Algebra and Logic, Vol. 40, No. 5, 2001

LARGE ABELIAN UNIPOTENT SUBGROUPS OF FINITE CHEVALLEY GROUPS

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UDC 512.542.5

We compute maximal orders of unipotent Abelian subgroups, estimate p-ranks, and describe the structure of Thompson subgroups of maximal unipotent subgroups of finite exceptional groups of Lie type.

INTRODUCTION

Subgroups of finite Chevalley groups have been extensively studied within last 40 years. The importance of research on Chevalley groups is explained primarily by the fact that they constitute the bulk of the known finite simple groups and play a significant part in a series of applications.

Of special interest to researchers are the structure and orders of maximal Abelian subgroups and, in particular, of large Abelian subgroups. In the 1960s, orders of maximal tori in all finite Chevalley groups were estimated. In [1], a versatile method was propounded for determining all non-conjugate maximal tori in every finite Chevalley group. In [2-5], orders were estimated and the structure determined for large Abelian unipotent subgroups in all finite classical groups; there, too, Thompson subgroups of maximal unipotent subgroups in finite classical groups were found. Upper bounds for orders of Abelian subgroups in all finite simple groups were obtained in [6].

In [7], orders of large unipotent Abelian subgroups in the groups $G_2(q)$ and ${}^3D_4(q^3)$ are estimated, and it is proved that the order of an arbitrary Abelian subgroup of a finite Chevalley group does not exceed an order of a large unipotent Abelian subgroup or order of a maximal torus. Therefore, to bring to a close the study of large unipotent Abelian subgroups, and of all large Abelian subgroups of finite Chevalley groups therewith, we need to find large unipotent Abelian subgroups in all finite exceptional groups of Lie type.

In [8], note, problems are posed which deal in finding orders of large unipotent Abelian subgroups and p-ranks of finite Lie-type groups.

In the present article, we compute orders of large unipotent Abelian subgroups, estimate p-ranks, and describe the structure of Thompson subgroups in maximal unipotent subgroups of finite exceptional groups of Lie type. The results obtained are collected in Table 4. As distinct from [2-5] where large Abelian unipotent subgroups of all groups of classical type are found by appealing to peculiarities of the structure of each of the classical groups in question, our Theorems 1-3 provide a simple and clear means for describing Abelian subgroups of maximal order in all split finite groups. These results can also be generalized to the linear algebraic groups defined over an algebraically closed field of arbitrary characteristic. In this case the number $m(A)$ yields an upper bound for a dimension of a closed unipotent subgroup.

*Supported by RFFR grants No. 99-01-00550 and 01-01-06184, by a SO RAN grant for Young Scientists (Presidium Decree No. 83 of 3.10.2000), and by a Fundamental Natural Science Research (FNSR) grant of RF Ministry of Education (code 2000.1.77).

Translated from *Algebra i Logika*, Vol. 40, No. 5, pp. 523-544, September-October, 2001. Original article submitted February 7, 2000.

1. NOTATION

In this article we use the notation adopted and the definitions couched in [9, 10]. If G is a group, by writing $H \leq G$ and $H \trianglelefteq G$ we mean that H is, respectively, a subgroup and a normal subgroup of G; $|G : H|$ is an index of H in G and $N_G(H)$ is a normalizer of H in G. If H is normal in G then G/H denotes a factor group of G w.r.t. H. If M is a subset of G then $\langle M \rangle$ is a subgroup generated by the set M and $|M|$ is the cardinality of M (or order of an element, if M consists of just one element). Denote by $[x, y] = x^{-1}y^{-1}xy$ a commutator of the elements x and y, and by $[A, B]$ the commutant of subgroups A and B in G.

Let φ be an homomorphism of G and g be an element of G. Then G^φ and g^φ are images of G and g, respectively. If φ is an endomorphism of G then we denote the set of φ-stable points by G_φ.

The notation related to Lie-type groups is borrowed from [10]. In dealing with Chevalley groups, we assume that $GF(q)$ is a field of order q, p its characteristic, and $GF(q)^*$ is a multiplicative group of $GF(q)$. A Chevalley group G corresponding to a root system Φ over $GF(q)$ is denoted by $\Phi(q)$, and we call $GF(q)$ the *base* field of G. Twisted groups are denoted by ${}^2A_n(q^2)$, ${}^2D_n(q^2)$, ${}^2E_6(q^2)$, ${}^3D_4(q^3)$, ${}^2B_2(q)$, ${}^2G_2(q)$, and ${}^2F_4(q)$, in which case the base field for G can be one of $GF(q)$, $GF(q^2)$, or $GF(q^3)$, depending on the degree of q in a group's denotation. If ρ is a symmetry of the Dynkin diagram for a root system Φ of G, which extends to a graph automorphism of G, then r^ρ is denoted by $\overline{r}$, and r^{ρ^2} — by $\overline{\overline{r}}$, where $r \in \Phi$. Also, an image of every element $t \in GF(q^\alpha)$ ($\alpha = 1, 2$, or 3) under a corresponding field automorphism λ is denoted by $\overline{t} = t^\lambda$ and $\overline{\overline{t}} = t^{\lambda^2}$, respectively. $GF(q)$ is a *definition field* for all groups of Lie type. An element x in a Chevalley group $\Phi(q)$ is said to be unipotent if its order is a degree of p. Similarly, a subgroup of $\Phi(q)$ is unipotent if its order is a degree of p.

Further, if G is a finite Chevalley group then $U(G)$ is a set of maximal unipotent subgroups of G and U is some element of $U(G)$. An Abelian subgroup of maximal order in a finite group G is referred to as *large*. A set of large Abelian (resp., of large elementary Abelian and of large normal Abelian) subgroups of G is denoted by $A(G)$ [resp., $A_e(G)$ and $A_n(G)$], and an order of some element in $A(G)$ [$A_e(G)$ and $A_n(G)$] — by $a(G)$ [$a_e(G)$ and $a_n(G)$]. If G is a finite group, then $J(G)$ is a Thompson subgroup of G (the subgroup generated by elements of $A(G)$); $J_e(G)$ is a subgroup generated by elementary Abelian subgroups of maximal order; $J_n(G)$ is a subgroup generated by normal Abelian subgroups of maximal order; $m_p(G)$ is a p-rank, that is, the maximum of ranks of Abelian p-subgroups, of G.

For every split finite group of Lie type, we assume that X_r is a root subgroup corresponding to some root $r \in \Phi$, $x_r(t)$ is an element of X_r, where t is an element of the definition field of G. It is well known that if some order is defined on a root system Φ then every element of a unipotent subgroup is uniquely represented as a product of elements $x_r(t)$, taken in the given order (see, e.g., [10]). Below is a so-called Chevalley commutator formula, which is valid for $x_r(t)$ and $x_s(u)$ (see [10, Sec. 5.2.2]).

