$

And, finally, the center can be of order 4 and then

$$|A| \leq 4 \cdot 2 \cdot 2 \cdot 3^2 \leq 4 \cdot 3^4.$$

If $m_1 = 1$, $m_2 = 3$, then, as in the case of ${}^2A_2(3^2)$, the group $M \cap S$ coincides with the center of M; hence $|A| \leq (q+1)q^2 \leq q^4$.

The last two cases ${}^2A_4(2^2)$ and ${}^2A_4(3^2)$ are studied in a similar way. Thus the case in which the dimension of the central torus is 1 has been examined.

Now suppose that the dimension of the torus S is greater than 1. For all m and k, we have the inequality

$$(q^k + (-1)^{k+1})(m, q^k + (-1)^{k+1})q^{k[m^2/4]} \leq (q+1)(mk, q+1)q^{[m^2k^2/4]}. \tag{2}$$

Besides, for $1 \leq m_1 \leq m_2$, except for the cases $m_1 = 1$, $m_2 = 1$; $m_1 = 1$, $m_2 = 2$; and $m_1 = 1$, $m_2 = 3$, $q = 2$, the following inequality holds:

$$(q+1)(m_1, q+1)q^{[m_1^2/4]}(q+1)(m_2, q+1)q^{[m_2^2/4]} \leq (q+1)(m_1+m_2, q+1)q^{[(m_1+m_2)^2/4]}. \tag{3}$$

By using inequalities (2) and (3), expression (1) can be reduced to one of the following forms:

$$\begin{gathered}(q+1)^2 3^{(n+1)/3}(m_1, q+1)q^{[m_1^2/4]}(m_2, q+1)q^{[m_2^2/4]}(m_3, q+1)q^{[m_3^2/4]} \\ (m_1+m_2+m_3 = n+1),\end{gathered} \tag{4}$$

$$(q+1)^{n-1}3^{(n+1)/3}(2, q+1)q, \tag{5}$$

$$(q+1)^{n-3}3^{(n+1)/3}(2, q+1)^2q^2, \tag{6}$$

$$3^{n+(n+1)/3}2^2, \tag{7}$$

$$3^{n+(n+1)/3-2}2^4. \tag{8}$$

The last two cases are possible only if $q = 2$. Expressions (4)–(8) do not exceed the order of a large Abelian subgroup for $n \geq 5$. Thus it is only necessary to consider the groups ${}^2A_n(q^2)$ for $n \leq 4$ (for $n = 4$, only the group ${}^2A_4(2^2)$). All cases are examined in the same way, so we shall consider only the case $n = 3$. For $n = 3$, there are only two possibilities for the connected reductive group R. The group R is either a maximal torus or the central product of a torus of dimension 2 and a group of type A_1.

Suppose that the group R is a maximal torus, i.e., $A = A_s$. Then R is the homomorphic image of the group of diagonal matrices under the natural homomorphism

$$SL_4(K) \to SL_4(K)/Z, \qquad \text{where} \quad Z \le Z(SL_4(K)).$$

Since the group A_s normalizes R, we can define the action of A_s/A_{s0} on R by the rule $s^{\bar{x}} = s^x$, where $s \in R$, $x \in A_s$, and $\bar{x} \in A_s/A_{s0}$. The elements of A_s/A_{s0} permute the diagonal elements of R. Under the action of the group A_s/A_{s0}, the set $\{1, 2, 3, 4\}$ (the set of diagonal positions) is partitioned into orbits. Since the group A_s is Abelian, the elements of A_s/A_{s0} centralize the elements of A_{s0}; hence elements in positions corresponding to one orbit can only differ by a factor belonging to Z. Hence the group A_{s0} contains a subgroup A_{s1} of index at most $|Z|$ (note that $|Z| \le 4$) which is contained in a torus of dimension less by one than the number of orbits of A_s/A_{s0}. If there exist four orbits, then $A_s = A_{s0}$ and, by Lemma 3.2, we have $|A| \le (q+1)^3$, which is not greater than the order of a large Abelian subgroup as given in Table 2. If there are three orbits, then the order of A_s/A_{s0} is 2,

$$|A_{s0}/A_{s_1}| \le (4, q+1), \quad |A_{s1}| \le (q+1)^2, \qquad \text{i.e.,} \quad |A_s| \le 2(4, q+1)(q+1)^2,$$

which is again not greater than the order of a large Abelian subgroup as given in Table 2. If there are two orbits, then

$$|A_s/A_{s0}| \le 4, \quad |A_{s0}/A_{s1}| \le (4, q+1), \quad |A_{s1}| \le q+1, \qquad \text{whence} \quad |A_s| \le 4(4, q+1)(q+1),$$

i.e., the order of A_s is again not greater than the order of a large Abelian subgroup. Finally, if there is only one orbit, then $|A_s/A_{s0}| \le 4$, $A_{s0}/A_{s1} = A_{s0} \le Z$, and hence $|A_s| \le 4(4, q+1)$.

