An adjacency criterion for the prime graph of a finite simple group[1]

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Abstracts: In the paper we give an exhaustive arithmetic criterion of adjacency in prime graph of every finite nonabelian simple group. By using this criterion for the prime graph of every finite simple groups an independence set with the maximal number of vertices, an independence set containing 2 with the maximal number of vertices, and the orders of these independence sets are given. We assemble this information in the tables at the end of the paper. Several applications of obtained results for various problems of finite group theory are considered, in particular, on the recognition problem of a finite group by spectrum.

Keywords: finite groups, finite simple groups, groups of Lie type, spectrum of finite group, recognition by spectrum, prime graph of finite group, independent number of prime graph, 2-independent number of prime graph

Let G be a finite group, $\pi(G)$ be the set of all prime divisors of its order and $\omega(G)$ be the spectrum of G, that is the set of all its element orders. A graph $GK(G) = \langle V(GK(G)), E(GK(G))\rangle$, where $V(GK(G))$ is the set of vertices and $E(GK(G))$ is the set of edges, is called the *Gruenberg-Kegel graph* (or the *prime graph*) of G if $V(GK(G)) = \pi(G)$ and the edge (r, s) is in $E(GK(G))$ if and only if $rs \in \omega(G)$. Primes $r, s \in \pi(G)$ are called *adjacent* if they are adjacent as vertices of $GK(G)$, that is $(r, s) \in E(GK(G))$. Otherwise r and s are called *non-adjacent*.

The properties of the prime graph $GK(G)$ yield a rich information on the structure of G (see [1–4] and Sections 5 and 7 of the paper). The main purpose of this article is to give an exhaustive arithmetic criterion of adjacency in prime graph $GK(G)$ for every finite nonabelian simple group G. Sections 1–4 are devoted to this goal. In Section 5 we discuss some recent results on prime graph of finite groups that gave us the motivation for the present work. Furthermore, we explain the importance of so-called independence numbers of prime graph for investigations on a group structure. In Section 6 we calculate those invariants for all finite nonabelian simple groups (the resulting tables are assembled in Section 8). The applications of our results are considered in Section 7.

1 Preliminaries

Our notations is standard. If n is a natural number, π a set of primes, then by $\pi(n)$ we denote the set of all prime divisors of n, by n_π we denote the maximal divisor t of n such

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that $\pi(t) \subseteq \pi$. Note that for a finite group G, $\pi(G) = \pi(|G|)$ by definition.

The adjacency criterion of two prime divisors for alternating groups is obvious and can be given as follows.

Proposition 1.1. *Let $G = Alt_n$ be an alternating group of degree n.*

(1) *Let $r, s \in \pi(G)$ be odd primes. Then r, s are non-adjacent if and only if $r + s > n$.*

(2) *Let $r \in \pi(G)$ be an odd prime. Then $2, r$ are non-adjacent if and only if $r + 4 > n$.*

The information on the adjacency of vertices in a prime graph for every sporadic group and Tits group ${}^2F_4(2)'$ can be extracted from [5] or [6]. Thus, we need to consider only simple groups of Lie type.

For groups of Lie type and linear algebraic groups our notations agrees with that in [7] and [8] respectively. Denote by G_{sc} a universal group of Lie type. Then every factor group G_{sc}/Z, where $Z \leq Z(G_{sc})$, we call a *group of Lie type*. Almost in all cases $G_{sc}/Z(G_{sc})$ is simple and we say that $G_{ad} = G_{sc}/Z(G_{sc})$ is of *adjoint type*. Some groups of Lie type over small fields are not simple. In Table 1 we assemble the information of [7, Theorems 11.1.2 and 14.4.1] and give all such exceptions. Sometimes we shall use notations $A_n^\varepsilon(q), D_n^\varepsilon(q)$, and $E_6^\varepsilon(q)$, where $\varepsilon \in \{+, -\}$, and $A_n^+(q) = A_n(q)$, $A_n^-(q) = {}^2A_n(q)$, $D_n^+(q) = D_n(q)$, $D_n^-(q) = {}^2D_n(q)$, $E_6^+(q) = E_6(q)$, $E_6^-(q) = {}^2E_6(q)$.

If G is isomorphic to ${}^2A_n(q)$, ${}^2D_n(q)$, or ${}^2E_6(q)$ we say that G is defined over $GF(q^2)$, if $G \simeq {}^3D_4(q)$ we say that G is defined over $GF(q^3)$ and we say that G is defined over $GF(q)$ for other finite groups of Lie type. The field $GF(q)$ in all cases is called the *base field* of G. If G is a universal finite group of Lie type with the base field $GF(q)$, then there are a natural number $N (= |\Phi^+|$ in most cases) and a polynomial $f(t) \in \mathbb{Z}[t]$ such that $|G| = f(q) \cdot q^N$ and $(q, f(q)) = 1$ (see [7, Theorems 9.4.10 and 14.3.1]). This polynomial we denote by $f_G(t)$. If G is not universal then there is a universal group K with $G = K/Z$, where $Z = Z(K)$, and we define $f_G(t) = f_K(t)$.

Assume that $\overline{G}$ is a connected simple linear algebraic group defined over an algebraically closed field of a positive characteristic p. Let σ be an endomorphism of $\overline{G}$ such that $\overline{G}_\sigma = C_{\overline{G}}(\sigma)$ is a finite set. Then σ is said to be a *Frobenius map* and $G = O^{p'}(\overline{G}_\sigma)$ is appeared to be a finite group of Lie type. Moreover all finite groups of Lie type both split and twisted can be derived in this way. Below we assume that for every finite group G of Lie type we fix (in some way) a linear algebraic group $\overline{G}$ and a Frobenius map σ such that $G = O^{p'}(\overline{G}_\sigma)$. If $\overline{G}$ is simply connected, then $G = \overline{G}_\sigma = O^{p'}(\overline{G}_\sigma)$; and if $\overline{G}$ is of adjoint type, then $\overline{G}_\sigma = \widehat{G}$ is the group of inner-diagonal automorphisms of G (see [9, § 12]).

If $\overline{R}$ is a σ-stable reductive subgroup of $\overline{G}$, then $\overline{R}_\sigma \cap G = \overline{R} \cap G$ is called a *reductive subgroup* of G. If $\overline{R}$ is of maximal rank, then $\overline{R}_\sigma \cap G$ is also said to be of *maximal rank*. Note that if $\overline{R}$ is σ-stable reductive of maximal rank, then $\overline{R} = \overline{T} \cdot \overline{G}_1 * \ldots * \overline{G}_k$, where $\overline{T}$ is some σ-stable maximal torus of $\overline{R}$ and $\overline{G}_1, \ldots, \overline{G}_k$ are subsystem subgroups of $\overline{G}$. Furthermore $\overline{G}_1 * \ldots * \overline{G}_k = \left[\overline{R}, \overline{R}\right]$. It is known that $\overline{R}_\sigma = \overline{T}_\sigma G_1 * \ldots * G_m$, subgroups $G_1, \ldots, G_m$ we call *subsystem subgroups* of G. In general $m \leqslant k$ and for all i the base field of G_i is equal to $GF(q^{\alpha_i})$, where $\alpha_i \geqslant 1$. There is a nice algorithm due to Borel and de Siebental determining all subsystem subgroups of $\overline{G}$ (see [10] and also [11]). One has to consider the extended Dynkin diagram of $\overline{G}$ and remove any number of nodes. Connected components of the remaining graph are Dynkin diagrams of subsystem subgroups of $\overline{G}$ and Dynkin diagrams of all subsystem subgroups can be derived in this way.

If $\overline{T}$ is a σ-stable torus of $\overline{G}$ then $T = \overline{T} \cap G = \overline{T}_\sigma \cap G$ is called a *torus* of G. If $\overline{T}$ is maximal, then T is a *maximal torus* of G. If G is neither a Suzuki group, nor a Ree group, then for every maximal torus T we have that $|\overline{T}_\sigma| = g(q)$, where $GF(q)$ is the base field of G, $g(t)$ is a polynomial of degree n dividing $f_G(t)$ and n is the rank of $\overline{G}$. For more details see [12, Chapter 1].

Table 1. Non simple groups of Lie type

Group	Properties
$A_1(2)$	Group is soluble
$A_1(3)$	Group is soluble
$B_2(2)$	$B_2(2) \simeq \mathrm{Sym}_6$
$G_2(2)$	$[G_2(2), G_2(2)] \simeq {}^2A_2(3)$
${}^2A_2(2)$	Group is soluble
${}^2B_2(2)$	Group is soluble
${}^2G_2(3)$	$[{}^2G_2(3), {}^2G_2(3)] \simeq A_1(8)$
${}^2F_4(2)$	$[{}^2F_4(2), {}^2F_4(2)] = {}^2F_4(2)'$ is the Tits group

In Lemmas 1.2 and 1.3 we assemble the information about maximal tori in finite simple groups of Lie type.

Lemma 1.2. (see [13, Propositions 7–10] and [14]) *Let $\overline{G}$ be a connected simple classical algebraic group of adjoint type and let $G = O^{p'}(\overline{G}_\sigma)$ be the finite simple classical group.*

1. *Every maximal torus T of $G = A^\varepsilon_{n-1}(q)$ has the order*

$$\frac{1}{(n, q-(\varepsilon 1))(q-(\varepsilon 1))}(q^{n_1} - (\varepsilon 1)^{n_1}) \cdot (q^{n_2} - (\varepsilon 1)^{n_2}) \cdot \ldots \cdot (q^{n_k} - (\varepsilon 1)^{n_k})$$

   for appropriate partition $n_1 + n_2 + \ldots + n_k = n$ of n. Moreover, for every partition there exists a torus of corresponding order.

2. *Every maximal torus T of G, where $G = B_n(q)$ or $G = C_n(q)$, has the order*

$$\frac{1}{(2, q-1)}(q^{n_1} - 1) \cdot (q^{n_2} - 1) \cdot \ldots \cdot (q^{n_k} - 1) \cdot (q^{l_1} + 1) \cdot (q^{l_2} + 1) \cdot \ldots \cdot (q^{l_m} + 1)$$

   for appropriate partition $n_1 + n_2 + \ldots + n_k + l_1 + l_2 + \ldots + l_m = n$ of n. Moreover, for every partition there exists a torus of corresponding order.

3. *Every maximal torus T of $G = D^\varepsilon_n(q)$ has the order*

$$\frac{1}{(4, q^n - \varepsilon 1)} \cdot (q^{n_1} - 1) \cdot (q^{n_2} - 1) \cdot \ldots \cdot (q^{n_k} - 1) \cdot (q^{l_1} + 1) \cdot (q^{l_2} + 1) \cdot \ldots \cdot (q^{l_m} + 1)$$

   for appropriate partition $n_1 + n_2 + \ldots + n_k + l_1 + l_2 + \ldots + l_m = n$ of n, where m is even if $\varepsilon = +$ and m is odd if $\varepsilon = -$. Moreover, for every partition there exists a torus of corresponding order.

Lemma 1.3. (see [14] and [15]) *Let $\overline{G}$ be a connected simple exceptional algebraic group of adjoint type and let $G = O^{p'}(\overline{G}_\sigma)$ be the finite simple exceptional group of Lie type.*

1. *Every maximal torus T of $G = G_2(q)$ has one of the following orders:*

$$(q \pm 1)^2, q^2 - 1, q^2 \pm q + 1.$$

   Moreover, for every number given above there exists a torus of corresponding order.

2. *Every maximal torus T of $G = F_4(q)$ has one of the following orders:*

$$(q \pm 1)^4, (q \pm 1)^2 \cdot (q^2 \pm 1), (q^2 \pm 1)^2, (q \pm 1)(q^3 \pm 1), q^4 \pm 1, (q^2 \pm q + 1)^2, q^4 - q^2 + 1.$$

   Here and below symbol $\pm$ means that we can choose either "+" or "−" independently for all multiplies, i. e., $(q \pm 1)^2(q^2 \pm 1)$ is equal to either $(q - 1)^2(q^2 - 1)$, or $(q + 1)^2(q^2 - 1)$, or $(q - 1)^2(q^2 + 1)$, or $(q + 1)^2(q^2 + 1)$. Moreover, for every number given above there exists a torus of corresponding order.