LEMMA 1. Let $x_r(t)$ and $x_s(u)$ be elements belonging to root subgroups X_r and X_s, respectively, and $r \neq \pm s$. Then

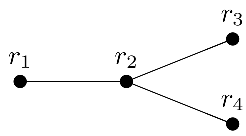
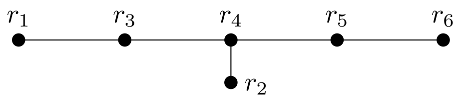
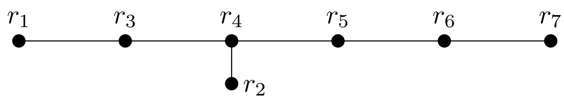
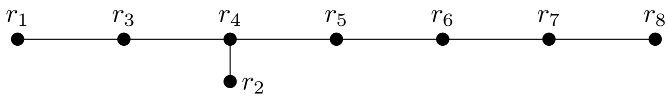
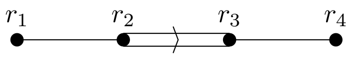
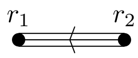
$$[x_r(t), x_s(u)] = \prod_{ir+js\in\Phi,\ i,j>0} x_{ir+js}(C_{ijrs}(-t)^i u^j),$$

where the constants C_{ijrs} do not depend on t and u.

From the formula above we see that the commutant of X_r and X_s lies in the group generated by root subgroups X_{ir+js}, where $i, j > 0$ and $ir + js \in \Phi$.

Let p be a characteristic of the definition field for a finite group G of Lie type. A subset Ψ of a root system Φ is called an *Abelian p-subset* if, for every two roots $r, s \in \Psi$, either $r + s$ does not belong to Ψ,

TABLE 1. Root Systems and Dynkin Diagrams

Type of Φ	Dynkin diagram
D_4	
E_6	
E_7	
E_8	
F_4	
G_2	

or $C_{11rs} = 0$ in characteristic p. For the root system Φ, $a(\Phi, p)$ denotes the maximum of orders of Abelian p-subsets of Φ^+. A subset Ψ of the root system Φ^+ is said to be *Abelian* if $r + s \notin \Phi$ for every two roots $r, s \in \Psi$. Abelian subsets of maximal order are presented in [11].

Recall that for every root system Φ there exists a set of roots $r_1, \dots, r_n$ such that every root of Φ is uniquely represented as $\sum\limits_{i=1}^{n} \alpha_i r_i$, where all α_i are integers, and they all are either non-negative or non-positive at a time. Such a set of roots is called a *fundamental system* of Φ. The fundamental system, in this case, is a basis for a space $\mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$. A dimension of $\mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$ is called the *rank* of Φ. Below, we assume that all fundamental roots are positive. A root r is positive iff it is representable as a linear combination of fundamental roots with non-negative coefficients. For a root system Φ, Φ^+ (Φ^-) denotes a set of positive (negative) roots.

We say that a partial (or linear) order is defined on a root system Φ (on a space $\mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$) if that order agrees with addition of roots and multiplication by a real scalar. In what follows, unless otherwise stated, we assume that on $\mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$ a fixed linear order $\leq$ (corresponding to a fundamental system $r_1, \dots, r_n$) is given via the following rule: either $v = 0$, or $0 \leq v = \sum\limits_{i=1}^{n} \alpha_i r_i$ iff the last non-zero coefficient α_i is greater than 0. Writing $r \leq s$ means that $0 \leq s - r$ and writing $r < s$ means that $r \leq s$ and $r \neq s$. We also fix a partial order $\preceq$ given by the following rule: $r \preceq s$ iff $s - r = \sum\limits_{i=1}^{n} \alpha_i r_i$, and all coefficients α_i are non-negative.

Let $x \in U$ be some unipotent element; then $x = \prod\limits_{r \in \Phi^+} x_r(t_r)$, where $t_r \in GF(q)$ and roots r agree with the order $\leq$, is called the *canonical form* of x. Such a form, as noted above, is unique for every

unipotent element x of U. Denote by $\Phi(x)$ a set of roots r such that $t_r \neq 0$. If L is a subgroup of U, put $\Phi(L) = \bigcup_{x\in H} \Phi(x)$. Assume that a linear order is defined on Φ; for a unipotent element x, $m(x)$ then denotes a minimal element of $\Phi(x)$. For a unipotent subgroup $L \leq U$, put $m(L) = \bigcup_{x\in L} \{m(x)\}$.

In Table 1, we give Dynkin diagrams of exceptional type root systems and of a root system of type D_4, as well as the labeling of vertices by fundamental roots in these.

2. MAIN THEOREMS

THEOREM 1. Let G be a finite split Chevalley group with a definition field $GF(q)$ and $L \leq U$ be some unipotent subgroup of G. Then $|L| \leqslant q^{|m(L)|}$.

Proof. For every $r \in m(L)$, choose some maximal subset $X(r) = \{x(r, t_r) \mid t_r \in GF(q)\}$ of L satisfying the following conditions:

(1) $m(x(r, t_r)) = r$ [this reflects on the use of a root r in the representation of $x(r, t_r)$];

(2) an element $x_r(t_r)$ occurs in the canonical form of $x(r, t_r)$ [this reflects on the use of an element t_r of $GF(q)$ in the representation of $x(r, t_r)$];

(3) for every two distinct elements $x(r, t_r)$ and $x(r, u_r)$ in a root subgroup $X(r)$, $t_r \neq u_r$ (hence, if x and y are two distinct elements of $X(r)$ then the factors of X_r too are different in the canonical forms of x and y);

(4) $e \in X(r)$ for all $r \in m(L)$, where e is unity in the group L.

The set $X(r)$ is an analog in L of a root subgroup X_r of U. Generally, $X(r)$ is not uniquely determined. Yet, as will be clear from our further reasoning, the cardinality of $X(r)$ is constant and does not exceed $|GF(q)| = q$. We claim the validity of the following:

> Fix some set $\{X(r) \mid r \in m(L)\}$. Then every element x of L can be written in the form $x = \prod_{r\in m(L), r\geq m(x)} x(r, t_r)$, taken in a given order $\leq$. $\quad (*)$

Assume that $(*)$ is invalid and that x is a maximal counterexample w.r.t. $m(x)$. Suppose that $r = m(x)$ and that an element $x_r(t_r)$ of the root subgroup X_r occurs in the canonical decomposition of x. By construction, the set $X(r)$ is non-empty and contains an element $x(r, u)$ such that $u = t_r$. It follows that $x = x_r \prod_{s>r} x_s(t_s)$ and $(x(r, t_r))^{-1} = x_r(-t_r) \prod_{s>r} x_s(u_s)$. Consequently, we have the following chain of equalities:

$$(x(r, t_r))^{-1} \cdot x = x_r(-t_r) \cdot \prod_{s>r} x_s(u_s) \cdot x_r(t_r) \cdot \prod_{s>r} x_s(t_s) =$$

$$x_r(-t_r) \cdot x_r(t_r) \cdot \prod_{s>r} x_s(u_s) \cdot \left[\prod_{s>r} x_s(u_s), x_r(t_r)\right] \cdot \prod_{s>r} x_s(t_s) =$$

$$\prod_{s>r} x_s(u_s) \cdot \left[\prod_{s>r} x_s(u_s), x_r(t_r)\right] \cdot \prod_{s>r} x_s(t_s).$$

In view of the Chevalley commutator formula, the commutator $\left[\prod_{s>r} x_s(u_s), x_r(t_r)\right]$ lies in the subgroup $\langle X_s \mid s > r\rangle$, just as do $\prod_{s>r} x_s(t_s)$ and $\prod_{s>r} x_s(u_s)$. Consequently, the element $x_1 = (x(r, t_r))^{-1} \cdot x$ belongs to

$\langle X_s \mid s > r\rangle$. In particular, $m(x_1) > m(x)$. Since $m(x_1) > m(x)$ and x is a maximal counterexample w.r.t. $m(x)$, x_1 is decomposed thus: $x_1 = \prod\limits_{s\in m(L), s\geq m(x_1)} x(s,t_s)$. Then

$$x = x(r,t_r) \prod_{s\in m(L), s\geq m(x_1)} x(s,t_s) = \prod_{s\in m(L), s\geq m(x)} x(s,t_s),$$

which is a contradiction with the choice of x.