Now suppose that R is a central product of a torus S of dimension 2 and a simple connected group M of type A_1. By Lemma 2.3, we have $A_s \le N(R)$. By [2, Lemma 1], it follows that $N(R)/R \cong K \ltimes (W_R^{\perp})$, where K is isomorphic to the symmetry group of the Dynkin diagram for the semisimple part of R, and $W_R^{\perp} = \langle w_s \mid (s, r) = 0 \text{ for all } r \in \Phi(R)\rangle$. Since the rank of M is 1, the group K is trivial, and $W_R^{\perp} \le S_3$. Therefore, $|A_s/A_{s0}| \le 3$. As in the proof of Lemma 2.3, in R we can find a maximal torus T which is normalized by A_s. The torus S is contained in all maximal tori; hence it is contained in T. As in the previous case, let us define an action of the group A_s on T. In some basis of the space K^4, the torus T is diagonalizable, and the group A_s/A_{s0} permutes the diagonal elements. If this quotient group is nontrivial, then in A_{s0} there is a subgroup of index not greater than the order of the group Z, which lies in a torus of dimension not greater than 1. Hence

$$|A| = |A_s| \cdot |A_u| \le 4(4, q+1)(q+1)q,$$

which is not greater than the order of a large Abelian subgroup as given in Table 2. If A_s coincides with A_{s0}, then $|A| \le (q+1)^2 q$, which is again not greater than the order of a large Abelian subgroup.

Type D_l

Applying Lemma 4.1 in the case of an algebraic group of type D_l yields the following result.

Lemma 4.3 [2, Proposition 10]. *In the notation of Lemma 4.1, suppose that G is a group of type D_l over an algebraically closed field of characteristic p, and G_1 is a σ-invariant reductive subgroup of maximal rank in G defined by a pair of partitions λ, μ such that $|\lambda|+|\mu|=l$. Let k_i be the number of parts in λ that are equal to i, and let n_i be the number of parts in μ that are equal to i $(n_1=0)$. Then*

$$\operatorname{Aut}_W(\Delta_1)\cong\begin{cases} S_{k_2}\times(\mathbb{Z}_2\wr S_{k_3})\times(\mathbb{Z}_2\wr S_{k_4})\times\cdots\times(\mathbb{Z}_2\wr S_{n_2})\times(\mathbb{Z}_2\wr S_{n_3})\times\cdots & \text{if } k_1>0,\\ \text{a subgroup of index 2 in it} \quad \text{if } k_1=0.\end{cases}$$

Let G_1^g be the σ-invariant subgroup of G obtained by twisting G_1 by the element $w\in W$ which is defined by the rule $\pi(g^\sigma g^{-1})=w$. Assume that w corresponds to $\tau\in\operatorname{Aut}_W(\Delta_1)$. Suppose that $\sigma_0\tau$ $(\sigma=q\sigma_0)$ yields pairs of partitions $\xi^{(i)},\eta^{(i)}$ with $|\xi^{(i)}|+|\eta^{(i)}|=k_i$ (where $\eta^{(i)}$ is empty if $i=2$) and a pair of partitions $\zeta^{(i)},\omega^{(i)}$ with $|\zeta^{(i)}|+|\omega^{(i)}|=n_i$. Then the simple components of the semisimple group $(M^g)_\sigma$ are of type

$$A_{i-1}(q^{\xi_j^{(i)}}),\quad {}^2A_{i-1}(q^{2\eta_j^{(i)}}),\quad D_i(q^{\zeta_j^{(i)}}),\quad {}^2D_i(q^{2\omega_j^{(i)}}),$$

with one component for each part of each of the partitions $\xi^{(i)},\eta^{(i)}$ and $\zeta^{(i)},\omega^{(i)}$.

If $k_1=0$, then the total number of components of type ${}^2A_{i-1}$ with odd i and of type ${}^2D_i$ is even for $\sigma_0=1$, and is odd for $\sigma_0\neq1$.