3. *For every maximal torus T of $G = E_6^\varepsilon(q)$, the number $(3, q - \varepsilon 1)|T|$ is equal to one of the following:*
   $(q - \varepsilon 1)^k \cdot (q + \varepsilon 1)^{6-k}$, $2 \leqslant k \leqslant 6$; $(q^k - (\varepsilon 1)^k) \cdot (q^{6-k} - (\varepsilon 1)^{6-k})$, $1 \leqslant k \leqslant 5$;
   $(q^k - (\varepsilon 1)^k) \cdot (q - \varepsilon 1)^{6-k}$, $3 \leqslant k \leqslant 6$; $(q^3 - \varepsilon 1)(q^2 - 1)(q \pm 1)$; $(q^5 - \varepsilon 1)(q + \varepsilon 1)$;
   $(q^3 + \varepsilon 1)(q^2 \pm 1)(q - \varepsilon 1)$; $(q^4 + 1)(q^2 - 1)$; $(q^2 + 1)^2(q - \varepsilon 1)^2$; $(q^2 + \varepsilon q + 1)^3$;
   $(q^2 + \varepsilon q + 1)^2(q^2 - 1)$; $(q^4 - 1)(q + \varepsilon 1)^2$; $(q^3 + \varepsilon 1)(q^2 + \varepsilon q + 1)(q + \varepsilon 1)$;
   $(q^4 - q^2 + 1)(q^2 + \varepsilon q + 1)$; $q^6 + \varepsilon q^3 + 1$; $(q^2 + \varepsilon q + 1)(q^2 - \varepsilon q + 1)^2$. *Moreover, for every number n given above there exists a torus T with $(3, q - \varepsilon 1)|T| = n$.*

4. *For every maximal torus T of $G = E_7(q)$, the number $(2, q - 1)|T|$ is equal to one of the following:*
   $(q^{n_1} \pm 1) \cdot \ldots \cdot (q^{n_k} \pm 1)$, $n_1 + \ldots + n_k = 7$ *and* $(2, q - 1)|T| \neq (q \pm 1)(q^6 + 1)$;
   $(q - \epsilon 1) \cdot (q^2 + \epsilon q + 1)^3$; $(q^5 - \epsilon 1) \cdot (q^2 + \epsilon q + 1)$; $(q^3 \pm 1) \cdot (q^4 - q^2 + 1)$; $(q - \epsilon 1) \cdot (q^6 + \epsilon q^3 + 1)$;
   $(q^3 - \epsilon 1) \cdot (q^2 - \epsilon q + 1)^2$, *where* $\epsilon = \pm$. *Moreover, for every number n given above there exists a torus T with $(2, q - 1)|T| = n$.*

5. *Every maximal torus T of $G = E_8(q)$ has one of the following orders:* $(q^{n_1} \pm 1) \cdot \ldots \cdot (q^{n_k} \pm 1)$, $n_1 + \ldots + n_k = 8$ $|T| \neq q^8 + 1$;
   $(q - \epsilon 1) \cdot (q^2 + \epsilon q + 1)^3 \cdot (q \pm 1)$; $(q^5 - \epsilon 1) \cdot (q^2 + \epsilon q + 1) \cdot (q \pm 1)$; $(q^3 \pm 1) \cdot (q^4 - q^2 + 1) \cdot (q \pm 1)$;
   $(q - \epsilon 1) \cdot (q^6 + \epsilon q^3 + 1) \cdot (q \pm 1)$; $(q^3 - \epsilon 1) \cdot (q^2 - \epsilon q + 1)^2 \cdot (q \pm 1)$; $q^8 - q^4 + 1$;
   $q^8 + q^7 - q^5 - q^4 - q^3 + q + 1$; $q^8 - q^6 + q^4 - q^2 + 1$; $(q^4 - q^2 + 1)^2$; $(q^6 + \epsilon q^3 + 1)(q^2 + \epsilon q + 1)$;
   $q^8 - q^7 + q^5 - q^4 + q^3 - q + 1$; $(q^4 + \epsilon q^3 + q^2 + \epsilon q + 1)^2$; $(q^4 - q^2 + 1)(q^2 \pm q + 1)^2$;
   $(q^2 - q + 1)^2 \cdot (q^2 + q + 1)^2$; $(q^2 \pm q + 1)^4$, *where* $\epsilon = \pm$. *Moreover, for every number given above there exists a torus of corresponding order.*

6. *Every maximal torus T of $G = {}^3D_4(q)$ has one of the following orders:*

$$(q^3 \pm 1)(q \pm 1); (q^2 \pm q + 1)^2; q^4 - q^2 + 1.$$

Moreover, for every number given above there exists a torus of corresponding order.

7. *Every maximal torus T of $G = {}^2B_2(2^{2n+1})$ has one of the following orders:*

$$q - 1; q \pm \sqrt{2q} + 1,$$

where $q = 2^{2n+1}$. Moreover, for every number given above there exists a torus of corresponding order.

8. *Every maximal torus T of $G = {}^2G_2(3^{2n+1})$ has one of the following orders:*

$$q \pm 1; q \pm \sqrt{3q} + 1,$$

where $q = 3^{2n+1}$. Moreover, for every number given above there exists a torus of corresponding order.

9. *Every maximal torus T of $G = {}^2F_4(2^{2n+1})$ with $n \geqslant 1$ has one of the following orders:*

$$(q\pm1)^2; q^2\pm1; q^2-q+1; (q\pm\sqrt{2q}+1)^2; q^2-\epsilon q\sqrt{2q}+\epsilon\sqrt{2q}-1; q^2+\epsilon q\sqrt{2q}+q+\epsilon\sqrt{2q}+1,$$

where $q = 2^{2n+1}$ and $\epsilon = \pm$. Moreover, for every number given above there exists a torus of corresponding order.

If q is a natural number, r is an odd prime and $(r, q) = 1$, then by $e(r, q)$ we denote the minimal natural number n with $q^n \equiv 1 (\mathrm{mod}\ r)$. If q is odd, let $e(2, q) = 1$ if $q \equiv 1 (\mathrm{mod}\ 4)$ and $e(2, q) = 2$ if $q \equiv -1 (\mathrm{mod}\ 4)$.

The main technical tools in Section 6 is the following statement.

Lemma 1.4. (Zsigmondy Theorem [16]) *Let q be a natural number greater than 1. Then for every $n \in \mathbb{N}$ there exists a prime r such that $e(r, q) = n$ but for the case where $n = 6$ and $q = 2$.*

The prime r with $e(r, q) = n$ is said to be a *primitive prime divisor* of $q^n - 1$. By Zsigmondy theorem it exists excepting the cases indicated above. If q is fixed, we denote by r_n some primitive prime divisor of $q^n - 1$ (obviously, $q^n - 1$ can have more than one such divisor). Note that according to our definition every prime divisor of $q - 1$ is a primitive prime divisor of $q - 1$ with sole exception: 2 is not a primitive prime divisor of $q - 1$ if $e(2, q) = 2$. In the last case 2 is a primitive prime divisor of $q^2 - 1$.

In view [7, Theorems 9.4.10 and 14.3.1] the order of any finite simple group of Lie type G of rank n over the field $GF(q)$ of characteristic p is given by

$$|G| = \frac{1}{d} q^N (q^{m_1} \pm 1) \cdot \ldots \cdot (q^{m_n} \pm 1).$$

It follows that any prime divisor r of $|G|$ distinct from the characteristic p is a primitive divisor of $q^m - 1$ for some natural m. Thus, the Zsigmondy Theorem allows us to “find” prime divisors of $|G|$. Moreover, if G is neither a Suzuki group nor a Ree group, Lemmas 1.2 and 1.3 imply that for a fixed m every two primitive prime divisors of q^m-1 are adjacent in $GK(G)$.

For Suzuki and Ree groups we use following

Lemma 1.5. *Let n be a natural number.*

(1) Let $m_1(B,n)=2^{2n+1}-1$, $m_2(B,n)=2^{2n+1}-2^{n+1}+1$, and $m_3(B,n)=2^{2n+1}+2^{n+1}+1$. Then $(m_i(B,n),m_j(B,n))=1$ if $i\neq j$.

(2) Let $m_1(G,n)=3^{2n+1}-1$, $m_2(G,n)=3^{2n+1}+1$, $m_3(G,n)=3^{2n+1}-3^{n+1}+1$, and $m_4(G,n)=3^{2n+1}+3^{n+1}+1$. Then $(m_1(G,n),m_2(G,n))=2$ and $(m_i(G,n),m_j(G,n))=1$ otherwise.

(3) Let $m_1(F,n)=2^{4n+2}-1$, $m_2(F,n)=2^{4n+2}+1$, $m_3(F,n)=2^{4n+2}-2^{2n+1}+1$, $m_5(F,n)=2^{4n+2}-2^{3n+2}+2^{2n+1}-2^{n+1}+1$, and $m_6(F,n)=2^{4n+2}+2^{3n+2}+2^{2n+1}+2^{n+1}+1$. Then $(m_1(F,n),m_3(F,n))=3$ and $(m_i(F,n),m_j(F,n))=1$ otherwise.

Proof. It easy to check by the direct computation. $\square$

By Lemma 1.3 every distinct from the characteristic prime divisor s of order of the Suzuki group ${}^2B_2(2^{2n+1})$ divides one of the numbers $m_i(B,n)$ defined in Lemma 1.5. The same is true for Ree groups ${}^2G_2(3^{2n+1})$, ${}^2F_2(2^{2n+1})$ and all prime divisors of the numbers $m_i(G,n)$, $m_i(F,n)$ respectively. Thus, Lemma 1.5 allow us to find prime divisors of orders of Suzuki and Ree groups. Moreover, Lemma 1.3 implies that for a fixed k every two prime divisors of $m_k(B,n)$ are adjacent in $GK({}^2B_2(2^{2n+1}))$. The same is also true for Ree groups and all prime divisors of $m_k(G,n)$ and $m_k(F,n)$.

2 Adjacent odd primes

In this section we consider whether two odd primes distinct from the characteristic are adjacent in the Gruenberg — Kegel graph of a finite group of Lie type.

Proposition 2.1. *Let $G=A_{n-1}(q)$ be a finite simple group of Lie type over a field of characteristic p. Let r,s be odd primes and $r,s\in\pi(G)\setminus\{p\}$. Denote $k=e(r,q)$, $l=e(s,q)$ and suppose that $2\leqslant k\leqslant l$. Then r and s are non-adjacent if and only if $k+l>n$ and k does not divide l.*

Proof. Note first that, for any odd prime $c\neq p$:

$$c \textit{ divides } q^x-1 \textit{ if and only if } e(c,q) \textit{ divides } x. \tag{1}$$

Indeed, by definition, c divides $q^{e(c,q)}-1$ and does not divide q^y-1 for all $y<e(c,q)$, i. e., $e(c,q)$ is the order of q in the multiplicative group $GF(c)^*$ of the finite field $GF(c)$. So if c divides q^z-1, then $q^z=1$ in $GF(c)^*$, hence $e(c,q)$ divides z. Now assume that $e(c,q)$ divides z. Then $q^z-1=(q^{e(c,q)}-1)f(q)$ for some $f(t)\in\mathbb{Z}[t]$, hence c divides q^z-1.

Assume that $k+l\leqslant n$. Consider a maximal torus T of G of order

$$\frac{1}{(n,q-1)(q-1)}(q^k-1)\cdot(q^l-1)\cdot(q-1)^{n-k-l}.$$

The torus T is an abelian subgroup of G and $r, s \in \pi(T)$. It follows that T contains an element of order rs, hence r, s are adjacent. If k divides l then both r and s divide $q^l - 1$, therefore a maximal torus of order $\frac{1}{(n,q-1)}(q^l-1)(q-1)^{n-l-1}$ contains an element of order rs.