Thus $|L| \leqslant \prod\limits_{r\in m(L)} |X(r)| \leqslant q^{|m(L)|}$.

THEOREM 2. Let G be some finite split Lie-type group with a definition field $GF(q)$ of characteristic p and let $U \in U(G)$. Assume that $x, y \in U$ are two unipotent elements and $[x,y] = 1$. Then $\{m(x), m(y)\}$ is an Abelian p-set. In particular, if $A \leq U$ is an Abelian unipotent subgroup then $m(A)$ is an Abelian p-set.

Proof. Let $m(x) = r$ and $m(y) = s$. By assumption, $x = x_r(t)v_1x_{r+s}(t_1)v_2$ and $y = x_s(u)w_1x_{r+s}(u_1)w_2$, where

$$v_1 = \prod_{f\in\Phi,\ r<f<r+s} x_f(t_f),\ v_2 = \prod_{f\in\Phi, f>r+s} x_f(t_f),$$

$$w_1 = \prod_{f\in\Phi, s<f<r+s} x_f(u_f),\ w_2 = \prod_{f\in\Phi, f>r+s} x_f(u_f),$$

and $t, t_1, t_f, u, u_1, u_f \in GF(q)$. Suppose $r \leq s$. Then, by the Chevalley commutator formula, the following equalities hold:

$$xy = x_r(t)v_1x_{r+s}(t_1)v_2 \cdot x_s(u)w_1x_{r+s}(u_1)w_2 =$$

$$x_r(t)\cdot\left(\prod_{f\in\Phi, r<f<r+s} x_f(t_f)\right)\cdot x_{r+s}(t_1)\cdot\left(\prod_{f\in\Phi, f>r+s} x_f(t_f)\right)\times$$

$$x_s(u)\cdot\left(\prod_{f\in\Phi, s<f<r+s} x_f(u_f)\right)\cdot x_{r+s}(u_1)\left(\prod_{f\in\Phi, f>r+s} x_f(u_f)\right) =$$

$$x_r(t)\cdot\left(\prod_{f\in\Phi, r<f<s} x_f(t_f)\right)\cdot x_s(u)\cdot\left(\prod_{f\in\Phi, s\leq f<r+s} x_f(t_f)\right)\cdot x_{r+s}(t_1)\cdot\left(\prod_{f\in\Phi, f>r+s} x_f(t_f)\right)\times$$

$$\left[\left(\prod_{f\in\Phi, s\leq f<r+s} x_f(t_f)\right)\cdot x_{r+s}(t_1)\cdot\left(\prod_{f\in\Phi, f>r+s} x_f(t_f)\right), x_s(u)\right]\times$$

$$\left(\prod_{f\in\Phi, s<f<r+s} x_f(u_f)\right)\cdot x_{r+s}(u_1)\cdot\left(\prod_{f\in\Phi, f>r+s} x_f(u_f)\right) =$$

$$x_r(t)\cdot\left(\prod_{f\in\Phi, r<f<s} x_f(t_f)\right)\cdot x_s(u+t_s)\cdot\left(\prod_{f\in\Phi, s<f<r+s} x_f(t_f)\right)\cdot x_{r+s}(t_1)\times$$

$$\left(\prod_{f\in\Phi, s<f<r+s} x_f(u_f)\right)\cdot x_{r+s}(u_1)\cdot\left(\prod_{f\in\Phi, f>r+s} x_f(t_f)\right)\times$$

$$\left[\left(\prod_{f\in\Phi, s\le f<r+s} x_f(t_f)\right)\cdot x_{r+s}(t_1)\cdot\left(\prod_{f\in\Phi, f>r+s} x_f(t_f)\right), x_s(u)\right]\times$$

$$\left[\left(\prod_{f\in\Phi, f>r+s} x_f(t_f)\right)\cdot\left[\left(\prod_{f\in\Phi, s\le f<r+s} x_f(t_f)\right)\cdot x_{r+s}(t_1)\times\right.\right.$$

$$\left.\left.\left(\prod_{f\in\Phi, f>r+s} x_f(t_f)\right), x_s(u)\right], \left(\prod_{f\in\Phi, s<f<r+s} x_f(u_f)\right)\cdot x_{r+s}(u_1)\right]\cdot\left(\prod_{f\in\Phi, f>r+s} x_f(u_f)\right)=$$

$$x_r(t)\cdot\left(\prod_{f\in\Phi, r<f<s} x_f(t_f)\right)\cdot x_s(u+t_s)\times$$

$$\left(\prod_{f\in\Phi, s<f<r+s} x_f(t_f+u_f)\right)\cdot x_{r+s}(t_1+u_1)\cdot z\cdot\left(\prod_{f\in\Phi, f>r+s} x_f(t_f)\right)\times$$

$$\left[\left(\prod_{f\in\Phi, s\le f<r+s} x_f(t_f)\right)\cdot x_{r+s}(t_1)\cdot\left(\prod_{f\in\Phi, f>r+s} x_f(t_f)\right), x_s(u)\right]\times$$

$$\left[\left(\prod_{f\in\Phi, f>r+s} x_f(t_f)\right)\cdot\left[\left(\prod_{f\in\Phi, s\le f<r+s} x_f(t_f)\right)\cdot x_{r+s}(t_1)\times\right.\right.$$

$$\left.\left.\left(\prod_{f\in\Phi, f>r+s} x_f(t_f)\right), x_s(u)\right], \left(\prod_{f\in\Phi, s<f<r+s} x_f(u_f)\right)\cdot x_{r+s}(u_1)\right]\cdot\left(\prod_{f\in\Phi, f>r+s} x_f(u_f)\right).$$

Here, z is defined as follows. First we take a minimal f_1 in $\Psi_1=\{s<f\le r+s \mid f\in\Phi\}$. Then

$$\left(\prod_{f\in\Phi, s<f<r+s} x_f(t_f)\right)\cdot x_{r+s}(t_1)\cdot x_{f_1}(u_{f_1})=x_{f_1}(u_{f_1})\cdot\left(\prod_{f\in\Phi, s<f<r+s} x_f(t_f)\right)\times$$

$$x_{r+s}(t_1)\cdot\left[\left(\prod_{f\in\Phi, s<f<r+s} x_f(t_f)\right)\cdot x_{r+s}(t_1), x_{f_1}(u_{f_1})\right].$$

Put

$$z_1=\left[\left(\prod_{f\in\Phi, s<f<r+s} x_f(t_f)\right)\cdot x_{r+s}(t_1), x_{f_1}(u_{f_1})\right].$$