The order of the torus $(S^g)_\sigma$ is given by

$$|(S^g)_\sigma|=\prod_{i,j}(q^{\xi_j^{(i)}}-1)\prod_{i,j}(q^{\eta_j^{(i)}}+1).$$

In view of Table 1 and Lemmas 3.3, 4.3, the following inequality holds:

$$\begin{aligned}|A|\le 2^l\prod_{i,j}(q^{\xi_j^{(i)}}-1)\prod_{i,j}(q^{\eta_j^{(i)}}+1)\prod_{i,j}(i,q^{\xi_j^{(i)}}-1)q^{\xi_j^{(i)}[i^2/4]}\prod_{i,j}(i,q^{\eta_j^{(i)}}+1)q^{\eta_j^{(i)}[i^2/4]}\\ \times\prod_{i,j}(4,q^{\zeta_j^{(i)}}-1)q^{\zeta_j^{(i)}\frac{i(i-1)}{2}}\prod_j(4,q^{\omega_j^{(4)}}+1)q^{6\omega_j^{(4)}}\prod_{i>5,j}(4,q^{\omega_j^{(i)}}+1)q^{\omega_j^{(i)}\frac{(i-1)(i-2)+2}{2}}.\end{aligned}\tag{9}$$

For any $m\ge3$, we have the inequality

$$(q^k-1)\left(\frac{m}{k},q^k-1\right)q^{k[m^2/4k^2]}\le q^{m(m-1)/2},\tag{10}$$

and for any $m\ge4$, we have the inequality

$$(q^k+1)\left(\frac{m}{k},q^k+1\right)q^{k[m^2/4k^2]}\le q^{m(m-1)/2}.\tag{11}$$

Further, for any m, we have

$$(4,q^k-1)q^{k\frac{(m/k)(m/k-1)}{2}}\le q^{m(m-1)/2},\tag{12}$$

$$(4,q^k+1)q^{k\frac{(m/k-1)(m/k-2)+2}{2}}\le q^{((m-1)(m-2)+2)/2}.\tag{13}$$

Besides, for any m_1,m_2, the following inequalities hold:

$$(4,q-1)q^{m_1(m_1-1)/2}q^{m_2(m_2-1)/2}\le q^{(m_1+m_2)(m_1+m_2-1)/2},\tag{14}$$

$$(4,q+1)q^{((m_1-1)(m_1-2)+2)/2}q^{((m_2-1)(m_2-2)+2)/2}\le q^{((m_1+n_2-1)(m_1+m_2-2)+2)/2},\tag{15}$$

$$(4,q+1)q^6\cdot q^{((m-1)(m-2)+2)/2}\le q^{((m+4-1)(m+4-2)+2)/2}.\tag{16}$$

Assume that $\sigma_0 = 1$, i.e., $O^{p'}((D_l(K))_\sigma)$ is a group of normal type. Then, by using inequalities (10)–(16), the right-hand side of (9) can be reduced to the form

$$2^l(2(q-1)q)^{m_1}(2(q+1)q)^{m_2}((3,q+1)(q+1)q^2)^{m_3}(4,q-1)q^{m_4(m_4-1)/2} \\ \times (4,q-1)q^{m_5(m_5-1)/2}(4,q+1)q^{((m_6-1)(m_6-2)+2)/2}, \qquad (17)$$

where $m_1+m_2+m_3+m_4+m_5+m_6=l$, $m_4, m_5, m_6 \ge 2$ (some of the m_i can be 0). For any m_1, m_2, m_3, m_4, m_5, and m_6, the expression (17) is not greater than the order of a large Abelian subgroup as given in Table 2.

Let $\sigma_0 \ne 1$, i.e., $O^{p'}((D_l(K))_\sigma)$ is a group of twisted type. By using inequalities (10)–(16), the right-hand side of (9) is reduced to (17), but by Lemma 4.3, we have $m_6 \ne 0$. Again, for any m_1, m_2, m_3, m_4, m_5, and m_6, the expression (17) is not greater than the order of a large Abelian subgroup as given in Table 2.

Type B_l, odd q

The following result can be obtained as a consequence of Lemma 4.1,

Lemma 4.4 [2, Proposition 11]. *Suppose that G is a group of type B_l over an algebraically closed field of characteristic $p \ne 2$. Let G_1 be the reductive subgroup of maximal rank in G determined by a triple (λ, μ, ν), where λ, μ are partitions, ν is a nonzero integer, and $|\lambda| + |\mu| + \nu = l$. Let k_i be the number of parts in λ that are equal to i, and let n_i be the number of parts in μ that are equal to i ($n_1 = 0$). Then*

$$\mathrm{Aut}_W(\Delta_1) \cong S_{k_2} \times (\mathbb{Z}_2 \wr S_{k_3}) \times (\mathbb{Z}_2 \wr S_{k_4}) \times \cdots \times (\mathbb{Z}_2 \wr S_{n_2}) \times (\mathbb{Z}_2 \wr S_{n_3}) \times \cdots .$$