Assume now that $k + l > n$, k does not divide l, and assume that $g \in G$ is an element of order rs. Then $(|g|, p) = 1$ hence g is semisimple. Therefore there exists a maximal torus T such that $g \in T$. By Lemma 1.2 the order of T is equal to

$$\frac{1}{(n,q-1)(q-1)}(q^{n_1}-1)\cdot(q^{n_2}-1)\cdot\ldots\cdot(q^{n_x}-1)$$

for appropriate partition $n_1+n_2+\ldots+n_x = n$ of n. Since r, s are prime, there exist n_i, n_j such that r divides $q^{n_i}-1$ and s divides $q^{n_j}-1$. In view of (1), it follows that $n_i = a\cdot k$, $n_j = b \cdot l$ for some $a, b \geqslant 1$. Moreover, since $k + l > n$ and $k \leqslant l$, we have that $b = 1$. Indeed, otherwise $n_j \geqslant l + l \geqslant k + l > n$, a contradiction with $n_1 + \ldots + n_x = n$. Since k does not divide l it follows that $n_i \neq n_j$. Hence $n_1+n_2+\ldots+n_x \geqslant n_i+n_j = a\cdot k+l > n$; a contradiction. □

Proposition 2.2. *Let $G = {}^2A_{n-1}(q)$ be a finite simple group of Lie type over a field of characteristic p. Define a function*

$$\nu(m) = \begin{cases} m & , \textit{if } m \equiv 0 (\mathrm{mod}\ 4), \\ \frac{m}{2} & , \textit{if } m \equiv 2 (\mathrm{mod}\ 4), \\ 2m & , \textit{if } m \equiv 1 (\mathrm{mod}\ 2). \end{cases}$$

Let r, s be odd primes and $r, s \in \pi(G) \setminus \{p\}$. Denote $k = e(r, q)$, $l = e(s, q)$ and suppose that $2 \leqslant \nu(k) \leqslant \nu(l)$. Then r and s are non-adjacent if and only if $\nu(k) + \nu(l) > n$ and $\nu(k)$ does not divide $\nu(l)$.

Proof. Note first that, for any odd prime $c \neq p$:

$$c \textit{ divides } q^x - (-1)^x \textit{ if and only if } \nu(e(c,q)) \textit{ divides } x. \qquad (2)$$

Assume first that $e(c, q)$ is odd. If c divides $q^z - (-1)^z$ then c divides $q^{2z} - 1$, hence, by (1), $e(c, q)$ divides $2z$. Since $e(c, q)$ is odd it follows that $e(c, q)$ divides z and therefore (again by (1)) c divides $q^z - 1$. Since c is odd we have that $q^z + 1$ is not divisible by c, so $q^z - (-1)^z = q^z - 1$, i. e., z is even. But $e(c, q)$ is odd hence $2e(c, q) = \nu(e(c, q))$ divides z. Now assume that $\nu(e(c, q))$ divides z. Then z is even, hence $q^z - (-1)^z = q^z - 1$ and c divides $q^z - (-1)^z$ by (1). So (2) is true in this case.

Assume that $e(c, q) \equiv 2(\mathrm{mod}\ 4)$. If c divides $q^z - (-1)^z$ then c divides $q^{2z} - 1$, hence $e(c, q)$ divides $2z$. But $\nu(e(c, q)) = \frac{e(c,q)}{2}$, therefore $\nu(e(c, q))$ divides z. If $\nu(e(c, q))$ divides z, and z is odd, then $q^z - (-1)^z = q^z + 1$. We have that $e(c, q)$ divides $2z$, hence c divides $q^{2z} - 1$. Now $q^{2z} - 1 = (q^z - 1)\cdot(q^z + 1)$. Since z is odd, $e(c, q)$ does not divide z, therefore, by (1), c does not divide $q^z - 1$, hence c divides $q^z + 1 = q^z - (-1)^z$. If $\nu(e(c, q))$ divides z, and z, is even then $2\nu(e(c, q)) = e(c, q)$ divides z, hence, by (1), c divides $q^z - (-1)^z = q^z - 1$. Thus (2) is true in this case as well.

At the end assume that $e(c, q) \equiv 0(\mathrm{mod}\ 4)$. If c divides $q^z - (-1)^z$ then, as above, c divides $q^{2z} - 1$, hence $e(c, q)$ divides $2z$ and $\frac{e(c,q)}{2}$ divides z. But $\frac{e(c,q)}{2}$ is even, hence,

z is even and $q^z - (-1)^z = q^z - 1$. It follows that $e(c,q) = \nu(e(c,q))$ divides z. If $\nu(e(c,q)) = e(c,q)$ divides z then z is even and, by (1), c divides $q^z - (-1)^z = q^z - 1$.

Assume that $\nu(k) + \nu(l) \leqslant n$. Consider a maximal torus T of G of order

$$\frac{1}{(n,q+1)(q+1)}(q^{\nu(k)} - (-1)^{\nu(k)}) \cdot (q^{\nu(l)} - (-1)^{\nu(l)}) \cdot (q+1)^{n-\nu(k)-\nu(l)}.$$

The torus T is an abelian subgroup of G and $r, s \in \pi(T)$. It follows that T contains an element of order rs, hence r, s are adjacent. If $\nu(k)$ divides $\nu(l)$ then both r and s divide $q^{\nu(l)} - (-1)^{\nu(l)}$, therefore a maximal torus of order $\frac{1}{(n,q+1)}(q^{\nu(l)} - (-1)^{\nu(l)})(q+1)^{n-\nu(l)-1}$ contains an element of order rs.

Assume now that $\nu(k) + \nu(l) > n$, $\nu(k)$ does not divide $\nu(l)$, and assume that $g \in {}^2A_{n-1}(q)$ is an element of order rs. Then $(|g|, p) = 1$ hence g is semisimple. Therefore there exists a maximal torus T such that $g \in T$. Using (2) and Lemma 1.2 we obtain a contradiction like in the proof of Proposition 2.1. $\square$

Proposition 2.3. *Let G be one of the finite simple groups of Lie type $B_n(q)$ or $C_n(q)$ over a field of characteristic p. Define the function*

$$\eta(m) = \begin{cases} m & \text{, if } m \text{ is odd,} \\ \frac{m}{2} & \text{, if } m \text{ is even.} \end{cases}$$

Let r, s be odd primes with $r, s \in \pi(G) \setminus \{p\}$. Denote $k = e(r,q)$, $l = e(s,q)$ and suppose that $1 \leqslant \eta(k) \leqslant \eta(l)$. Then r and s are non-adjacent if and only if $\eta(k) + \eta(l) > n$ and $\eta(k)$, $\eta(l)$ satisfy (3):

$$\textit{Either } \frac{\eta(l)}{\eta(k)} \textit{ is not an odd integer, or } \eta(k) = \eta(l) \text{ and } k \neq l. \tag{3}$$

Note that (3) is true in the following cases:

1. Both k and l are even and either $\eta(k)$ does not divide $\eta(l)$ or $\frac{\eta(l)}{\eta(k)}$ is even. In this case $q^{\eta(k)} + (-1)^k = q^{\eta(k)} + 1$ does not divide $q^{\eta(l)} + (-1)^l = q^{\eta(l)} + 1$.
2. Both k and l are odd and $\eta(k)$ does not divide $\eta(l)$. In this case $q^{\eta(k)} + (-1)^k = q^k - 1$ does not divide $q^{\eta(l)} + (-1)^l = q^l - 1$.
3. k is odd and l is even. In this case $q^{\eta(k)} + (-1)^k = q^k - 1$ does not divide $q^{\eta(l)} + (-1)^l = q^{\eta(l)} + 1$.
4. k is even, l is odd and either $\eta(k)$ does not divide $\eta(l)$ or $\frac{\eta(l)}{\eta(k)}$ is either even, or is equal to 1. In this case $q^{\eta(k)} + (-1)^k = q^{\eta(k)} + 1$ does not divide $q^{\eta(l)} + (-1)^l = q^l - 1$.

This means that $\eta(k)$, $\eta(l)$ satisfy (3) if and only if $q^{\eta(k)} + (-1)^k$ does not divide $q^{\eta(l)} + (-1)^l$.

Proof. First we prove that, for any odd prime $c \neq p$:

$$\textit{if } c \textit{ divides } q^x \pm 1 \textit{ then } \eta(e(c,q)) \textit{ divides } x. \tag{4}$$

Assume first that $e(c,q)$ is odd and c divides $q^z \pm 1$. By (1) it follows that $e(c,q)$ divides $2z$. But $e(c,q)$ is odd hence $e(c,q) = \eta(e(c,q))$ divides z. Assume now that $e(c,q)$ is even and c divides $q^z \pm 1$. It follows that c divides $q^{2z}-1$, hence $e(c,q)$ divides $2z$ and $\frac{e(c,q)}{2} = \eta(e(c,q))$ divides z.

Now if $\eta(k)+\eta(l) \leqslant n$ we may consider a maximal torus T of order $\frac{1}{(2,q-1)}(q^{\eta(k)}+(-1)^k)\cdot(q^{\eta(l)}+(-1)^l)\cdot(q-1)^{n-\eta(k)-\eta(l)}$. We have that T is an abelian group and $r,s \in \pi(T)$. Hence T contains an element of order rs. If $\eta(k)$, $\eta(l)$ does not satisfy (3), then $q^{\eta(k)}+(-1)^k$ divides $q^{\eta(l)}+(-1)^l$, therefore both r and s divide $q^{\eta(l)}+(-1)^l$. Thus a maximal torus of order $\frac{1}{(2,q-1)}(q^{\eta(l)}+(-1)^l)(q-1)^{n-\eta(l)}$ contains an element of order rs.

Assume that $\eta(k)+\eta(l) > n$, $\eta(k)$ and $\eta(l)$ satisfy (3), and assume that there exists an element $g \in G$ of order rs. Since $(|g|,p)=1$, it follows that g is semisimple. So there exists a maximal torus T containing g. In view of Lemma 1.2, the order $|T|$ is equal to

$$\frac{1}{(2,q-1)}(q^{n_1}\pm 1)\cdot(q^{n_2}\pm 1)\cdot\ldots\cdot(q^{n_x}\pm 1)$$

for appropriate partition $n_1+n_2+\ldots+n_x = n$ of n. Using (4) we obtain a contradiction like in Proposition 2.1. □

Proposition 2.4. *Let $G = D_n^{\varepsilon}(q)$ be a finite simple group of Lie type over a field of characteristic p and the function $\eta(m)$ is defined in Proposition 2.3. Let r,s be odd primes with $r,s \in \pi(D_n^{\varepsilon}(q))\setminus\{p\}$. Denote $k = e(r,q)$, $l = e(s,q)$ and suppose that $1 \leqslant \eta(k) \leqslant \eta(l)$. Then r and s are non-adjacent if and only if $2\cdot\eta(k)+2\cdot\eta(l) > 2n-(1-\varepsilon(-1)^{k+l})$ and $\eta(k)$, $\eta(l)$ satisfy (3).*

Proof. Using (4) and Lemma 1.2 we prove the proposition as above. □

Proposition 2.5. *Let G be a finite simple exceptional group of Lie type over a field of characteristic p, let r,s be odd primes, $r,s \in \pi(G)\setminus\{p\}$, $k = e(r,q)$, $l = e(s,q)$, and $1 \leqslant k \leqslant l$. Then r and s are non-adjacent if and only if $k \neq l$ and one of the following holds:*

1. *$G = G_2(q)$ and either $r \neq 3$ and $l \in \{3,6\}$, or $r = 3$ and $l = 9-3k$.*
2. *$G = F_4(q)$ and either $l \in \{8,12\}$, or $k \in \{3,4\}$ and $l = 6$, or $k = 3$ and $l = 4$.*
3. *$G = E_6(q)$ and either $l = 5$ and $k \geqslant 3$, or $l = 6$ and $k = 5$, or $l = 8$ and $k \geqslant 3$, or $l = 8$ and $r = 3$ for $(q-1)_3 = 3$, or $l = 9$, or $l = 12$, $k \neq 3$.*
4. *$G = {}^2E_6(q)$ and either $l = 8$ and $k \geqslant 3$, or $l = 8$ and $r = 3$ for $(q+1)_3 = 3$, or $l = 10$ and $k \geqslant 3$, or $l = 12$ and $k \neq 6$, or $l = 18$.*
5. *$G = E_7(q)$ and either $l \in \{14,18\}$ and $k \neq 2$, or $l \in \{7,9\}$ and $k \geqslant 2$, or $l = 8$ and $k = 7$, or $l = 10$ and $k \geqslant 3, k \neq 4,6$, or $l = 12$ and $k \geqslant 4, k \neq 6$.*
6. *$G = E_8(q)$ and either $l \in \{7,14\}$ and $k \geqslant 3$, or $l = 9$ and $k \geqslant 4$, or $l \in \{8,10,12\}$ and $k \geqslant 5, k \neq 6$, or $l = 18$ and $k \neq 1,2,6$, or $l = 20$ and $r\cdot k \neq 20$, or $l \in \{15,24,30\}$.*

7. $G = {}^3D_4(q)$ *and either* $l = 6$ *and* $k = 3$, *or* $l = 12$.