Take then a minimal f_2 in $\Psi_2=\{f_1<f\le r+s \mid f\in\Phi\}$. Now

$$\left(\prod_{f\in\Phi, f_1<f<r+s} x_f(t_f)\right)\cdot x_{r+s}(t_1)\cdot z_1\cdot x_{f_2}(u_{f_2})=x_{f_2}(u_{f_2})\cdot\left(\prod_{f\in\Phi, f_1<f<r+s} x_f(t_f)\right)\times$$

$$x_{r+s}(t_1)\cdot z_1\cdot\left[\left(\prod_{f\in\Phi, s<f<r+s} x_f(t_f)\right)\cdot x_{r+s}(t_1)\cdot z_1, x_{f_2}(u_{f_2})\right].$$

Put

$$z_2 = z_1 \cdot \left[\left(\prod_{f\in\Phi, s<f<r+s} x_f(t_f) \right) \cdot x_{r+s}(t_1) \cdot z_1, x_{f_2}(u_{f_2}) \right].$$

The given procedure carries on and on until we arrive at some step yielding the empty set Ψ_i. Since Ψ_1 is finite and elements of Ψ_k decrease in number with every step taken, the empty set appears necessarily in response. Set z equal to z_{i-1}. In view of the Chevalley commutator formula, z lies in the group $\langle X_f \mid f \geq f_1 + f_2 \rangle$. Since $f_1 > r$ and $f_2 > s$, we have $m(z) > r + s$. Consequently, $z \in \langle X_f \mid f > r + s \rangle$.

By the Chevalley commutator formula, therefore, all factors of the element

$$z \left(\prod_{f\in\Phi, f>r+s} x_f(t_f) \right) \cdot \left[\left(\prod_{f\in\Phi, s\leq f<r+s} x_f(t_f) \right) \cdot x_{r+s}(t_1) \times \right.$$

$$\left. \left(\prod_{f\in\Phi, f>r+s} x_f(t_f) \right), x_s(u) \right] \cdot \left[\left(\prod_{f\in\Phi, f>r+s} x_f(t_f) \right) \times \right.$$

$$\left[\left(\prod_{f\in\Phi, s\leq f<r+s} x_f(t_f) \right) \cdot x_{r+s}(t_1) \cdot \left(\prod_{f\in\Phi, f>r+s} x_f(t_f) \right), x_s(u) \right] \times$$

$$\left. \left(\prod_{f\in\Phi, s<f<r+s} x_f(u_f) \right) \cdot x_{r+s}(u_1) \right] \cdot \left(\prod_{f\in\Phi, f>r+s} x_f(u_f) \right)$$

are in the group $\langle X_f \mid f > r + s \rangle$; hence, the given element belongs to it, too. We can therefore represent it as $\prod\limits_{f>r+s} x_f(a_f)$, where all elements a_f are taken from $GF(q)$. Thus the element xy can be written in the canonical form as follows:

$$xy = x_r(t) \cdot \left(\prod_{f\in\Phi, r<f<s} x_f(t_f) \right) \cdot x_s(u + t_s) \cdot \left(\prod_{f\in\Phi, s<f<r+s} x_f(t_f + u_f) \right) \times$$

$$x_{r+s}(t_1 + u_1) \cdot \prod_{f>r+s} x_f(a_f). \tag{1}$$

Similarly yx assumes the canonical form

$$yx = x_r(t) \left(\prod_{f\in\Phi, r<f<s} x_f(t_f) \right) x_s(u + t_s) \left(\prod_{f\in\Phi, s<f<r+s} x_f(t_f + u_f) \right) \times$$

$$x_{r+s}(t_1 + u_1 - C_{11rs}tu) \left(\prod_{f\in\Phi, f>r+s} x_f(b_f) \right), \tag{2}$$

where all elements b_f are in $GF(q)$. From equalities (1) and (2), it follows that if $\{m(x), m(y)\}$ is not an Abelian p-set, then xy and yx have different canonical forms, since $x_{r+s}(t_1 + u_1)$ is distinct from $x_{r+s}(t_1 + u_1 - C_{11rs}tu)$. Since every unipotent element has a unique canonical form, the elements xy and yx do not coincide, which contradicts the condition that $[x, y] = 1$.

COROLLARY. Let G be a finite split Lie-type group with a definition field $GF(q)$ of characteristic p, A be a unipotent Abelian subgroup of G, and Φ be its root system. Then $|A| \leqslant q^{a(\Phi,p)}$.

Proof. Since A is unipotent, we may assume that $A \leq U$. By Theorem 2, $m(A)$ is an Abelian p-set. By Theorem 1, therefore, $|A| \leqslant q^{m(A)} \leqslant q^{a(\Phi,p)}$.

THEOREM 3. Let G be a finite split group of Lie type, A_1 and A_2 be two unipotent subgroups of U, and $B = N_G(U)$ be a Borel subgroup of G. Then $m(A_1) = m(A_2)$ if A_1 and A_2 are conjugate in B.

Proof. Let $A_1^g = A_2$ for some $g \in B$. Then $g = hu$, where h is an element of a Cartan subgroup H of B, and $u \in U$. For any root $r \in \Phi^+$, $X_r^L \subseteq X_r$; consequently, $X_r^h \subseteq X_r$. By the Chevalley commutator formula, $X_r^u \subseteq X_r\langle X_s \mid s > r\rangle$ for any $u \in U$. Let $x \in A_1$ and $r = m(x)$. By the above argument, $x^g \in X_r\langle X_s \mid s > r\rangle$. Consequently, $m(x^g) = m(x) \in m(A_2)$. Hence $m(A_1) \subseteq m(A_2)$. Since $A_2^{g^{-1}} = A_1$, the reverse inclusion holds also. Thus $m(A_1) = m(A_2)$.

3. LARGE ABELIAN UNIPOTENT SUBGROUPS OF FINITE EXCEPTIONAL GROUPS OF LIE TYPE

Under this section, we compute orders of large unipotent Abelian subgroups, estimate p-ranks, and describe the structure of Thompson subgroups in maximal unipotent subgroups of finite exceptional groups of Lie type. In view of the corollary to Theorem 2, in order to describe large Abelian subgroups, we need to find Abelian p-subsets of maximal order in Φ^+. A TurboPascal program was specially created to tackle this problem. In order to check the program for correctness, we give a full list of maximal Abelian p-subsets for sets G_2^+, D_4^+, and F_4^+. In root systems E_6, E_7, and E_8, an Abelian p-subset will be Abelian for any p. As noted above, Abelian subsets of maximal order are listed in [11].

Groups $G_2(q)$. The structure and orders of large Abelian unipotent subgroups, and also of Thompson subgroups, will depend on the characteristic p of a definition field $GF(q)$ for $G_2(q)$.