Let G_1^g be the σ-invariant subgroup of G obtained by twisting G_1 by the element $w \in W$ defined by the rule $\pi(g^\sigma g^{-1}) = w$. Assume that w corresponds to $\tau \in \mathrm{Aut}_W(\Delta_1)$. Suppose that τ yields a pair of partitions $\xi^{(i)}, \eta^{(i)}$ such that $|\xi^{(i)}| + |\eta^{(i)}| = k_i$, and a pair of partitions $\zeta^{(i)}, \omega^{(i)}$ such that $|\zeta^{(i)}| + |\omega^{(i)}| = n_i$, and the parts of these partitions are lengths of positive and negative cycles on the components of τ. Then the simple components of the semisimple group $(M^g)_\sigma$ are of type

$$A_{i-1}(q^{\xi_j^{(i)}}), \quad {}^2A_{i-1}(q^{2\eta_j^{(i)}}), \quad D_i(q^{\zeta_j^{(i)}}), \quad {}^2D_i(q^{2\omega_j^{(i)}}), \quad B_\nu(q),$$

with one component for each part of the partitions $\xi^{(i)}, \eta^{(i)}$ and $\zeta^{(i)}, \omega^{(i)}$.

The order of the torus $(S^g)_\sigma$ is given by

$$|(S^g)_\sigma| = \prod_{i,j}(q^{\xi_j^{(i)}}-1)\prod_{i,j}(q^{\eta_j^{(i)}}+1).$$

Table 1 and Lemmas 3.3, 4.4 imply the following inequality:

$$|A| \le 2^l \prod_{i,j}(q^{\xi_j^{(i)}}-1)\prod_{i,j}(q^{\eta_j^{(i)}}+1)\prod_{i,j}(i, q^{\xi_j^{(i)}}-1)q^{\xi_j^{(i)}[i^2/4]}\prod_{i,j}(i, q^{\eta_j^{(i)}}+1)q^{\eta_j^{(i)}[i^2/4]} \\ \times \prod_{i,j}(4, q^{\zeta_j^{(i)}}-1)q^{\zeta_j^{(i)}i(i-1)/2}\prod_j(4, q^{\omega_j^{(4)}}+1)q^{6\omega_j^{(i)}} \\ \times \prod_{i>5,j}(4, q^{\omega_j^{(i)}}+1)q^{\omega_j^{(i)}((i-1)(i-2)+2)/2}2q^{(\nu(\nu-1)+1)/2}. \qquad (18)$$

By using inequalities (10)–(16), the right-hand side of (18) can be reduced to the following form:

$$2^l(2(q-1)q)^{m_1}(2(q+1)q)^{m_2}((3,q+1)(q+1)q^2)^{m_3}(4,q-1)q^{m_4(m_4-1)/2} \\ \times (4,q+1)q^{((m_5-1)(m_5-2)+2)/2}2q^{(m_6(m_6-1)+1)/2}, \qquad (19)$$

where $m_1+m_2+m_3+m_4+m_5+m_6=l$, $m_4, m_5 \ge 2$. For any m_1, m_2, m_3, m_4, m_5, and m_6, the expression (19) is not greater than the order of a large Abelian subgroup as given in Table 2.

Groups $B_2(2^n)$, $n \geq 2$

Since the characteristic of the field K is 2, in $B_2(K)$ there are no semisimple elements whose order is a bad number, and the center of this group is trivial. Besides, $W(B_2)$ is a 2-group; hence $A_0 = A$. Therefore, the central torus S of the group R has dimension at least 1, and $A \leq R$. Then the following cases are possible.

Suppose that $\dim S = 1$. Then $R = S * A_1(K)$, and $Z(A_1(K))$ is trivial. Therefore, by Lemmas 3.2 and 3.3, we have $|A| \leq (q+1)q \leq q^3$, i.e., this number is not greater than the order of a large Abelian subgroup as given in Table 2.

Suppose that $\dim S = 2$. Then $R = S$, and by Lemma 3.2, we have $|A| \leq (q+1)^2 < q^3$, i.e., this number is again less than the order of a large Abelian subgroup as given in Table 2.

Type G_2

The proof of Lemma 3.1 implies that the group A lies in the centralizer of a certain semisimple element. Since the group $G_2(K)$ is simply connected, the centralizer of any of its semisimple elements is connected (see, for example, [12, Theorem 2.11]). Therefore, the group A is contained in the fixed point set for the Frobenius morphism σ of a certain maximal σ-invariant connected reductive subgroup of maximal rank in $G_2(K)$. By the algorithm of Borel and de Siebenthal, there exist only two maximal connected reductive subgroups R of maximal rank in $G_2(K)$, i.e., the groups $A_2(K)$ and $A_1(K) * A_1(K)$.