Proof. As for classical groups of Lie type, prime divisors $r, s \in \pi(G)$ satisfying the conditions of the proposition are adjacent if and only if rs divides the order of some maximal torus of G. Thus, using Lemma 1.3 instead of Lemma 1.2 we prove the proposition as above. $\square$

Proposition 2.6. *Let G be a finite simple Suzuki or Ree group over a field of characteristic p, let r, s be odd primes $r, s \in \pi(G) \setminus \{p\}$. Then r, s are non-adjacent if and only if one of the following holds:*

1. $G = {}^2B_2(2^{2n+1})$, *r divides $m_k(B, n)$, s divides $m_l(B, n)$ and $k \neq l$.*

2. $G = {}^2G_2(3^{2n+1})$, *r divides $m_k(G, n)$, s divides $m_l(G, n)$ and $k \neq l$.*

3. $G = {}^2F_4(2^{2n+1})$, *r divides $m_k(F, n)$, s divides $m_l(F, n)$, $k \neq l$, and if $\{k, l\} = \{1, 3\}$, then $r \neq 3 \neq s$.*

Numbers $m_i(B, n)$, $m_i(G, n)$, and $m_i(F, n)$ are defined in Lemma 1.5.

Proof. We use Lemma 1.3, Lemma 1.5 and arguments as in the previous propositions of the section. $\square$

3 Adjacency with the characteristic

In this section we consider whether a prime r and the characteristic p of the base field of a finite group of Lie type are adjacent.

Proposition 3.1. *Let $G = O^{p'}(\overline{G}_\sigma)$ be a finite simple classical group of Lie type defined over a field of characteristic p. Let $r \in \pi(G)$ and $r \neq p$. Then r and p are non-adjacent if and only if one of the following holds:*

1. $G = A_{n-1}(q)$, *r is odd, and $e(r, q) > n - 2$.*

2. $G = {}^2A_{n-1}(q)$, *r is odd, and $\nu(e(r, q)) > n - 2$. The function $\nu(m)$ is defined in Proposition* 2.2.

3. $G = C_n(q)$, $\eta(e(r, q)) > n - 1$. *The function $\eta(m)$ is defined in Proposition* 2.3.

4. $G = B_n(q)$, $\eta(e(r, q)) > n - 1$. *The function $\eta(m)$ is defined in Proposition* 2.3.

5. $G = D_n^\varepsilon(q)$, $\eta(e(r, q)) > n - 2$. *The function $\eta(m)$ is defined in Proposition* 2.3.

6. $G = A_1(q)$, $r = 2$.

7. $G = A_2^\varepsilon(q)$, $r = 3$ *and* $(q - \varepsilon 1)_3 = 3$.

Proof. It is evident that 2 and p are adjacent in all classical simple groups except $A_1(q)$. Hence we may assume that r is odd.

First we outline the general idea of the proof. We shall use [13] as a technical tool in our proof, so we shall keep notations of [13]. In order to prove that r and p are adjacent we find a connected reductive subgroup R of maximal rank in G such that $R = T(G_1 * G_2)$, where T is a maximal torus, both G_1 and G_2 are non-trivial groups of Lie type such that $r \in \pi(G_1)$. Then G_1 contains an element of order r and it centralizes G_2. Since G_2 is non-trivial, it contains an element of order p, so $G \geq R$ contains an element of order rp.

In order to prove that r and p are non-adjacent we shall consider arbitrary element g of order r and its connected centralizer $G \cap C_{\overline{G}}(g)^0$. Recall that $C_{\overline{G}}(g)^0 = \overline{S} * \overline{L}$, where $\overline{S} = Z(C_{\overline{G}}(g)^0)$ is a central torus and $\overline{L}$ is a semisimple part. Clearly $g \in \overline{S} \cap G \leq \overline{S}_\sigma$, hence r divides $|\overline{S}_\sigma|$. This condition would imply that $\overline{L}_\sigma$ is trivial, hence $C_{\overline{G}}(g)^0$ does not contain unipotent elements. But every unipotent element of $C_{\overline{G}}(g)$ is contained in $C_{\overline{G}}(g)^0 \leq N_{\overline{G}}(\overline{T})^0 = \overline{T}$. So none unipotent element of G centralizes g. Now consider all classical groups case by case.

$A_1(q)$. It is known that if g is an element of order $r \neq p$, then $(|C_{A_1(q)}(g)|, p) = 1$ (see, [13, Proposition 7]). So r, p are non-adjacent for every $r \in \pi(A_1(q)) \setminus \{p\}$.

$A_2(q)$. By using [13, Proposition 7] we obtain that only prime divisors of $\frac{q-1}{(3,q-1)}$ are adjacent to p and the proposition is true in this case.

$A_{n-1}(q)$, and $n \geqslant 4$. In our case T is a Cartan subgroup, $G_1 = A_{n-3}(q)$ and $G_2 = A_1(q)$. The existence of such a subgroup can be obtained by using [13, Proposition 7]. Thus every r with $e(r,q) \leqslant n-2$ divides $|G_1|$, hence is adjacent to p. Now let $e(r,q) = n-1$ and let g be an element of order r. In view of (1) we obtain that $q^{n-1} - 1$ must divide $|\overline{S}_\sigma| \cdot (q-1)$. It follows from [13, Proposition 7] that $G \cap C_{\overline{G}}(g)^0$ is a maximal torus of order $\frac{1}{(n,q-1)}(q^{n-1} - 1)$. Hence $|\overline{L}_\sigma| = 1$ and r, p are non-adjacent. If $e(r,q) = n$ and g is an element of order r, then by (1), $q^n - 1$ must divide $|\overline{S}_\sigma| \cdot (q-1)$. Again by using [13, Proposition 7] we obtain that $G \cap C_{\overline{G}}(g)^0$ is a maximal torus of order $\frac{1}{(n,q-1)}\frac{q^n-1}{q-1}$. So $|\overline{L}_\sigma| = 1$ in this case and every r with $e(r,q) > n-2$ is non-adjacent to p.

${}^2A_2(q)$. By using [13, Proposition 8] we obtain that only prime divisors of $\frac{q+1}{(3,q+1)}$ are adjacent to p and the proposition is true in this case.

${}^2A_{n-1}(q)$ and $n \geqslant 4$. Again T is a Cartan subgroup, $G_1 = {}^2A_{n-3}(q)$ and $G_2 = A_1(q)$. The existence of such a subgroup can be obtained by using [13, Proposition 8]. Thus every r with $\nu(e(r,q)) \leqslant n-2$ divides $|G_1|$, hence is adjacent to p. Now let $\nu(e(r,q)) = n-1$ and g is an element of order r. Then in view of (2) we obtain that $q^{n-1} - (-1)^{n-1}$ divides $|\overline{S}| \cdot (q+1)$. It follows from [13, Proposition 8] that $G \cap C_{\overline{G}}(g)^0$ is a maximal torus of order $\frac{1}{(n,q+1)}(q^{n-1} - (-1)^{n-1})$. Hence $|\overline{L}_\sigma| = 1$ and r, p are non-adjacent. If $\nu(e(r,q)) = n$ and g is an element of order r, then by (2), $q^n - (-1)^n$ divides $|\overline{S}_\sigma| \cdot (q+1)$. By using [13, Proposition 8] we obtain that $G \cap C_{\overline{G}}(g)^0$ is a maximal torus of order $\frac{1}{(n,q+1)}\frac{q^n-(-1)^n}{q+1}$. Therefore $|\overline{L}_\sigma| = 1$ and every r with $\nu(e(r,q)) > n-2$ is non-adjacent to p.

$C_n(q)$. Take for R torus T as a Cartan subgroup, $G_1 = C_{n-1}(q)$, $G_2 = A_1(q)$. Such a subgroup R exists in view of [13, Propositions 9 and 12]. Again every r with $\eta(e(r,q)) \leqslant n-1$ divides $|G_1|$, hence is adjacent to p. If $\eta(e(r,q)) = n$ and g is an element of order r, then (4) implies that either $q^n - 1$, or $q^n + 1$ divides $|\overline{S}_\sigma|$. From [13, Propositions 9 and 12] we obtain that $G \cap C_{\overline{G}}(g)^0$ is a maximal torus of order $\frac{1}{(2,q-1)}(q^n \pm 1)$ respectively, hence $|\overline{L}_\sigma| = 1$ and r, p are non-adjacent.

$B_n(q)$. Since $B_2(q) \simeq C_2(q)$ and $B_n(2^t) \simeq C_n(2^t)$, we may assume that p is odd and $n \geqslant 3$. We can take T to be a Cartan subgroup, $G_1 = B_{n-2}(q)$, and $G_2 = D_2(q)$. Such a subgroup exists in view of [13, Proposition 11]. Since every prime r with $\eta(e(r,q)) \leqslant n-2$ divides the order of G_1, we obtain that r and p are adjacent if $\eta(e(r,q)) \leqslant n-2$. If $\eta(e(r,q)) = n-1$, then, again by [13, Proposition 11], there exists a reductive subgroup R such that $|\overline{S}_\sigma| = q^{\eta(e(r,q))} + (-1)^{e(r,q)}$ and $\overline{L}_\sigma = B_1(q) \simeq A_1(q)$. Hence r, p are adjacent in G. Assume now that $\eta(e(r,q)) = n$ and that g is an element of order r of G. Then the order $|\overline{S}_\sigma|$ is given in [13, Proposition 11] and is equal to $\prod_i (q^{n_i} \pm 1)$, where $\sum_i n_i \leqslant n$. By using (4) we obtain that $\eta(e(r,q)) = n$ divides n_i for some i. Hence either $q^n - 1$, or $q^n + 1$ divides $|S_\sigma|$. In view of [13, Proposition 11] this implies that $|\overline{L}_\sigma| = 1$, hence r, p are non-adjacent.

$D_n^\varepsilon(q)$. We can take T to be a Cartan subgroup, $G_1 = D_{n-2}^\varepsilon(q)$, $G_2 = A_1(q)$. The existence of such a subgroup R can be obtained by using [13, Proposition 10]. So r, p are adjacent for all r with $\eta(e(r,q)) \leqslant n-2$, except one case: $\eta(e(r,q)) = n-2$ and r divides $q^{n-2} + \varepsilon 1$. In this last case we obtain that there exists a reductive subgroup R such that $|\overline{S}_\sigma| = q^{n-2} + \varepsilon 1$ and $\overline{L}_\sigma \simeq D_2^\varepsilon(q)$. Hence in this exceptional case r, p are adjacent also. Now if $\eta(e(r,q)) \geqslant n-1$ and g is an element of G of order r, then as above we obtain that $q^{\eta(e(r,q))} + (-1)^{e(r,q)}$ divides $|\overline{S}_\sigma|$ and [13, Proposition 10] implies that $\overline{L}_\sigma$ is trivial. Therefore r, p are non-adjacent in this case. □

Proposition 3.2. *Let G be a finite simple exceptional group of Lie type over a field of characteristic p. Let $r \in \pi(G)$, $k = e(r,q)$, and $r \neq p$. Then r, p are non-adjacent if and only if one of the following holds:*

1. $G = G_2(q)$, $k \in \{3, 6\}$.
2. $G = F_4(q)$, $k \in \{8, 12\}$.
3. $G = E_6(q)$, $k \in \{8, 9, 12\}$.
4. $G = {}^2E_6(q)$, $k \in \{8, 12, 18\}$.
5. $G = E_7(q)$, $k \in \{7, 9, 14, 18\}$.
6. $G = E_8(q)$, $k \in \{15, 20, 24, 30\}$.
7. $G = {}^3D_4(q)$, $k = 12$.

Proof. All statements are obtained by using information about conjugacy classes and centralizers of semisimple elements given in [15] and [17]. □

Proposition 3.3. *Let G be a finite simple Suzuki or Ree group over a field of characteristic p, let $r \in \pi(G) \setminus \{p\}$. Then r, p are non-adjacent if and only if one of the following holds:*

1. $G = {}^2B_2(2^{2n+1})$, *r divides $m_k(B, n)$.*
2. $G = {}^2G_2(3^{2n+1})$, *r divides $m_k(G, n)$ and $r \neq 2$.*

3. *$G = {}^2F_4(2^{2n+1})$, r divides $m_k(F,n)$, $r \neq 3$, and $k > 2$.*

Numbers $m_i(B,n)$, $m_i(G,n)$, and $m_i(F,n)$ are defined in Lemma 1.5.