(1) $p = 2$. All maximal Abelian 2-subsets of Φ^+ fall in the following list: $\{r_1, r_1+r_2, 3r_1+r_2, 3r_1+2r_2\}$, $\{r_2, r_1+r_2, 3r_1+2r_2\}$, $\{r_2, 2r_1+r_2, 3r_1+2r_2\}$, and $\{2r_1+r_2, 3r_1+r_2, 3r_1+2r_2\}$. Consider the first set. It is not hard to verify that $xy \neq yx$ if $x, y \in U$, $m(x) = r_1$, and $m(y) = r_1+r_2$. Thus if $m(A) \subseteq \{r_1, r_1+r_2, 3r_1+r_2, 3r_1+2r_2\}$ for some Abelian subgroup A of U then either $m(A) \subseteq \{r_1, 3r_1+r_2, 3r_1+2r_2\}$ or $m(A) \subseteq \{r_1+r_2, 3r_1+r_2, 3r_1+2r_2\}$. In the former case, up to conjugation, A coincides with one of the following groups: $\langle X_{r_1}, X_{3r_1+r_2}, X_{3r_1+2r_2}\rangle$, $\langle\{x_{r_1}(a)x_{r_1+r_2}(a) \mid a \in GF(q)\}, X_{3r_1+r_2}, X_{3r_1+2r_2}\rangle$. By Theorem 1, $|A| \leqslant q^3$ in any case. For the sake of space, here we do not give a full list of large Abelian subgroups. Note only that $\langle X_{r_1}, X_{3r_1+r_2}, X_{3r_1+2r_2}\rangle$ and $\langle X_{r_2}, X_{r_1+r_2}, X_{3r_1+2r_2}\rangle$ are large elementary Abelian subgroups of U, and they generate the whole group U. Moreover, the sole normal Abelian subgroup is $\langle X_{2r_1+r_2}, X_{3r_1+r_2}, X_{3r_1+2r_2}\rangle$. Consequently, $a(U) = a_e(U) = a_n(U) = q^3$, $J(U) = J_e(U) = U$, $J_n(U) = \langle X_{2r_1+r_2}, X_{3r_1+r_2}, X_{3r_1+2r_2}\rangle$, and $m_2(G_2(2^\alpha)) = 3\alpha$.

(2) $p = 3$. Maximal Abelian 3-subsets of Φ^+ are as follows: $\{r_1, 2r_1+r_2, 3r_1+r_2, 3r_1+2r_2\}$, $\{r_1+r_2, 2r_1+r_2, 3r_1+r_2, 3r_1+2r_2\}$, and $\{r_2, r_1+r_2, 2r_1+r_2, 3r_1+2r_2\}$. Thus a large Abelian unipotent subgroup A of $U \in U(G_2(q))$ is conjugate (in U) to $\langle X_{r_1}, X_{2r_1+r_2}, X_{3r_1+r_2}, X_{3r_1+2r_2}\rangle$ or $\langle\{x_{r_1}(a)x_{r_1+r_2}(a) \mid a \in GF(q)\}, X_{2r_1+r_2}, X_{3r_1+r_2}, X_{3r_1+2r_2}\rangle$, if $m(A)$ coincides with the first set. A is conjugate to $\langle\{X_{r_1+r_2}, X_{2r_1+r_2}, X_{3r_1+r_2}, X_{3r_1+2r_2}\rangle$ if $m(A)$ coincides with the second set. Lastly, A is conjugate to $\langle X_{r_2}, X_{r_1+r_2}, X_{2r_1+r_2}, X_{3r_1+2r_2}\rangle$ or $\langle\{x_{r_2}(a)x_{3r_1+r_2}(a) \mid a \in GF(q)\}, X_{r_1+r_2}, X_{2r_1+r_2}, X_{3r_1+2r_2}\rangle$, if $m(A)$ coincides with the third set. Obviously, $a(U) = a_e(U) = a_n(U) = q^4$, $J(U) = J_e(U) = U$, $J_n(U) = \langle\{X_{r_1+r_2}, X_{2r_1+r_2}, X_{3r_1+r_2}, X_{3r_1+2r_2}\rangle$, and $m_3(G_2(3^\alpha)) = 4\alpha$.

(3) $p > 3$. Maximal Abelian p-subsets of Φ^+ are Abelian and fall in the following list: $\{r_1, 3r_1+r_2, 3r_1+2r_2\}$, $\{r_2, r_1+r_2, 3r_1+2r_2\}$, $\{r_2, 2r_1+r_2, 3r_1+2r_2\}$, $\{r_1+r_2, 3r_1+r_2, 3r_1+2r_2\}$, and $\{2r_1+r_2, 3r_1+r_2, 3r_1+2r_2\}$. Here, we abstain from giving a full list of unipotent Abelian subgroups. Note only that

TABLE 2. Maximal Abelian Subsets of D_4^+

1000 0010 0001 1111 1211	1000 0010 1110 1111 1211	1000 0001 1101 1111 1211	1000 1100 1110 1101 1111 1211	0100 1100 0110 0101 1211	0100 1100 0110 1110 1211	0100 1100 0101 1101 1211	0100 1100 1110 1101 1211	0100 0110 0101 0111 1211	0100 0110 1110 0111 1211
0100 1110 1101 0111 1211	0010 0001 0111 1111 1211	0010 0110 1110 0111 1111 1211	0001 0101 1101 0111 1111 1211	1100 0110 0101 1111 1211	1100 0110 1110 1111 1211	1100 0101 1101 1111 1211	0110 0101 0111 1111 1211	1110 1101 0111 1111 1211	0100 0101 1101 0111 1211

$\langle X_{r_1}, X_{3r_1+r_2}, X_{3r_1+2r_2}\rangle$ and $\langle X_{r_2}, X_{r_1+r_2}, X_{3r_1+2r_2}\rangle$ are large unipotent elementary Abelian subgroups, and they generate the whole group U. Moreover, the only Abelian normal subgroup of maximal order is $\langle X_{2r_1+r_2}, X_{3r_1+r_2}, X_{3r_1+2r_2}\rangle$. Consequently, $a(U) = a_e(U) = a_n(U) = q^3$, $J(U) = J_e(U) = U$, $J_n(U) = \langle X_{2r_1+r_2}, X_{3r_1+r_2}, X_{3r_1+2r_2}\rangle$, and $m_p(G_2(p^\alpha)) = 3\alpha$.

Groups $^3D_4(q^3)$. Since all roots of a root system D_4 are equal in length, all constants C_{ijrs} are equal to 1. Consequently, an Abelian p-subset of Φ is Abelian for any p. In Table 2, we give a list of maximal Abelian subsets of D_4^+. (In the table, the root $\alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3 + \alpha_4 r_4$ is in correspondence with the quadruple $\alpha_1\alpha_2\alpha_3\alpha_4$.)

Using the first two Abelian subsets from Table 2 as an example, we show how to treat the unipotent Abelian subgroups of $^3D_4(q^3)$.

Let A be a unipotent Abelian subgroup of $^3D_4(q^3)$. Then A is Abelian in $D_4(q^3)$ and consists of σ-stable points w.r.t. a graph automorphism σ corresponding to the symmetry ρ in the root system D_4. By Theorem 2, $m(A)$ is a subset of one of the sets in Table 2.