Assume that $R = A_2(K)$. Then, by Lemma 2.2, the group A contains a subgroup A_0 of index 3 which lies in $A_2(q)$ or in ${}^2A_2(q^2)$ (by [3, Table 4, p. 138], both these cases are possible). Note that the estimate of the index given in Lemma 2.2 is rather crude (especially for the case in which the number of prime factors is small), since we divide by some factors several times. In our case as well, if the group $A_2(q)$ or ${}^2A_2(q^2)$ is simply connected, then it coincides with R_σ, and hence the subgroup lies entirely inside $A_2(q)$ (or ${}^2A_2(q^2)$). On the other hand, if $A_2(q)$ (or ${}^2A_2(q^2)$) is of adjoint type, then A contains a subgroup of index at most 3 which lies in $A_2(q)$ (or ${}^2A_2(q^2)$), but the latter, in turn, has trivial center. The above arguments and the results obtained earlier for classical groups show that in any case the order of A is not greater than one of the following numbers: $3q^2$ or $(q+1)^2$, which, clearly, do not exceed the number $a(G_2(q))$ given in Table 2. In what follows, we shall omit the detailed analysis of similar situations, referring to Lemma 2.2 and meaning by this not only the assertion of the lemma but rather the method of its proof, which can be used to obtain a much sharper estimate than the one stated in the lemma.

Assume that $R = A_1(K) * A_1(K)$. Then, by Lemma 2.2, we have either $|A| \leq (2, q-1)q^2$ or $|A| \leq (q+1)^2$, which does not exceed the number $a(G_2(q))$ given in Table 2.

Type F_4

Since the group $F_4(K)$ is simply connected, the centralizer of any semisimple element is connected; hence the group A is contained in a proper connected reductive subgroup of maximal rank in $F_4(K)$.

By the algorithm of Borel and de Siebenthal, the maximal (with respect to inclusion) proper σ-invariant reductive subgroups of maximal rank in $F_4(K)$ are

$$B_4(K), \quad A_2(K) * A_2(K), \quad A_1(K) * A_3(K), \quad C_3(K) * A_1(K).$$

Assume that the group A is contained in $R = B_4(K)$. Then, by Lemma 2.2, we shall have $|R_\sigma : O^{p'}(R_\sigma)| \leq 2$. Thus the group A contains a subgroup of index at most 2 which lies in the group $B_4(q)$. Since the number $a(B_4(q))$ has already been found, we have $|A| \leq 2q^7$ if q is odd, and $|A| \leq q^{10}$ if q is even. In both cases, the order of A does not exceed the lower estimate of $a(F_4(q))$.

Now assume that A lies in $R = A_1(K)*C_3(K)$. Then it follows from Lemma 2.2 that $|A| \le 4q^7$, i.e., the order of A is less than the lower estimate of $a(F_4(q))$.

Assume that A lies in a group of type $R = A_1(K) * A_3(K)$. Then $O^{p'}(R_\sigma)$ is isomorphic to one of the following groups: $A_1(q) * A_3(q)$ or $A_1(q) * {}^2A_3(q^2)$ (it follows from [3, Table 2, p. 133] that the two cases are both possible). Hence (again by using Lemma 2.2 and well-known results concerning classical groups) we obtain $|A| \le 8q^5$, which is less than the lower estimate of $a(F_4(q))$.

Assume that A lies in a group of type $R = A_2(K) * A_2(K)$. Then $O^{p'}(R_\sigma)$ is isomorphic either to the group $A_2(q) * A_2(q)$ or to the group ${}^2A_2(q^2) * {}^2A_2(q^2)$ (again we use information from [3, Table 2, p. 133]). Hence we obtain the estimate $|A| \le 9q^4$, which is less than the lower estimate of $a(F_4(q))$.

Type E_6

It is well known [21, lecture of Iwahori, Proposition 5] that for any semisimple element s of a connected simple algebraic group G, the group $C_G(s)/C_G^0(s)$ is isomorphically embedded in the group $\Gamma_{sc}/\Gamma_\pi = \Omega_\pi$. Since for a root system of type E_6 we have $|\Gamma_{sc} : \Gamma_\pi| = 3$, there exists a subgroup A_0 of index at most 3 in A which is contained in a certain connected reductive subgroup of maximal rank in $E_6(K)$. The maximal (with respect to inclusion) connected reductive subgroups of maximal rank in $E_6(K)$ are

$$A_1(K) * A_5(K), \quad A_2(K) * A_2(K) * A_2(K), \quad S * D_5(K).$$

Assume that A_0 lies in $R = A_1(K) * A_5(K)$. Then $O^{p'}(R_\sigma) = A_1(q) * A_5(q)$ (see [3, case of $E_6(q)$]). Then, by Lemma 2.2, we have $|A_0| \le 12q^{10}$, whence $|A| \le 36q^{10}$, which is less than the lower estimate of $a(E_6(q))$.