Proof. All statements are obtained by using information about conjugacy classes and centralizers of semisimple elements given in [15]. □

4 Adjacency of 2 and an odd prime r

In this section we consider whether 2 and an odd prime r are adjacent if both of them are not equal to the characteristic p of the base field. We start from groups $A_n^{\varepsilon}(q)$. Recall that for those groups we did not consider adjacency criterion for prime divisors of $q - \varepsilon 1$ in Section 2. It is natural to consider such a criterion together with a criterion for 2, hence we give it in the following two propositions.

Proposition 4.1. *Let $G = A_{n-1}(q)$ be a finite simple group of Lie type. Let r be a prime divisor of $q-1$ and s be an odd prime distinct from the characteristic. Denote $k = e(s,q)$. Then s and r are non-adjacent if and only if one of the following holds:*

1. *$k = n$, $n_r \leqslant (q-1)_r$, and if $n_r = (q-1)_r$, then $2 < (q-1)_r$.*

2. *$k = n-1$ and $(q-1)_r \leqslant n_r$.*

Proof. First we prove the following statement

$$\left(\frac{q^n-1}{q-1}\right)_r = n_r, \text{ if } (q-1)_r \geqslant n_r \text{ and } (q-1)_r > 2,$$
$$\left(\frac{q^n-1}{q-1}\right)_2 > 2, \text{ if } (q-1)_2 = n_2 = 2. \quad (5)$$

Assume that r is odd. Then $n = r^k \cdot l$, where $(l,r) = 1$ and $q = r^k \cdot m + 1$. Now $q^n - 1 = (q^{r^k} - 1) \cdot (q^{n-r^k} + q^{n-2r^k} + \ldots + q^{r^k} + 1)$. The second multiplier is the sum of l numbers of type q^i and $q^i \equiv 1 (\mathrm{mod}\ r)$ for all i. Since $(l,r) = 1$ it follows that the second multiplier is coprime to r. Thus $\left(\frac{q^n-1}{q-1}\right)_r = \left(\frac{q^{r^k}-1}{q-1}\right)_r$. Now remember that $q = r^k \cdot m + 1$, hence

$$\frac{q^{r^k}-1}{q-1} = (r^k \cdot m)^{r^k-1} + r^k \cdot (r^k \cdot m)^{r^k-2} + \ldots + \frac{1}{2} r^k (r^k - 1)(r^k \cdot m) + r^k.$$

Since r^{k+1} divides all summands, except the last, and r^{k+1} does not divide r^k we obtain that r^k divides $\left(\frac{q^{r^k}-1}{q-1}\right)_r$, but r^{k+1} does not divide $\left(\frac{q^{r^k}-1}{q-1}\right)_r$. Therefore $\left(\frac{q^{r^k}-1}{q-1}\right)_r = n_r$ in this case.

Assume now that $r = 2$, $n = 2^k \cdot l$, where l is odd, and $q = 2^k \cdot m + 1$. Then

$$\left(\frac{q^n-1}{q-1}\right)_2 = \left(\frac{q^l-1}{q-1}\right)_2 \cdot (q^l+1) \cdot (q^{2l}+1) \cdot \ldots \cdot (q^{2^{k-1}l}+1).$$

Since l is odd we have that $\left(\frac{q^l-1}{q-1}\right)_2=1$. If $(q-1)_2>2$ we have that $(q^i+1)_2=2$ for all i. Therefore

$$\left(\frac{q^n-1}{q-1}\right)_2=(q^l+1)_2\cdot(q^{2l}+1)_2\cdot\ldots\cdot(q^{2^{k-1}l}+1)_2=2^k=n_2.$$

The second case $(q-1)_2=n_2=2$ is evident.

Assume that $k\leqslant n-2$. By Lemma 1.2 it follows that there exists a maximal torus T of G of order $\frac{1}{(n,q-1)}(q^k-1)(q-1)^{n-k-1}$. We have that T is an abelian subgroup of G, and both r,s divide $|T|$. Therefore T contains an element of order rs.

Assume that $k=n$. By (1) and Lemma 1.2 every element of order s is contained in a maximal torus T of order $\frac{1}{(n,q-1)}\frac{q^n-1}{q-1}$. In view of (5) r does not divide $|T|$ if and only if condition 1 of the proposition holds.

Assume now that $k=n-1$. By (1) and Lemma 1.2 every element of order s is contained in a maximal torus T of order $\frac{1}{(n,q-1)}(q^{n-1}-1)$. We have that $(q^{n-1}-1)=(q-1)(q^{n-2}+q^{n-3}+\ldots+q+1)$ and $(q^{n-2}+q^{n-3}+\ldots+q+1)_r=1$. Therefore r does not divide $|T|$ if and only if $(q-1)_r\leqslant n_r$ and we obtain condition 2 of the proposition. □

Proposition 4.2. *Let $G={}^2A_{n-1}(q)$ be a finite simple group of Lie type. Let r be a prime divisor of $q+1$ and s be an odd prime distinct from the characteristic. Denote $k=e(s,q)$. Then s and r are non-adjacent if and only if one of the following holds:*

1. *$\nu(k)=n$, $n_r\leqslant(q+1)_r$, and if $n_r=(q+1)_r$, then $2<(q+1)_r$.*

2. *$\nu(k)=n-1$ and $(q+1)_r\leqslant n_r$.*

The function $\nu(m)$ is defined in Proposition 2.2.

Proof. Like in the previous proposition first we show that

$$\left(\frac{q^n-(-1)^n}{q+1}\right)_r=n_r,\text{ if }(q+1)_r\geqslant n_r\text{ and }(q+1)_r>2,$$
$$\left(\frac{q^n-(-1)^n}{q+1}\right)_2>2,\text{ if }(q+1)_2=n_2=2.\quad(6)$$

Assume that r is odd. Then $n=r^k\cdot l$, where $(l,r)=1$ and $q=r^k\cdot m-1$. Now $q^n-(-1)^n=(q^{r^k}+1)\cdot(q^{n-r^k}-q^{n-2r^k}+\ldots+(-1)^{l-1}q^{r^k}+(-1)^l)$. The second multiplier is the sum of l numbers of type $(-1)^tq^{n-(t+1)r^k}$ and $q^{n-(t+1)r^k}\equiv(-1)^{n+t-1}(\mathrm{mod}\ r)$ for all t. Hence the second multiplier is equivalent $(-1)^{n-1}l$ modulo r. Since $(l,r)=1$ it follows that the second multiplier is coprime to r. Thus $\left(\frac{q^n-(-1)^n}{q+1}\right)_r=\left(\frac{q^{r^k}+1}{q+1}\right)_r$. Now remember that $q=r^k\cdot m-1$, hence

$$\frac{q^{r^k}+1}{q+1}=(r^k\cdot m)^{r^k-1}-r^k\cdot(r^k\cdot m)^{r^k-2}+\ldots+(-1)^{k-2}\frac{1}{2}r^k(r^k-1)(r^k\cdot m)+(-1)^{k-1}r^k.$$

Since r^{k+1} divides all summands, except the last, and r^{k+1} does not divide r^k we obtain that r^k divides $\left(\frac{q^{r^k}+1}{q+1}\right)_r$, but r^{k+1} does not divide $\left(\frac{q^{r^k}+1}{q+1}\right)_r$. Therefore $\left(\frac{q^{r^k}+1}{q+1}\right)_r = n_r$ in this case.

Assume now that $r = 2$, $n = 2^k \cdot l$, where l is odd, and $q = 2^k \cdot m - 1$. Then

$$\left(\frac{q^n - (-1)^n}{q+1}\right)_2 = \left(\frac{q^l + 1}{q+1}\right)_2 \cdot (q^l - 1) \cdot (q^{2l} + 1) \cdot \ldots \cdot (q^{2^{k-1}l} + 1).$$

Since l is odd we have that $\left(\frac{q^l+1}{q+1}\right)_2 = 1$. If $(q+1)_2 > 2$ we have that $(q^{2i}+1)_2 = 2$ for all $i \geqslant 1$ and $(q^l - 1)_2 = 2$. Therefore

$$\left(\frac{q^n - (-1)^n}{q+1}\right)_2 = (q^l - 1)_2 \cdot (q^{2l} + 1)_2 \cdot \ldots \cdot (q^{2^{k-1}l} + 1)_2 = 2^k = n_2.$$

The second case $(q+1)_2 = n_2 = 2$ is evident.

Assume that $\nu(k) \leqslant n - 2$. By Lemma 1.2 it follows that there exists a maximal torus T of G of order $\frac{1}{(n,q+1)}(q^{\eta(k)} - (-1)^{\eta(k)})(q+1)^{n-k-1}$. We have that T is an abelian subgroup of G and both r, s divide $|T|$. Therefore T contains an element of order rs.

Assume that $\nu(k) = n$. By (2) and Lemma 1.2 every element of order s is contained in a maximal torus T of order $\frac{1}{(n,q+1)} \frac{q^n - (-1)^n}{q+1}$. In view of (6) r does not divide $|T|$ if and only if condition 1 of the proposition holds.

Assume now that $\nu(k) = n - 1$. By (2) and Lemma 1.2 every element of order s is contained in a maximal torus T of order $\frac{1}{(n,q+1)}(q^{n-1} - (-1)^{n-1})$. We have that $(q^{n-1} - (-1)^{n-1}) = (q+1)(q^{n-2} - q^{n-3} + \ldots + (-1)^{n-3}q + (-1)^{n-2})$ and $(q^{n-2} - q^{n-3} + \ldots + (-1)^{n-3}q + (-1)^{n-2})_r = 1$. Therefore r does not divide $|T|$ if and only if $(q+1)_r \leqslant n_r$ and we obtain condition 2 of the proposition. $\square$

Proposition 4.3. *Let $G = B_n(q)$ or $G = C_n(q)$ be a finite simple group of Lie type over a field of odd characteristic p. Let r be an odd prime divisor of $|G|$, $r \neq p$, and $k = e(r,q)$. Then $2, r$ are non-adjacent if and only if $\eta(k) = n$ and one of the following holds:*

1. *n is odd and $k = (3 - e(2,q))n$.*

2. *n is even and $k = 2n$.*

The function $\eta(m)$ is defined in Proposition 2.3.

Proof. If $\eta(k) \leqslant n - 1$, then by Lemma 1.2 there exists a maximal torus T of order $\frac{1}{2}(q^{\eta(k)} + (-1)^k)(q-1)^{n-\eta(k)}$. We have that T is an abelian subgroup of G and $2, r$ both divide $|T|$. Hence T contains an element of order $2r$ and $2, r$ are adjacent.

Thus we may assume that $\eta(k) = n$. In view of (4) and Lemma 1.2 we have that every element g of order r is contained in a maximal torus T of order $\frac{1}{2}(q^n + (-1)^k)$. Therefore $2, r$ are non-adjacent if and only if 2 does not divide $|T|$ and the proposition follows. $\square$

Proposition 4.4. *Let $G = D_n^{\varepsilon}(q)$ be a finite simple group of Lie type over a field of odd characteristic p. Let r be an odd prime divisor of $|G|$, $r \neq p$, and $k = e(r,q)$. Then 2 and r are non-adjacent if and only if one of the following holds:*

1. *$\eta(k) = n$ and $(4, q^n - \varepsilon 1) = (q^n - \varepsilon 1)_2$.*
2. *$\eta(k) = k = n - 1$, n is even, $\varepsilon = +$, and $e(2, q) = 2$.*
3. *$\eta(k) = \frac{k}{2} = n - 1$, $\varepsilon = +$ and $e(2, q) = 1$.*
4. *$\eta(k) = \frac{k}{2} = n - 1$, n is odd, $\varepsilon = -$, and $e(2, q) = 2$.*

The function $\eta(m)$ is defined in Proposition 2.3.

Proof. First note that if $\eta(k) \leqslant n-2$, then by Lemma 1.2 there exists a maximal torus T of order $\frac{1}{(4,q^n-\varepsilon 1)}(q^{\eta(k)} + (-1)^k)(q + (-\varepsilon 1)^k)(q-1)^{n-\eta(k)-1}$. We have that $2, r$ both divide $|T|$, hence $2, r$ are adjacent.

Now consider the case $G = {}^2D_n(q)$ (i. e. $\varepsilon = -$). If $\eta(k) = n - 1$ then, by (4) and Lemma 1.2, we have that every element of order r is contained in a maximal torus T of order $\frac{1}{(4,q^n+1)}(q^{n-1} + (-1)^k)(q - (-1)^k)$. Then 2 does not divide $|T|$ if and only if condition 4 of the proposition holds. If $\eta(k) = n$, then, by (4) and Lemma 1.2, every element of order r is contained in a maximal torus T of order $\frac{1}{(4,q^n+1)}(q^n+1)$ (in particular, $k = 2\eta(k)$). Then 2 does not divide $|T|$ if and only if $(4, q^n + 1) = (q^n + 1)_2$ and we obtain the proposition for twisted groups.