Suppose $m(A) \subseteq \{r_1, r_3, r_4, r_1+r_2+r_3+r_4, r_1+2r_2+r_3+r_4\}$. Define sets $X(r)$ and their elements $x(r,t)$ in the same way as in the proof of Theorem 1. Then $|A| \leqslant \prod\limits_{r\in m(A)} |X(r)|$. Assume $r_1 \in m(A)$. Then there exists an element $x \in A$ such that $m(x) = r_1$. Since x is σ-stable, the factors $x_{r_3}(t_{r_3})$ and $x_{r_4}(t_{r_4})$ in the canonical form of x are distinct from unity. The converse is also true: if $x_{r_3}(t_{r_3})$ or $x_{r_4}(t_{r_4})$ in the canonical form of x are distinct from unity then $x_{r_1}(t_{r_1})$ is also distinct from 1. Consequently, $r_3, r_4 \notin m(A)$. Further, note that $|X(r_1)| \leqslant q^3$. Roots $r_1+r_2+r_3+r_4$ and $r_1+2r_2+r_3+r_4$ are ρ-stable; so, $t = \bar{t} = \bar{\bar{t}} \in GF(q)$ for any element $x(r,t)$ (where $r = r_1+r_2+r_3+r_4$ or $r = r_1+2r_2+r_3+r_4$). Thus $|X(r)| \leqslant q$ (for $r = r_1+r_2+r_3+r_4$ or $r = r_1+2r_2+r_3+r_4$); hence, $|A| \leqslant q^5$. On the other hand, the group $\langle \{x_{r_1}(t), x_{r_3}(\bar{t}), x_{r_4}(\bar{\bar{t}}) \mid t \in GF(q^3)\}, \{x_{r_1+r_2+r_3+r_4}(t), x_{r_1+2r_2+r_3+r_4}(s) \mid t, s \in GF(q)\}\rangle$ is Abelian, and its order is equal to q^5.

Assume $m(A) \subseteq \{r_1, r_3, r_1+r_2+r_3, r_1+r_2+r_3+r_4, r_1+2r_2+r_3+r_4\}$. Direct computations show that $xy \neq yx$ for every two elements $x \in X(r_1)$ and $y \in X(r_1+r_2+r_3)$. Consequently, either r_1 or $r_1+r_2+r_3$ will not be in $m(A)$. Then $|A| \leqslant q^5$ as before.

Here, again we refrain from giving a list of large unipotent Abelian subgroups of $^3D_4(q^3)$ for it would

be to lengthy. Note only that $\langle\{x_{r_1}(t), x_{r_3}(\bar{t}), x_{r_4}(\bar{\bar{t}}) \mid t \in GF(q^3)\}, \{x_{r_1+r_2+r_3+r_4}(t), x_{r_1+2r_2+r_3+r_4}(s) \mid t, s \in GF(q)\}\rangle$ and $\langle\{x_{r_1+r_2}(t), x_{r_2+r_3}(\bar{t}), x_{r_2+r_4}(\bar{\bar{t}}) \mid t \in GF(q^3)\}, \{x_{r_2}(t), x_{r_1+2r_2+r_3+r_4}(s) \mid t, s \in GF(q)\}\rangle$ are large elementary unipotent Abelian subgroups of ${}^3D_4(q^3)$, and they generate the whole unipotent subgroup U. Moreover, the sole normal Abelian subgroup of maximal order in U is $\langle\{x_{r_1+r_2+r_3}(t)x_{r_1+r_2+r_4}(\bar{t})x_{r_2+r_3+r_4}(\bar{\bar{t}}) \mid t \in GF(q^3)\}\{x_{r_1+r_2+r_3+r_4}(t), x_{r_1+2r_2+r_3+r_4}(s) \mid t, s \in GF(q)\}\rangle = A$. Consequently, $a(u) = a_e(U) = a_n(U) = q^5$, $J(U) = J_e(U) = U$, $J_n(U) = A$, and $m_p({}^3D_4(p^{3\alpha})) = 5\alpha$.

Groups $F_4(q)$ **and** ${}^2F_4(q)$**.** The structure and orders of large unipotent Abelian subgroups of $F_4(q)$ depend on the parity of q, and we so consider two cases.

(1) q is even. In Table 3, we give a list of all maximal Abelian 2-subsets of F_4^+. As in Table 2, in correspondence with the root $\alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3 + \alpha_4 r_4$ is $\alpha_1\alpha_2\alpha_3\alpha_4$.

Clearly, $F_4(q)$ has quite a lot of large Abelian subgroups (in correspondence with every Abelian 2-subset of maximal order is at least one subgroup), and we so abstain from giving a full list of these. Denote by $\Psi_{i,j}$ a maximal Abelian 2-subset of F_4^+ which lies at the intersection of the ith row and jth column in Table 3. Then $\langle X_r \mid r \in \Psi_{2,1}\rangle$, $\langle X_r \mid r \in \Psi_{2,9}\rangle$, $\langle X_r \mid r \in \Psi_{3,7}\rangle$, and $\langle X_r \mid r \in \Psi_{5,3}\rangle$ are large unipotent elementary Abelian subgroups, and they generate the whole group U. In U, there are two normal Abelian subgroups of maximal order — $A_1 = \langle X_r \mid r \in \Psi_{2,12}\rangle$ and $A_2 = \langle X_r \mid r \in \Psi_{2,13}\rangle$. Consequently, $a(U) = a_e(U) = a_n(U) = q^{11}$, $J(U) = J_e(U) = U$, $J_n(U) = \langle A_1, A_2\rangle$, and $m_2(F_4(2^\alpha)) = 11\alpha$.

Now we turn to a group ${}^2F_4(q)$ and prove that $a({}^2F_4(q)) = 2q^5$. Let A be some Abelian subgroup of ${}^2F_4(q)$. Then A is an Abelian unipotent subgroup of $F_4(q)$ consisting of σ-stable elements, where σ is a graph automorphism. By Theorem 2, the set $m(A)$ is contained in some maximal Abelian 2-subset of Φ^+, given in Table 3. Consider the case where $m(A) \subseteq \Psi_{1,1}$; all other possibilities are treated in the same way.

Let $r_1 \in m(A)$. Since A consists of σ-stable elements, for any element $x(r_1, t)$, the factor $x_{r_4}(t)$ in the canonical form of $x(r_1, t)$ is distinct from unity. Furthermore, if $r_3 \in m(A)$ then for any element $x(r_3, t)$ its factor $x_{r_2}(t_{r_2})$ is distinct from 1. Since $r_2 < r_3$, we have $r_2 \in m(A)$, which is impossible. Hence $r_3 \notin m(A)$. By the same argument, roots $r_3 + r_4$, $r_1 + r_2 + 2r_3 + 2r_4$, $r_1 + 2r_2 + 4r_3 + 2r_4$, and $2r_1 + 3r_2 + 4r_3 + 2r_4$ do not belong to $m(A)$, either. Moreover, direct computations show that if $r_1 + r_2 + 2r_3 \in m(A)$ then $xy \neq yx$ for all elements $x \in X(r_1)$ and $y \in X(r_1 + r_2 + 2r_3)$. Hence only one of r_1, $r_1 + r_2 + 2r_3$ may be in $m(A)$. Consequently, $|m(A)| \leqslant 4$, and by Theorem 1, $|A| \leqslant q^4$.