Assume that A_0 lies in $R = A_2(K) * A_2(K) * A_2(K)$. Then $O^{p'}(R_\sigma)$ is isomorphic to one of the following groups:

$$A_2(q) * A_2(q) * A_2(q), \qquad A_2(q^2) * {}^2A_2(q^2), \qquad A_2(q^3)$$

(see [3, case of $E_6(q)$]). In all cases, Lemma 2.2 gives the estimate $|A_0| \le 27q^9$ (for $q = 2$, we have the estimate $9 \cdot 2^6$). Hence $|A|$ is less than the lower estimate of $a(E_6(q))$.

Finally, assume that A_0 lies in $R = S * D_5(K)$. Then $O^{p'}(R_\sigma) = (S_\sigma) * D_5(q)$. By Lemma 2.2, we have $|A_0| \le 4(q-1)q^{10}$. Hence $|A|$ is less than the lower estimate of $a(E_6(q))$.

Type ${}^2E_6$

The index $|\Gamma_{sc} : \Gamma_{ad}| = 3$; hence the group A contains a subgroup A_0 of index at most 3 which lies in a certain proper connected σ-invariant reductive subgroup R of maximal rank in $E_6(K)$. In this case, the maximal σ-invariant connected reductive subgroups of $E_6(K)$ are

$$A_1(K) * A_5(K), \quad S * D_5(K), \quad A_2(K) * A_2(K) * A_2(K).$$

If the characteristic of K is even, then there is no semisimple element in $E_6(K)$ whose centralizer coincides with $A_1(K)*A_5(K)$; in this case we must consider two additional connected reductive σ-invariant subgroups of maximal rank in $E_6(q)$, i.e., the groups $A_5(K)*S$ and $A_4(K)*A_1(K)*S$.

Assume that A_0 lies in $R = A_1(K) * A_5(K)$ (recall that in this case q is odd). Then

$$O^{p'}(R_\sigma) = A_1(q) * {}^2A_5(q^2)$$

(see [3, Table 1, p. 127]). By Lemma 2.2, we have $|A| \le 24q^{10}$, which is less than the lower estimate of $a({}^2E_6(q^2))$ for $q > 3$. For $q = 3$, we note that the group $E_6(K)$ can be assumed to be simply connected (since in this case the groups ${}^2E_6(3^2)_{sc}$ and ${}^2E_6(3^2)_{ad}$ are isomorphic). Hence the group A itself (and not only its subgroup of index 3) is contained in R. Besides, in this case Lemma 2.2 gives the estimate $|A| \le 4q^{10}$, which is less than the lower estimate of $a({}^2E_6(q^2))$.

Assume that the group A_0 lies in $R = S * D_5(K)$. Then $O^{p'}(R_\sigma) = (S_\sigma) * {}^2D_5(q^2)$ (see [3, Table 1, p. 127]). Therefore, $|A| \le 12(q+1)q^8$ ($|A| \le 3q^8$ if q is even), thus the order of A does not exceed the lower estimate of $a({}^2E_6(q^2))$.

Assume that the group A_0 lies in $R = A_2(K) * A_2(K) * A_2(K)$. Then $O^{p'}(R_\sigma)$ coincides with one of the following groups:

$$A_2(q^2) * A_2(q), \qquad {}^2A_2(q^6), \qquad {}^2A_2(q^2) * {}^2A_2(q^2) * {}^2A_2(q^2)$$

(see [3, Table 1, p. 127]). In any case, we have the estimate $|A| \le 27q^6$, which is less than the lower estimate of $a({}^2E_6(q^2))$.

Assume that the group A_0 lies in $R = A_5(K) * S$ (we consider this case only for a field K of characteristic 2). Then $O^{p'}(R_\sigma)$ coincides with ${}^2A_5(q^2) * S_\sigma$. Hence the order of A does not exceed $6(q+1)q^9$ ($3q^9$ for $q = 2$), which is less than the lower estimate of $a({}^2E_6(q^2))$.

Assume that the group A_0 lies in $R = A_4(K) * A_1(K) * S$ (it is again assumed that the characteristic of K is even). Then $O^{p'}(R_\sigma)$ coincides with ${}^2A_4(q^2) * A_1(q) * S_\sigma$, whence

$$|A| \le 15(q+1)^2 q^6 \qquad (|A| \le 3^2 \cdot 2^6 \text{ for } q = 2),$$

which is less than the lower estimate of $a({}^2E_6(q^2))$.