Assume now that $G = D_n(q)$. If $\eta(k) = n - 1$, then, by (4) and Lemma 1.2 we obtain that every element of order r is contained in a maximal torus T of order $\frac{1}{(4,q^n-1)}(q^{n-1} + (-1)^k)(q + (-1)^k)$ and $k = n - 1$ if k is odd. Then 2 does not divide $|T|$ if and only if condition 2 or condition 3 of the proposition hold. If $\eta(k) = n$, then n is odd and, by (4) and Lemma 1.2, every element of order r is contained in a maximal torus T of order $\frac{1}{(4,q^n-1)}(q^n - 1)$. Clearly, 2 does not divide $|T|$ if and only if $(4, q^n - 1) = (q^n - 1)_2$ and the proposition follows. ☐

Proposition 4.5. *Let G be a finite simple exceptional group of Lie type over a field of odd characteristic p. Let r be an odd prime divisor of $|G|$, $r \neq p$, and $k = e(r, q)$. Then 2 and r are non-adjacent if and only if one of the following holds*:

1. *$G = G_2(q)$, $k \in \{3, 6\}$.*
2. *$G = F_4(q)$, $k = 12$.*
3. *$G = E_6(q)$, $k \in \{9, 12\}$.*
4. *$G = {}^2E_6(q)$, $k \in \{12, 18\}$.*
5. *$G = E_7(q)$, and either $k \in \{7, 9\}$ and $e(2, q) = 2$, or $k \in \{14, 18\}$ and $e(2, q) = 1$.*
6. *$G = E_8(q)$, $k \in \{15, 20, 24, 30\}$.*
7. *$G = {}^3D_4(q)$, $k = 12$.*
8. *$G = {}^2G_2(3^{2n+1})$, and r divides $m_3(G, n)$ or $m_4(G, n)$, where $m_i(G, n)$ are defined in Lemma* 1.5.

Proof. The proposition is immediate from Lemmas 1.3 and 1.5. ☐

5 On connection between the properties of the prime graph $GK(G)$ and the structure of G

Let $GK(G)$ be a prime graph of a finite group G. Obviously, the spectrum $\omega(G)$ uniquely determines the structure of $GK(G)$. Denote by $s(G)$ the number of connected components of $GK(G)$ and by $\pi_i(G)$, $i = 1, \ldots, s(G)$, the ith connected component of $GK(G)$. If G has even order then put $2 \in \pi_1(G)$. Denote by $\omega_i(G)$ the set of numbers $n \in \omega(G)$ such that each prime divisor of n belongs to $\pi_i(G)$.

Gruenberg and Kegel obtained the following description of finite groups with disconnected prime graph.

Theorem 5.1. (Gruenberg — Kegel Theorem) (see [1]) *If G is a finite group with disconnected graph $GK(G)$, then one of the following statements holds*:

(a) *$s(G) = 2$ and G is a Frobenius group*;

(b) *$s(G) = 2$ and G is a 2-Frobenius group, i.e., $G = ABC$, where A, AB are normal subgroups of G; AB, BC are Frobenius groups with cores A, B and complements B, C respectively*;

(c) *there exists a nonabelian simple group S such that $S \leq \overline{G} = G/K \leq \mathrm{Aut}(S)$, where K is a maximal normal soluble subgroup of G. Furthermore, K and $\overline{G}/S$ are $\pi_1(G)$-subgroups, the graph $GK(S)$ is disconnected, $s(S) \geqslant s(G)$, and for every i, $2 \leqslant i \leqslant s(G)$, there is j, $2 \leqslant j \leqslant s(S)$, such that $\omega_i(G) = \omega_j(S)$.*

Together with the classification of finite simple groups with disconnected prime graph obtained by Williams and Kondrat'ev (see [1] and [2]) the Gruenberg — Kegel theorem implied a series of important corollaries (see, for example, [1, Theorems 3–6] and [2, Theorems 2–3]). In last years this theorem is used for the proving of recognizability of finite groups by spectrum (for details see [3] and [4]).

The proof of the Gruenberg — Kegel Theorem substantially used the fact that in a group G (if its order is even) there exists an element of odd prime order disconnected in $GK(G)$ with a prime 2. It turns out that disconnectedness of $GK(G)$ could be successfully replaced in most cases by a weaker condition for the prime 2 to be nonadjacent to at least one odd prime.

Denote by $t(G)$ the maximal number of prime dividers of the order of G pairwise non-adjacent in $GK(G)$. In other words, $t(G)$ is a maximal number of vertices in independent sets of $GK(G)$ (a set of vertices is called *independent* if its elements are pairwise non-adjoint). In graph theory this number is usually called an *independence number* of the graph. By analogy we denote by $t(r, G)$ the maximal number of vertices in independent sets of $GK(G)$ containing the prime r. We call this number an *r-independence number*.

Theorem 5.2. (see [4]) *Let G be a finite group satisfying two conditions*:

(a) *there exist three primes in $\pi(G)$ whose are pairwise non-adjacent in $GK(G)$, i. e., $t(G) \geqslant 3$;*

(b) *there exists an odd prime in $\pi(G)$ which is non-adjacent to prime 2 in $GK(G)$, i. e., $t(2, G) \geqslant 2$.*

Then there exists a finite nonabelian simple group S such that $S \leq \overline{G} = G/K \leq$ $\mathrm{Aut}(S)$ *for maximal normal soluble subgroup K of G. Furthermore, $t(S) \geqslant t(G) - 1$ and one of the following statements holds:*

(1) *$S \simeq Alt_7$ or $A_1(q)$ for some odd q and $t(S) = t(2, S) = 3$.*

(2) *For every prime $r \in \pi(G)$ non-adjacent in $GK(G)$ to 2 the Sylow r-subgroup of G is isomorphic to the Sylow r-subgroup of S. In particular, $t(2, S) \geqslant t(2, G)$.*

Note that the condition (a) of Theorem 5.2 easily implies the insolubility of group G and so, by the Feit — Thompson Odd Theorem, implies that G is of even order. Moreover, the condition (a) can be replaced by the weaker condition of the insolubility of G (see [4, Propositions 2–3]). Together with [18, Theorem 2] Theorem 5.2 implies the following statement.

Theorem 5.3. (see [4, Proposition 4]) *Let L be a finite simple group with $t(2, L) \geqslant 2$ non-isomorphic to $A_2(3)$, ${}^2A_3(3)$, $C_2(3)$ and Alt_{10}. Let G be a finite group with $\omega(G) = \omega(L)$. Then for a group G the conclusion of Theorem 5.2 holds true. In particular, G has a unique nonabelian composition factor.*

As a matter of fact, these recent results (Theorem 5.2 and Theorem 5.3) gave the motivation for the present work. In Section 6 using results of Sections 1–4 we calculate the values of the independence numbers and the 2-independence numbers for all finite nonabelian simple groups. Furthermore, for every finite nonabelian simple group of Lie type over the field of characteristic p we also determine p-independence number $t(p, G)$.

6 Independence numbers

For a finite group G denote by $\rho(G)$ (by $\rho(r, G)$) some independence set in $GK(G)$ (containing r) with maximal number of vertices. Thus, $|\rho(G)| = t(G)$ and $|\rho(r, G)| = t(r, G)$. For a given finite nonabelian simple group we point out some $\rho(G)$ and $\rho(2, G)$ (obviously, they may be not uniquely defined) and so determine $t(G)$ and $t(2, G)$.

Proposition 6.1. *Let G be a sporadic group. Then $\rho(G)$, $t(G)$, $\rho(2, G)$ and $t(2, G)$ are listed in Table 2. Furthermore, $\rho(2, G)$ is uniquely determined.*

Proof. The results are easy to obtain using [5] or [6]. □

Now we deal with simple alternating groups.

Proposition 6.2. *Let $G = Alt_n$ be an alternating group of degree $n \geqslant 5$. Let s'_n be the largest prime which does not exceed $n/2$ and s''_n be the smallest prime greater than $n/2$. Denote by $\tau(n)$ the set $\{s \mid s \text{ is a prime}, n/2 < s \leqslant n\}$ and by $\tau(2, n)$ the set $\{s \mid s \text{ is a prime}, n - 3 \leqslant s \leqslant n\}$. Then $\rho(G)$, $t(G)$, $\rho(2, G)$ and $t(2, G)$ are listed in Table 3. Furthermore, $\rho(2, G)$ is uniquely determined.*

Proof. The result is immediate corollary of Proposition 1.1. □

Now we consider groups of Lie type. Since the problem here is more complicated we divide the process of its solution into several natural steps. The main tools will be a Zsigmondy Theorem, Lemma 1.5 and our results from Sections 2-4. Recall that for a given natural number q we denote by r_n some primitive prime divisor of $q^n - 1$ (if it exists). Note that a primitive prime divisor r_{2m} of $q^{2m} - 1$ divides $q^m + 1$ and does not divide $q^k + 1$ for every natural $k < m$. Since arguments in even characteristic of the field of definition are suitable for every characteristic p we start with determination of $t(p, G)$.

Proposition 6.3. *Let G be a finite simple group of Lie type over a field of characteristic p and order q. Then $\rho(p,G)$ and $t(p,G)$ are listed in Table 4 for classical groups and Table 5 for exceptional groups.*

Proof. (1) $G = A^{\varepsilon}_{n-1}(q)$.

Let $G = A_1(q)$. Since $A_1(2)$ and $A_1(3)$ are not simple we may assume that $q > 3$. By Zsigmondy Theorem, if $q \neq 2^t - 1$, then there exist primitive prime divisors r_1 and r_2 of $q-1$ and $q+1$. They are non-adjacent in $GK(G)$. By Proposition 3.1, they are non-adjacent to p. Since $q > 3$, for any $q = 2^t - 1$ there exists odd primitive prime divisor r_1 of $q-1$ which is not adjacent to 2 in $GK(G)$. Since in this case $e(2,q) = 2$ and $q+1 = 2^t$, the prime 2 is a unique primitive prime divisor of $q^2 - 1$. Hence $\rho(p,G) = \{p, r_1, r_2\}$ in this case too.

Suppose that $n > 2$. In order to consider groups $A_{n-1}(q)$ and ${}^2A_{n-1}(q)$ together we define new function:

$$\nu_{\varepsilon}(m) = \begin{cases} m & \text{, if either } \varepsilon = +, \text{ or } \varepsilon = - \text{ and } m \equiv 0 (\text{mod } 4), \\ \frac{m}{2} & \text{, if } \varepsilon = - \text{ and } m \equiv 2 (\text{mod } 4), \\ 2m & \text{, if } \varepsilon = - \text{ and } m \equiv 1 (\text{mod } 2). \end{cases}$$

Obviously, that $\nu_{\varepsilon}(m)$ is the identity function if $\varepsilon = +$, and $\nu_{\varepsilon}(m) = \nu(m)$ if $\varepsilon = -$. It is easy to check that ν_{ε} is a bijection on $\mathbb{N}$, thus ν_{ε}^{-1} is well defined.