In order to illuminate the structure of large Abelian subgroups, we introduce some new notation. If $r + \bar{r} \notin F_4$ then $X_{\{r\}}$ denotes a group $(X_r X_{\bar{r}})_\sigma$. If $r + \bar{r} \in F_4$ and r and $\bar{r}$ are, respectively, a long and a short roots in F_4 then $X_{\{r\}}$ stands for a group $(X_r X_{\bar{r}} X_{r+\bar{r}} X_{r+2\bar{r}})_\sigma$. In the former case $X_{\{r\}}$ is an Abelian group of order q; in the latter case $X_{\{r\}}$ is isomorphic to a unipotent subgroup of ${}^2B_2(q)$ and contains $q - 1$ large Abelian subgroups of order $2q$. In this instance large Abelian unipotent subgroups of $U \in U({}^2F_4(q))$ are conjugate in U to one of the following: $\langle X_{\{r_2+r_3\}}, A_1, X_{\{r_1+2r_2+2r_3\}}, X_{\{r_1+2r_2+2r_3+r_4\}}, X_{\{r_1+2r_2+3r_3+2r_4\}}\rangle$ and $\langle A_2, X_{\{r_1+r_2+r_3+r_4\}}, X_{\{r_1+r_2+2r_3+r_4\}}, X_{\{r_1+2r_2+2r_3+r_4\}}, X_{\{r_1+2r_2+3r_3+r_4\}}\rangle$, where $A_1 \in A(X_{\{r_1+r_2+2r_3\}})$ and $A_2 \in A(X_{\{r_1+r_2+r_3\}})$. Clearly, these groups are normal in U. It follows that $a(U) = a_n(U) = 2q^5$, $a_e(U) = q^5$, $J(U) = J_n(U) = \langle X_{\{r\}} \mid r \succeq r_2 + r_3\rangle$, $J_e(q) = \langle X_{\{r\}} \mid r \succeq r_2 + r_3, r \neq r_1 + r_2 + r_3, r \neq r_1 + r_2 + 2r_3\rangle$, and $m_2({}^2F_4(2^\alpha)) = 5\alpha$.

(2) q is odd. In this case there exist 25 Abelian subsets of order 9 in F_4^+ (see [11]). In correspondence with such subsets each are different large Abelian subgroups. Moreover, the only normal Abelian subgroup of maximal order is $A_1 = \langle X_{r_2+2r_3+2r_4}, X_{r_1+2r_2+2r_3+r_4}, X_{r_1+r_2+2r_3+2r_4}, X_{r_1+2r_2+3r_3+r_4}, X_{r_1+2r_2+2r_3+2r_4}, X_{r_1+2r_2+3r_3+2r_4}, X_{r_1+2r_2+4r_3+2r_4}, X_{r_1+3r_2+4r_3+2r_4}, X_{2r_1+3r_2+4r_3+2r_4}\rangle$. Thus $a(U) = a_e(U) = a_n(U) = q^9$, $J(U) = J_e(U) = \langle X_r \mid r \neq r_1\rangle$, $J_n(U) = A_1$, and $m_p(F_4(p^\alpha)) = 9\alpha$.

TABLE 3. Abelian 2-Subsets of F_4^+

1000	1000	1000	1000	1000	1000	1000	1000	0100	0100	0100	0120	0111
0010	0010	0010	0001	0001	0001	0011	0011	0001	0001	0001	0121	1111
0011	1110	1110	1100	0011	1111	1120	1111	1100	0111	0111	1121	1220
1120	1120	1120	1111	1111	1121	1111	1121	0111	1111	0121	0122	1221
1121	1220	1121	1121	1121	1221	1121	1122	1111	0122	0122	1221	1231
1122	1121	1122	1221	1122	1122	1122	1231	1221	1221	0122	1231	1222
1231	1231	1231	1122	1222	1222	1231	1222	1222	1222	1222	1232	1232
1232	1232	1232	1222	1232	1232	1232	1232	1232	1232	1232	1242	1242
1242	1242	1242	1232	1242	1242	1242	1242	1342	1342	1342	1342	1342
2342	2342	2342	2342	2342	2342	2342	2342	2342	2342	2342	2342	2342
1000	1000	1000	1000	1000	1000	1000	1000	0100	0100	0100	0121	1111
1100	1100	1100	1100	1110	1110	1110	1110	1100	1100	1100	1121	1121
1110	1110	1110	1110	1120	1120	1111	1111	0110	0110	1110	0122	0122
1120	1120	1111	1111	1111	1111	1220	1121	1110	0111	1111	1221	1221
1111	1111	1220	1121	1220	1121	1121	1221	1220	1220	1220	1122	1122
1220	1121	1121	1221	1121	1221	1221	1122	1221	1221	1221	1231	1231
1121	1221	1221	1122	1221	1122	1231	1231	1231	1231	1231	1222	1222
1221	1122	1231	1231	1231	1231	1222	1222	1222	1222	1222	1232	1232
1231	1231	1222	1222	1232	1232	1232	1232	1232	1232	1232	1242	1242
1232	1232	1232	1232	1242	1242	1242	1242	1342	1342	1342	1342	1342
2342	2342	2342	2342	2342	2342	2342	2342	2342	2342	2342	2342	1342
0100	0100	0100	0100	0100	0010	0010	0010	0010	0010	0010	0111	0111
1100	0110	0110	0110	0110	0110	0110	0011	0011	0011	0011	1111	0121
0111	0120	0120	0111	0111	1110	0120	0120	0120	1120	0121	0122	0122
1111	0111	0111	0121	0121	0120	1120	1120	0121	0121	1121	1221	1221
1220	0121	0121	1220	0122	1120	0121	0121	1121	1121	0122	1122	1122
1221	1220	0122	1221	1221	1220	1220	1121	0122	1122	1122	1231	1231
1231	1221	1221	1231	1231	1231	1231	1231	1231	1231	1231	1222	1222
1222	1231	1231	1222	1222	1232	1232	1232	1232	1232	1232	1232	1232
1232	1232	1232	1232	1232	1242	1242	1242	1242	1242	1242	1242	1242
1342	1342	1342	1342	1342	1342	1342	1342	1342	1342	1342	1342	1342
2342	2342	2342	2342	2342	2342	2342	2342	2342	2342	2342	2342	2342
0100	0100	0010	0010	0001	0001	1100	1100	0110	0011	0011	1120	0121
0110	0111	0110	1110	1100	1100	0110	0111	1110	0120	1120	0121	1220
1110	1111	0120	1120	0111	1111	1110	1111	1220	0111	1111	1121	1121
0120	0122	0121	1121	1111	1121	1120	1221	1221	0121	1121	1121	1121
1220	1221	0122	1122	1221	1221	1220	1122	1231	0122	1122	1122	1231
1221	1231	1231	1231	1122	1122	1221	1231	1222	1231	1231	1122	1231
1231	1222	1232	1232	1222	1222	1231	1222	1232	1232	1232	1232	1232
1232	1232	1242	1242	1232	1232	1232	1232	1242	1242	1242	1242	1242
1342	1342	1342	1342	1342	1342	1342	1342	1342	1342	1342	1342	1342

Table 3 (Continued)