Type E_7

The index $|\Gamma_{sc} : \Gamma_{ad}| = 2$; hence the group A contains a subgroup A_0 of index 2 which lies in a proper connected reductive subgroup R of $E_7(K)$. A maximal (with respect to inclusion) connected reductive subgroup of maximal rank in $E_7(K)$ is one of the following groups:

$$A_7(K), \quad A_1(K) * D_6(K), \quad A_1(K) * A_3(K) * A_3(K), \quad A_2(K) * A_5(K), \quad S * E_6(K).$$

Assume that the group A_0 lies in $R = A_7(K)$. Then we see that either $O^{p'}(R_\sigma) = A_7(q)$ or $O^{p'}(R_\sigma) = {}^2A_7(q^2)$ (see [4, Table 1]). It follows from Lemma 2.2 that $|A| \le 16q^{16}$, thus the order of A is not greater than the lower estimate of $a(E_7(q))$.

Assume that the group A_0 lies in $R = A_1(K) * D_6(K)$. Then $O^{p'}(R_\sigma)$ coincides with the group $A_1(q) * D_6(q)$ (see [4, Table 1]). In any case, $|A| \le 16q^{16}$, which is less than the lower estimate of $a(E_7(q))$.

Assume that the group A_0 lies in $R = A_1(K) * A_3(K) * A_3(K)$. Then the group $O^{p'}(R_\sigma)$ is isomorphic to one of the following groups: $A_1(q) * A_3(q) * A_3(q)$ or $A_1(q) * A_3(q^2)$ (see [4, Table 1]). In any case, Lemma 2.2 yields $|A| \le 64q^9$, which is not greater than the lower estimate of $a(E_7(q))$.

Assume that the group A_0 lies in $A_2(K) * A_5(K)$. Then the group $O^{p'}(R_\sigma)$ is isomorphic to one of the following groups: $A_2(q) * A_5(q)$ or ${}^2A_2(q^2) * {}^2A_5(q^2)$ (see [4, Table 1]). Therefore, we have $|A| \le 36q^{11} < a(E_7(q))$.

Finally, assume that the group A lies in $R = S * E_6(K)$. Then $O^{p'}(R_\sigma)$ is isomorphic to $(S_\sigma) * E_6(q)$ or to $(S_\sigma) * {}^2E_6(q^2)$. In any case we have $|A| \le 6(q-1)q^{20} < a(E_7(q))$.

Type E_8

Since $\Gamma_{sc} = \Gamma_{ad}$, the centralizer of any semisimple element is connected; hence the group A is contained in a proper connected reductive subgroup of maximal rank in $E_8(K)$. A maximal connected reductive subgroup of $E_8(K)$ is one of the following groups:

$$D_8(K), \quad A_8(K), \quad A_1(K) * A_2(K) * A_5(K),$$
$$A_4(K) * A_4(K), \quad A_3(K) * D_5(K), \quad A_2(K) * E_6(K), \quad A_1(K) * E_7(K).$$

Assume that the group A lies in $R = D_8(K)$. Then $O^{p'}(R_\sigma) = D_8(q)$ (see [4, Table 2]). It follows from Lemma 2.2 that $|A| \le 4q^{27} < a(E_8(q))$. Here and in what follows, for $a(E_8(q))$ we take the lower estimate of this number (see Table 2).

Assume that the group A lies in $R = A_8(K)$. Then we see that either $O^{p'}(R_\sigma) = A_8(q)$ or $O^{p'}(R_\sigma) = {}^2A_8(q^2)$ (see [4, Table 2]). By applying Lemma 2.2, we obtain $|A| \le 9q^{20} < a(E_8(q))$.

Assume that the group A lies in $R = A_1(K) * A_2(K) * A_5(K)$. Then $O^{p'}(R_\sigma)$ is one of the following groups: $A_1(q) * A_2(q) * A_5(q)$ or $A_1(q) * {}^2A_2(q^2) * {}^2A_5(q^2)$ (see [4, Table 2]). Again by using Lemma 2.2 we conclude that $|A| \le 36q^{12} < a(E_8(q))$.

Assume that the group A lies in $R = A_4(K) * A_4(K)$. Then $O^{p'}(R_\sigma)$ is one of the following groups: $A_4(q) * A_4(q)$, ${}^2A_4(q^2) * {}^2A_4(q^2)$, or ${}^2A_4(q^4)$ (see [4, Table 2]). It follows from Lemma 2.2 that $|A| \le 25q^{12} < a(E_8(q))$.

Assume that the group A lies in $R = A_3(K) * D_5(K)$. Then $O^{p'}(R_\sigma)$ is isomorphic to one of the following groups: $A_3(q) * D_5(q)$ or ${}^2A_3(q^2) * {}^2D_5(q^2)$ (see [4, Table 2]). By applying Lemma 2.2, we obtain $|A| \le 16q^{19} < a(E_8(q))$.

Assume that the group A lies in $R = A_2(K) * E_6(K)$. Then $O^{p'}(R_\sigma)$ is one of the following groups: $A_2(q) * E_6(q)$ or ${}^2A_2(q^2) * {}^2E_6(q^2)$. We have $|A| \le 9q^{22} < a(E_8(q))$.