Let $n = 3$ and $G = A^{\varepsilon}_2(q)$. Then, by Proposition 3.1, it follows that $r_{\nu_{\varepsilon}^{-1}(2)}$ and $r_{\nu_{\varepsilon}^{-1}(3)}$ are non-adjacent to p. Moreover, if $(q - \varepsilon 1)_3 = 3$, then, by Proposition 3.1, we have that 3 is non-adjacent to p and, by Propositions 4.1 and 4.2, in this case 3 is non-adjacent to $r_{\nu_{\varepsilon}^{-1}(2)}$ and $r_{\nu_{\varepsilon}^{-1}(3)}$. By Zsigmondy Theorem $r_{\nu_{\varepsilon}^{-1}(3)}$ exists for all q, except $\varepsilon = -$ and $q = 2$. But ${}^2A_2(2)$ is not simple and we do not consider this group. Therefore $r_{\nu_{\varepsilon}^{-1}(3)}$ exists for all simple groups in this case. Again by Zsigmondy Theorem, $r_{\nu_{\varepsilon^{-1}(2)}}$ exists if and only if $q + \varepsilon 1$ is not a power of 2, i.e., it is not a Mersenne prime in case $\varepsilon = +$, and it is not a Fermat prime or 9 in case $\varepsilon = -$. Thus, we have

$$\rho(p,G) = \begin{cases} \{p, 3, r_{\nu_{\varepsilon}^{-1}(2)}, r_{\nu_{\varepsilon}^{-1}(3)}\}, & \text{if } (q - \varepsilon 1)_3 = 3 \text{ and } q + \varepsilon \neq 2^k; \\ \{p, r_{\nu_{\varepsilon}^{-1}(2)}, r_{\nu_{\varepsilon}^{-1}(3)}\}, & \text{if } (q - \varepsilon 1)_3 \neq 3 \text{ and } q + \varepsilon \neq 2^k; \\ \{p, 3, r_{\nu_{\varepsilon}^{-1}(3)}\}, & \text{if } (q - \varepsilon 1)_3 = 3 \text{ and } q + \varepsilon = 2^k; \\ \{p, r_{\nu_{\varepsilon}^{-1}(3)}\}, & \text{if } (q - \varepsilon 1)_3 \neq 3 \text{ and } q + \varepsilon = 2^k. \end{cases}$$

Note that we avoid to use the functions $e(r,q)$ and ν_{ε} in the tables in Section 8.

Let $G = A_4(2), A_5(2)$ or ${}^2A_3(2)$. Since there are no primitive prime divisors of $2^6 - 1$, by Proposition 3.1 we have that $\rho(2, A_5(2)) = \{2, 31\}$, $\rho(2, A_6(2)) = \{2, 127\}$, and $\rho(2, {}^2A_3(2)) = \{2, 5\}$.

In all other cases by Zsigmondy Theorem there exist primitive prime divisors $r_{\nu_{\varepsilon}^{-1}(n-1)}$ and $r_{\nu_{\varepsilon}^{-1}(n)}$. By Lemma 1.2 we have $r_{\nu_{\varepsilon}^{-1}(n-1)}, r_{\nu_{\varepsilon}^{-1}(n)} \in \pi(G)$. By Propositions 2.1 and 2.2 they are non-adjacent in $GK(G)$. Now Proposition 3.1 yields that for $G = A^{\varepsilon}_{n-1}(q)$ we have $\rho(p,G) = \{p, r_{\nu_{\varepsilon}^{-1}(n-1)}, r_{\nu_{\varepsilon}^{-1}(n)}\}$.

(2) $G = C_n(q)$ or $B_n(q)$.

In view of Proposition 2.3, Proposition 3.1, and Proposition 4.3, we have that the prime graphs of $C_n(q)$ and $B_n(q)$ coincide. So we consider these groups together and, for brevity, use the symbol $C_n(q)$ in both cases.

Let $G = C_3(2)$. Since there are no primes r with $e(r, 2) = 6$, only 7 as the primitive prime divisor of $2^3 - 1$ is not adjacent to 2.

Let $G = C_n(q)$, $n \geqslant 2$ and $(n, q) \neq (3, 2)$. If n is even, by Proposition 3.1 only primitive prime divisors of $q^n + 1$ are non-adjacent to p. Thus, in this case $\rho(G) = \{p, r_{2n}\}$. If n is odd, the Propositions 2.3 and 3.1 yield $\rho(G) = \{p, r_n, r_{2n}\}$.

(3) $G = D_n^\varepsilon(q)$.

With respect to well-known isomorphisms of groups of small Lie rank, we may suppose that $n \geqslant 4$. Let $n = 4$ and $q = 2$. Since there are no primitive prime divisors of $2^6 - 1$ we have that only 7 as a primitive prime divisor of $2^3 - 1$ is non-adjacent to 2 in case $G = D_4(2)$ and only 7 and 17 as primitive divisors of $2^3 - 1$ and $2^8 - 1$ are non-adjacent to 2 in case $G = {}^2D_4(2)$. All others possibilities could be easily described in the direct accordance to Proposition 2.4 and Proposition 3.1. The results of such consideration one can see in Table 4.

(4) The result for exceptional groups distinct from Suzuki and Ree groups can be obtained by using Proposition 2.5, Proposition 3.2 and the Zsigmondy Theorem.

(5) G is a finite simple Suzuki or Ree group.

Let $G = {}^2B_2(2^{2m+1})$, and s_i be a prime divisor of $m_i(B, n)$, where $i = 1, 2, 3$ (see Lemma 1.5). By Proposition 2.6 primes s_i and s_j are adjacent if and only if $i = j$. On the other hand, every s_i, where $i = 1, 2, 3$, is non-adjacent to $p = 2$. Thus, $\rho(p, G) = \{p, s_1, s_2, s_3\}$ and $t(p, G) = 4$ in this case. The same arguments one can apply to Ree groups and prime divisors of $m_i(G, n)$ and $m_i(F, n)$. □

In general, a primitive prime divisor r_m of $q^m - 1$ could be chosen by several ways. Thus, the set $\rho(p, G)$ could be not uniquely determined for a finite simple group G of Lie type. However, it turns out that the values of $e(r, q)$ for all primitive prime divisors r from $\rho(p, G)$ are invariants for a given group G of Lie type.

Proposition 6.4. *Let G be a finite simple group of Lie type over a field of characteristic p and order q. Assume that G is not isomorphic to ${}^2B_2(q)$, ${}^2G_2(q)$, ${}^2F_4(2)'$, and ${}^2F_4(q)$. Let $\rho(p, G) = \{p, s_1, s_2, \ldots, s_m\}$ be an independence set in $GK(G)$ containing p with maximal number of vertices and $k_i = e(s_i, q)$. Then the set $\{k_1, \ldots, k_m\}$ is uniquely determined.*

Proof. It follows from the results of Sections 2-4. □

If we change primitive prime divisors on divisors of numbers defined in Lemma 1.5, we obtain a similar statement for Suzuki and Ree groups.

Proposition 6.5. *Let G be a finite simple Suzuki or Ree group over a field of characteristic p. Let $\rho(p, G) = \{p, s_1, \ldots, s_k\}$.*

1. *If $G = {}^2B_2(2^{2n+1})$, then, up to reodering, s_i divides $m_i(B, n)$. In particular, $k = 3$.*
2. *If $G = {}^2G_2(3^{2n+1})$, then, all of s_i are odd and, up to reodering, s_i divides $m_i(G, n)$. In particular, $k = 4$.*
3. *If $G = {}^2F_4(2^{2n+1})$, then, up to reodering, s_i divides $m_{i+2}(F, n)$ and $s_1 \neq 3$. In particular, $k = 3$.*

Numbers $m_i(B,n)$, $m_i(G,n)$, and $m_i(F,n)$ are defined in Lemma 1.5.

Now we determine $t(2,G)$. Obviously, for a group of Lie type over a field of even characteristic p-independence number and 2-independence number coincide. Thus, we may assume that G is defined over the field of odd characteristic.

Proposition 6.6. *Let G be a finite simple group of Lie type over the field of odd characteristic p and order q. Then $\rho(2,G)$ and $t(2,G)$ are listed in Table* 6 *for classical groups and in Table* 7 *for exceptional groups.*

Proof. (1) $G = A^{\varepsilon}_{n-1}(q)$.

Let $G = A_1(q)$ and $q > 3$. Since q is odd, every prime divisor $r \neq p$ of $|G|$ divides $(q-1)/2$ or $(q+1)/2$. If $e(2,q) = 1$, then 2 is non-adjacent to some prime divisor r_2 of $(q+1)/2$ and $\tau(G) = \{2, p, r_2\}$. If $e(2,q) = 2$, then 2 is non-adjacent to some divisor r_1 of $(q-1)/2$ and $\tau(G) = \{2, p, r_1\}$.

Let $G = A^{\varepsilon}_{n-1}(q)$ and $n \geqslant 3$. If $(q-\varepsilon 1)_2 < n_2$, then by Propositions 4.1 and 4.2 only primitive prime divisor $r_{\nu_{\varepsilon}^{-1}(n-1)}$ is non-adjacent to 2. Note that in this case $n_2 \geqslant 4$, since for $n_2 \leqslant 2$ the inequality $(q-\varepsilon 1)_2 < n_2$ is impossible. By Zsigmondy Theorem primitive prime divisor $r_{\nu_{\varepsilon}^{-1}(n-1)}$ always exists. Thus, $\rho(2,G) = \{2, r_{\nu_{\varepsilon}^{-1}(n-1)}\}$.

If $(q-\varepsilon 1)_2 > n_2$ or $(q-\varepsilon 1)_2 = n_2 = 2$, then by Propositions 4.1 and 4.2 every primitive prime divisor $r_{\nu_{\varepsilon}^{-1}(n)}$ is non-adjacent to 2. Therefore, $\rho(2,G) = \{2, r_{\nu_{\varepsilon}^{-1}(n)}\}$.

At last, let $(q-\varepsilon 1)_2 = n_2 > 2$. By Propositions 4.1 and 4.2 only primitive prime divisors $r_{\nu_{\varepsilon}^{-1}(n-1)}$ and $r_{\nu_{\varepsilon}^{-1}(n)}$ are non-adjacent to 2. On the other hand, by Propositions 2.1 and 2.2, primes $r_{\nu_{\varepsilon}^{-1}(n-1)}$ and $r_{\nu_{\varepsilon}^{-1}(n)}$ are non-adjacent. Thus, $\rho(2,G) = \{2, r_{\nu_{\varepsilon}^{-1}(n-1)}, r_{\nu_{\varepsilon}^{-1}(n)}\}$.

(2) $G = C_n(q)$ or $B_n(q)$.

The results from the table are directly following from Proposition 4.3.

(3) $G = D^{\varepsilon}_n(q)$.

The results again are the direct corollary of Proposition 4.4. Note that the equality $t(2,G) = 2$ is true for most groups of type D_n over fields of odd order. Exceptions are following: n is odd and $q \equiv 5 \pmod 8$ for $G = D_n(q)$, and $q \equiv 3 \pmod 8$ for $G = {}^2D_n(q)$.

(4) Now to complete the proof of the proposition one can use Proposition 4.5 and Lemma 1.5 for Suzuki and Ree groups, and the Zsigmondy Theorem for other exceptional groups. □

Now we consider the uniqueness of $\rho(2,G)$. The situation here is very similar to the situation with $\rho(p,G)$.

Proposition 6.7. *Let G be a finite simple group of Lie type over a field of odd characteristic p and order q. Assume that G is not isomorphic to $A_1(q)$, and ${}^2G_2(q)$. Let $\rho(2,G) = \{2, s_1, s_2, \ldots, s_m\}$ be an independence set in $GK(G)$ containing* 2 *with maximal number of vertices and $k_i = e(s_i,q)$. Then the set $\{k_1, \ldots, k_m\}$ is uniquely determined.*

Proof. It follows from the results of Sections 2–4. □

Proposition 6.8. *Let G be either $A_1(q)$ with q odd, or ${}^2G_2(3^{2n+1})$ and let $\rho(2,G) = \{2, s_1, s_2\}$.*

1. *If $G = A_1(q)$, q odd, then, up to renumbering, $s_1 = p$ and $e(s_2,q) = 3 - e(2,q)$.*

2. *If* $G = {}^2G_2(3^{2n+1})$, *then, up to renumbering,* $s_i = m_{i+2}(G,n)$.

Numbers $m_i(G,n)$ *are defined in Lemma* 1.5.

Proof. It follows from the results of Sections 2–4. □

Our last task is to determine for every finite simple group G of Lie type some independent set $\rho(G)$ in $GK(G)$ with maximal number of vertices.

Proposition 6.9. *Let* G *be a finite simple group of Lie type over the field of characteristic* p *and order* q. *Then* $\rho(G)$ *and* $t(G)$ *are listed in Table* 8 *for classical groups and in Table* 9 *for exceptional groups.*

Proof. (1) $G = A^{\varepsilon}_{n-1}(q)$.

If $G = A_1(q)$ or $G = A^{\varepsilon}_2(q)$, then arguing as in proof of Proposition 6.3 we obtain $\rho(G) = \rho(p,G)$.

Suppose $n = 4$. In view of Proposition 6.3 and Table 4 we have that $t(p,G) = 3$ in all cases, except ${}^2A_3(2)$. By Proposition 6.6 and Table 6 we obtain that $t(2,G) \leqslant 3$. By Propositions 4.1 and 4.2 it follows that $t(r_{\nu_\varepsilon^{-1}(1)},G) \leqslant 3$. Furthermore, there exist at most three other primitive divisors $r_{\nu_\varepsilon^{-1}(2)}$, $r_{\nu_\varepsilon^{-1}(3)}$, and $r_{\nu_\varepsilon^{-1}(4)} = r_4$. So $t(G) = t(p,G) = 3$ except the case: $t({}^2A_3(2)) = t(2, {}^2A_3(2)) = 2$.