2342	2342	2342	2342	2342	2342	2342	2342	2342	2342	2342	2342	2342
0010	0010	0001	0001	0001	0001	0001	0001	0001	0001	1100	1110	0120
1110	0120	0011	0011	0011	0011	0111	0111	1111	0121	1110	1111	1120
0120	1120	0111	0111	1111	0121	1111	0121	1121	1121	1120	1121	0121
1120	0121	1111	0121	1121	1121	0122	0122	0122	0122	1111	1221	1220
1220	1220	0122	0122	0122	0122	1221	1221	1221	1221	1220	1122	1121
1121	1121	1122	1122	1122	1122	1122	1122	1122	1122	1121	1231	1221
1231	1231	1222	1222	1222	1222	1222	1222	1222	1222	1221	1222	1231
1232	1232	1232	1232	1232	1232	1232	1232	1232	1232	1231	1232	1232
1242	1242	1242	1242	1242	1242	1242	1242	1242	1242	1232	1242	1242
1342	1342	1342	1342	1342	1342	1342	1342	1342	1342	1342	1342	1342
2342	2342	2342	2342	2342	2342	2342	2342	2342	2342	2342	2342	2342
1100	1100	1100	0110	0110	0110	0110	0110	0110	0011	0011	1110	1110
1110	1110	1110	1110	0120	0120	0120	0111	0111	0111	0111	1120	1111
1120	1111	1111	0120	0111	0111	1120	0121	0121	1111	0121	1111	1220
1111	1220	1121	1120	0121	0121	0121	1220	0122	0122	0122	1121	1121
1121	1121	1221	1220	1220	0122	1220	1221	1221	1122	1122	1221	1221
1221	1221	1122	1221	1221	1221	1221	1231	1231	1231	1231	1122	1231
1122	1231	1231	1231	1231	1231	1231	1222	1222	1222	1222	1231	1222
1231	1222	1222	1232	1232	1232	1232	1232	1232	1232	1232	1232	1232
1232	1232	1232	1242	1242	1242	1242	1242	1242	1242	1242	1242	1242
1342	1342	1342	1342	1342	1342	1342	1342	1342	1342	1342	1342	1342
2342	2342	2342	2342	2342	2342	2342	2342	2342	2342	2342	2342	2342
0011	0011	1110	1110									
1111	0121	0120	1120									
1121	1121	1120	1111									
0122	0122	1220	1220									
1122	1122	1121	1121									
1231	1231	1221	1221									
1222	1222	1231	1231									
1232	1232	1232	1232									
1242	1242	1242	1242									
1342	1342	1342	1342									
2342	2342	2342	2342									

Groups $E_6(q)$ and $^2E_6(q^2)$. Since all roots in a root system E_6 are equal in length, all coefficients C_{ijrs} are equal to 1. Consequently, an Abelian p-subset of E_6 is Abelian for any p. There exist only two Abelian subsets of E_6^+ whose order is equal to 16 (see [11]). In correspondence with these are $\langle X_r \mid r \succeq r_1\rangle$ and $\langle X_r \mid r \succeq r_6\rangle$, which are elementary normal Abelian subgroups of U. Thus $J(U) = J_e(U) = J_n(U) = \langle X_r \mid r \succeq r_1 \text{ or } r \succeq r_6\rangle$, $a(U) = a_e(U) = a_n(U) = q^{16}$, and $m_p(E_6(p^\alpha) = 16\alpha$.

Abelian subgroups of $^2E_6(q^2)$ can be treated in the same way as were the Abelian subgroups of $^3D_4(q^3)$. There exist about 2000 distinct Abelian subsets of E_6^+. For the sake of space, we refrain from listing them

here. The final result is this. We have $a(U) = a_e(U) = a_n(U) = q^{12}$, $J(U) = J_e(U) = \langle X_{\{r\}} | r \succeq r_3$ or $r \succeq r_4\rangle$, and $J_n(U) = \langle X_{\{r\}} \mid r \in \Psi\rangle$, where $X_{\{r\}} = (X_r X_{\bar{r}})_\sigma$ and $m_p({}^2E_6(p^{2\alpha})) = 12\alpha$. Here $\Psi = \{r_1+r_3+r_4+r_5+r_6, r_1+r_2+r_3+2r_4+r_5, r_1+r_2+r_3+r_4+r_5+r_6, r_2+r_3+2r_4+r_5+r_6, r_1+r_2+2r_3+2r_4+r_5, r_1+r_2+r_3+2r_4+r_5+r_6, r_2+r_3+2r_4+2r_5+r_6, r_1+r_2+2r_3+2r_4+r_5+r_6, r_1+r_2+r_3+2r_4+2r_5+r_6, r_1+r_2+2r_3+2r_4+2r_5+r_6, r_1+r_2+2r_3+3r_4+2r_5+r_6, r_1+2r_2+2r_3+3r_4+2r_5+r_6\}$.

Groups $E_7(q)$**.** Again an Abelian p-subset of a root system E_7 is Abelian for any p. There exists just one Abelian subset of maximal order 27 (see [11]). In correspondence with this is the unique elementary normal Abelian subgroup A of order q^{27}. Thus $J(U) = J_e(U) = J_n(U) = A = \langle X_r \mid r \succeq r_7\rangle$, $a(U) = a_e(U) = a_n(U) = q^{27}$, and $m_p(E_7(p^\alpha)) = 27\alpha$.

Groups $E_8(q)$**.** Since all roots of E_8 are equal in length, all constants C_{11rs} are equal to 1 for every two roots r and s. Thus an Abelian p-subset of E_8 is Abelian for any prime p. Abelian subsets of maximal order are given in [11]. Here, we abstain from listing all Abelian subsets of maximal order in E_8^+. Note only that $a(U) = a_e(U) = a_n(U) = q^{36}$, $J(U) = J_e(U) = \langle X_r \mid r \succeq r_1, r \succeq r_3, r \succeq r_4, r \succeq r_5$ or $r \succeq r_6\rangle$, and $m_p(E_8(p^\alpha)) = 36\alpha$. Nor do we describe the structure of $J_n(U)$, for it would be too lengthy.

Tally table. We give orders of large unipotent Abelian subgroups in finite exceptional groups of Lie type [except ${}^2B_2(q)$ and ${}^2G_2(q)$], and also the structure of Thompson subgroups. (Note that $q = p^\alpha$ throughout Table 4.)

TABLE 4. Orders of Large Abelian Subgroups, p-Ranks, and Thompson Subgroups of Maximal Unipotent Subgroups of Finite Exceptional Groups of Lie Type

Group G	$a(U)$	$J(U)$	$m_p(G)$
$G_2(q)$, $p \neq 3$	q^3	U	3α
$G_2(q)$, $p = 3$	q^4	U	4α
${}^3D_4(q^3)$	q^5	U	5α
$F_4(q)$, q is odd	q^9	$\langle X_r \mid r \neq r_1\rangle$	9α
$F_4(q)$, q is even	q^{11}	U	11α
${}^2F_4(q)$	$2q^5$	$\langle X_{\{r\}} \mid r \succeq r_2 + r_3\rangle$	5α
$E_6(q)$	q^{16}	$\langle X_r \mid r \succeq r_1$ or $r \succeq r_6\rangle$	16α
${}^2E_6(q^2)$	q^{12}	$\langle X_{\{r\}} \mid r \succeq r_3$ or $r \succeq r_4\rangle$	12α
$E_7(q)$	q^{27}	$\langle X_r \mid r \succeq r_7\rangle$	27α
$E_8(q)$	q^{36}	$\langle X_r \mid r \succeq r_1, r \succeq r_3, r \succeq r_4, r \succeq r_5$, or $r \succeq r_6\rangle$	36α

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