Finally, assume that the group A lies in $R = A_1(K) * E_7(K)$. Then $O^{p'}(R_\sigma) = A_1(q) * E_7(q)$. By Lemma 2.2, we have $|A| \le 4q^{33} < a(E_8(q))$.

Type ${}^3D_4$

There are only two maximal connected σ-invariant subgroups of maximal rank in $D_4(K)$:

$$A_1(K) * A_1(K) * A_1(K) * A_1(K) \quad \text{and} \quad T * A_2(K),$$

where T is a torus of dimension 2.

Assume that the group A lies in $R = A_1(K) * A_1(K) * A_1(K) * A_1(K)$. Then $O^{p'}(R_\sigma) = A_1(q) * A_1(q^3)$ (see [3, Table 7, p. 140]). It follows from Lemma 2.2 that $|A| \le 4q^4$ (therefore, $|A| \le 27$ if $q = 2$, and $|A| \le 2q^4$ if $q = 3$), which is less than $q^5 = a({}^3D_4(q^3))$.

Assume that the group A lies in $R = T * A_2(K)$. Then $O^{p'}(R_\sigma) = (T_\sigma) * A_2(q)$ or $O^{p'}(R_\sigma) = (T_\sigma) * {}^2A_2(q^2)$ (see [3, Table 7, p. 140]). It follows from Lemma 2.2 that $|A| \le 3(q^2 + q + 1)q^2$ ($|A| \le 28$ for $q = 2$ and $|A| \le 117$ for $q = 3$), which is again not greater than $q^5 = a({}^3D_4(q^3))$.

The orders $a(G)$ of large Abelian subgroups or their estimates for all simple finite groups of Lie type except for ${}^2F_4(q)$ are given in Table 2. In the "Source" column of this table, we indicate the paper in which the groups of the corresponding order were studied. If the order is given precisely, then the structure and the number of conjugacy classes of large Abelian subgroups is known. If the lower and upper estimates are given, in the present paper it is proved that if a certain Abelian subgroup A of G contains a noncentral semisimple element, then its order is less than the order of some unipotent Abelian subgroup. Thus a large Abelian subgroup in any finite simple group of Lie type (except for ${}^2F_4(q)$) coincides with a large semisimple or large unipotent Abelian subgroup. If the center of a group of Lie type G is nontrivial, i.e., the group G is not simple, then the value

Table 2

Group G	$a(G)$	Source
$A_n(q)$, except for $A_1(q)$, q even and $A_2(q)$, $(3, q-1) = 1$	$q^{[(n+1)^2/4]}$	[5]
$A_1(q)$, q even	$q+1$	[17]
$A_2(q)$, $(3, q-1) = 1$	q^2+q+1	[17]
$B_n(q)$, $n \geq 4$, q odd	$q^{n(n-1)/2+1}$	[5] and [7]
$B_3(q)$, q odd	q^5	[5] and [7]
$C_n(q)$, except for $C_2(2)$	$q^{n(n+1)/2}$	[5], [6], and [8]
$D_n(q)$	$q^{n(n-1)/2}$	[5] and [7]
${}^2A_n(q^2)$, except for ${}^2A_2(q^2)$, $(3, q+1) = 1$ and ${}^2A_3(2^2)$	$q^{[(n+1)^2/4]}$	[8]
${}^2A_2(q^2)$, $(3, q+1) = 1$	$(q+1)^2$	[ ]
${}^2A_3(2^2)$	27	[ ]
${}^2D_n(q^2)$, $n \geq 5$	$q^{(n-1)(n-2)/2+2}$	[7]
${}^2D_4(q^2)$	q^6	[7]
$G_2(q)$, q not divisible by 3, except for $G_2(2)$	q^3	[ ]
$G_2(q)$, q a power of 3	q^4	[ ]
$F_4(q)$, q even	$q^{11} \leq a(G) \leq q^{17}$	[ ]
$F_4(q)$, q odd	$q^9 \leq a(G) \leq q^{14}$	[17]
$E_6(q)$	$q^{16} \leq a(G) \leq q^{20}$	[17]
$E_7(q)$	$q^{27} \leq a(G) \leq q^{32}$	[17]
$E_8(q)$	$q^{36} \leq a(G) \leq q^{61}$	[17]
${}^2B_2(2^{2n+1})$	2^{2n+2}	[19]
${}^2G_2(q)$	q^2	[20]
${}^3D_4(q^3)$	q^5	[ ]
${}^2E_6(q^2)$	$q^{12} \leq a(G) \leq q^{20}$	[ ] and [17]

given in the table must be multiplied by the order of the center. By empty brackets we denote a reference to the results of the present paper.

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