Let $G = A^{\varepsilon}_{n-1}(q)$, $n \geqslant 5$ and $q \neq 2$. By Zsigmondy Theorem there exist primitive prime divisors $r_{\nu_\varepsilon^{-1}(k)}$ for every $k > 2$. Denote by m the number $[n/2]$, i.e., the integral part of $n/2$. By Propositions 2.1 and 2.2 the set

$$\rho = \{r_{\nu_\varepsilon^{-1}(m+1)}, r_{\nu_\varepsilon^{-1}(m+2)}, \ldots, r_{\nu_\varepsilon^{-1}(n)}\}$$

is an independent set in $GK(G)$. On the other hand, by Propositions 2.1, 2.2, 3.1, 4.1 and 4.2, every two prime divisors from $\pi\left(q\prod_{i=1}^{m}(q^i - (\varepsilon 1)^i)\right)$ are adjacent in $GK(G)$. Furthermore, each of them is adjacent to at least one number from ρ. Since every prime $s \in \pi(G) \setminus \left(\rho \cup \pi\left(q\prod_{i=1}^{m}(q^i - (\varepsilon 1)^i)\right)\right)$ is of the form r_i for some $i > m$, it follows that every independent set in $GK(G)$ with prime divisor from $\pi\left(q\prod_{i=1}^{m}(q^i - 1)\right)$ contains at most $|\rho|$ vertices. Thus, $\rho(G) = \rho$ and $t(G) = |\rho| = [(n+1)/2]$.

Let $q = 2$. Since $\nu_+^{-1}(6) = \nu_-^{-1}(3) = 6$, the results of previous paragraph hold true for $A_{n-1}(2)$ with $n \geqslant 12$ and ${}^2A_{n-1}(2)$ with $n \geqslant 6$. If $n = 5$ and $\varepsilon = -$, then for ${}^2A_4(2)$ one can check that $\rho(G) = \rho(2,G) = \{2,5,11\}$ and $t(G) = 3$. So we can suppose that $G = A_{n-1}(2)$. If $n = 5,6$ then $\rho(G) = \{r_3, r_4, r_5\} = \{5,7,31\}$ and $t(G) = 3$. If $7 \leqslant n \leqslant 11$ then we need to eliminate divisors of $2^6 - 1$ from $\rho(G)$ and at rest argue like in previous paragraph. In this case $\rho(G) = \{r_i \mid i \neq 6, [n/2] < i \leqslant n\}$ and $t(G) = [(n-1)/2]$.

(2) $G = C_n(q)$ or $B_n(q)$. Recall that $GK(C_n(q)) = GK(B_n(q))$ and we consider these groups together.

$G = C_2(q)$, $q > 2$. Since every two prime divisors of $q(q^2-1)$ are adjacent, we obtain $\rho(G) = \rho(p,G) = \{p, r_4\}$.

Let $n \geqslant 3$ if $q > 2$, and $n \geqslant 7$ if $q = 2$. Define the set ρ as follows:

$$\rho = \{r_{2i} \mid [n+1/2] < i \leqslant n\} \cup \{r_i \mid [n/2] < i \leqslant n, i \equiv 1 (\mathrm{mod}\ 2)\}.$$

Using results of Sections 1–4 and arguments as in proof for groups $A^{\varepsilon}_{n-1}(q)$ we obtain that $\rho(G) = \rho$ and as easy to verify $t(G) = [(3n+5)/4]$.

If $q = 2$ and $3 \leqslant n \leqslant 5$, all arguments stand the same but we have to eliminate the divisors of type r_6 from ρ. Thus, in this case $t(G) = [(3n+1)/4]$. At last, if $(n, q) = (6, 2)$ we have to eliminate a divisor of type r_6 from ρ, but instead it we can add a primitive prime divisor 7 of $2^3 - 1$. Thus, $t(G) = [(3n + 5)/4]$ as in the common situation.

(3) $G = D_n^{\varepsilon}(q)$.

We argue like in two previous parts of proof. Using Proposition 2.4 we obtain a set $\rho(G)$ in common situation and then consider some exceptions arising with respect to exceptions in the Zsigmondy Theorem.

(4) We obtain results for exceptional groups by using Propositions 2.5 and 2.6, Tables 5 and 6, and the Zsigmondy Theorem and Lemma 1.5. ☐

7 Applications

Here we apply our results in a spirit of Section 5. First of all our investigations show that the condition $t(2, G) > 1$ is realized for an extremely wide class of finite simple groups.

Theorem 7.1. *Let G be a finite nonabelian simple group with $t(2, G) = 1$, then G is an alternating group Alt_n with $\tau(2, n) = \{s \mid s \text{ is a prime}, n - 3 \leqslant s \leqslant n\} = \varnothing$.*

Proof. See Propositions 6.1–6.9 and corresponding tables in Section 8. ☐

Thus, we may apply results of Theorems 5.2 and 5.3 as follows.

Corollary 7.2. *Let L be a finite nonabelian simple group distinct from $A_2(3)$, ${}^2A_3(3)$, $C_2(3)$, Alt_{10} and Alt_n with $\tau(2, n) = \varnothing$. Let G be a finite group with $\omega(G) = \omega(L)$. Then there exists a finite nonabelian simple group S such that $S \leq \overline{G} = G/K \leq \mathrm{Aut}(S)$ for maximal normal soluble subgroup K of G. Furthermore, $t(S) \geqslant t(G) - 1$ and one of the following statements holds*:

(1) *$S \simeq Alt_7$ or $A_1(q)$ for some odd q and $t(S) = t(2, S) = 3$.*

(2) *For every prime $r \in \pi(G)$ non-adjacent in $GK(G)$ to 2 the Sylow r-subgroup of G is isomorphic to the Sylow r-subgroup of S. In particular, $t(2, S) \geqslant t(2, G)$.*

Proof. It is direct corollary of Theorem 5.3 and Theorem 7.1. ☐

If G is one of the groups $A_2(3)$, ${}^2A_3(3)$, $C_2(3)$, Alt_{10}, then $t(2, G) \geqslant 2$ and we also have

Corollary 7.3. *Let L be a finite simple group distinct from Alt_n with $\tau(2, n) = \varnothing$. Let G be a finite group with $\omega(G) = \omega(L)$. Then G has at most one nonabelian composition factor.*

For an arbitrary subset ω of the set $\mathbb{N}$ of natural numbers denote by $h(\omega)$ the number of pairwise nonisomorphic finite groups G such that $\omega(G) = \omega$. We say that for a finite group G *the recognition problem is solved* if we know the value of $h(\omega(G))$ (for brevity, $h(G)$). In particular, the group G is said to be *recognizable by spectrum* (briefly, recognizable) if $h(G) = 1$. In last twenty years the recognition problem is solved for many finite nonabelian simple and almost simple groups (for the review of results in this field see [3]).

But most of those groups have the disconnected prime graph, since the Gruenberg — Kegel Theorem and classification of finite simple groups with disconnected prime graph obtained by Williams and Kondrat'ev were an important part of proof. The main theorem of [4] (Theorem 5.2 here) and the results of the present work give a possibility to deal with a groups whose prime graph is connected. A finite nonabelian simple group L is said to be *quasirecognizable by spectrum* if every finite group with the same spectrum has exactly one nonabelian composition factor S and S is isomorphic to L. So the investigations of quasirecognizability is an important step in the determination, whether the given group is recognizable by spectrum. In [4] the first author gave a sketch of a proof for the following statement.

Theorem 7.4. ([4, Proposition 5]) *Let $L = {}^2D_n(q)$, $q = 2^k$, k, n are natural numbers, n is even and $n \geqslant 16$. Then L is quasirecognizable by spectrum.*

Actually, this result was proven modulo the results of Section 6 of the present article. Thus, now it is completely proven.

Now we would like to emphasize one statement that we have mentioned before.

Proposition 7.5. *Let $G = B_n(q)$ and $H = C_n(q)$. Then the prime graphs $GK(G)$ and $GK(H)$ coincide.*

Proof. It follows from the results of Sections 2–4. □

At last, we mention one recent result on prime graphs of finite groups. Recall that a set of vertices of a graph is said to be a clique if all vertices from this set are pairwise adjacent. In [18, Theorem 1] Lucido and Moghaddamfar describe all finite nonabelian simple groups whose prime graph connected components are cliques. We check their list of such groups using results of the present article. Unfortunately, there are some mistakes in the list. As a matter of fact, for groups $G = A_2^{\varepsilon}(q)$, where $q = 2^k - \varepsilon 1$ and $(q - \varepsilon 1)_3 = 3$, we have $3, p \in \pi_1(G)$ and $3p \notin \omega(G)$. So the component $\pi_1(G)$ is not a clique for those groups contrary to the statement of Theorem 1 in [18]. Below in Corollary 7.6 we give a revised list of such groups.

Corollary 7.6. *Let G be a finite nonabelian simple group and all connected components of its prime graph $GK(G)$ are cliques. Then G is one of the following groups*:

1. *Sporadic groups M_{11}, M_{22}, J_1, J_2, J_3, HiS.*
2. *Alternating groups Alt_n, where $n = 5, 6, 7, 9, 12, 13$.*
3. *Groups of Lie type $A_1(q)$, where $q > 3$; $A_2(4)$; $A_2(q)$, where $(q-1)_3 \neq 3$, $q+1 = 2^k$; ${}^2A_3(3)$; ${}^2A_5(2)$; ${}^2A_2(q)$, where $(q+1)_3 \neq 3$, $q-1 = 2^k$; $C_3(2)$, $C_2(q)$, where $q > 2$; $D_4(2)$; ${}^3D_4(2)$; ${}^2B_2(q)$, where $q = 2^{2k+1}$; $G_2(q)$, where $q = 3^k$.*

Proof. All connected component of prime graph $GK(G)$ of a finite group G are cliques if and only if the number $s(G)$ of the components is equal to the independence number $t(G)$ of $GK(G)$. For every finite nonabelian simple group the values of $s(G)$ and $t(G)$ are now known. Thus, using Tables 2a–2c in [3] for the values of $s(G)$ and Tables 2–9 in Section 8 of the present article for $t(G)$, we obtain the result of the corollary. □

8 Resulting Tables

In the tables below n, k are assumed to be naturals. By $[x]$ we denote the integral part of x. For a finite group G we denote by $\rho(G)$ (by $\rho(r, G)$) some independence set in $GK(G)$ (containing r) with maximal number of vertices and put $t(G) = |\rho(G)|$, $t(r, G) = |\rho(r, G)|$. In Table 3 by $\tau(n)$ we denote the set $\{s \mid s \text{ is a prime}, n/2 < s \leqslant n\}$ and by $\tau(2, n)$ we denote the set $\{s \mid s \text{ is a prime}, n-3 \leqslant s \leqslant n\}$. We denote s'_n to be the largest prime which does not exceed $n/2$ and s''_n to be the smallest prime greater than $n/2$. In Tables 4–9 we assume G to be a finite nonabelian simple group of Lie type over a field of characteristic p and order q. By r_m we define the primitive prime divisor of $q^m - 1$. If p is odd then we say that 2 is a primitive prime divisor of $q - 1$ if $q \equiv 1 (\text{mod } 4)$ and that 2 is a primitive prime divisor of $q^2 - 1$ if $q \equiv -1 (\text{mod } 4)$.

Table 2. Sporadic groups

G	$t(G)$	$\rho(G)$	$t(2,G)$	$\rho(2,G)$
M_{11}	3	$\{3,5,11\}$	3	$\{2,5,11\}$
M_{12}	3	$\{3,5,11\}$	2	$\{2,11\}$
M_{22}	4	$\{3,5,7,11\}$	4	$\{2,5,7,11\}$
M_{23}	4	$\{3,7,11,23\}$	4	$\{2,5,11,23\}$
M_{24}	4	$\{5,7,11,23\}$	3	$\{2,11,23\}$
J_1	4	$\{5,7,11,19\}$	4	$\{2,7,11,19\}$
J_2	2	$\{5,7\}$	2	$\{2,7\}$
J_3	3	$\{5,17,19\}$	3	$\{2,17,19\}$
J_4	7	$\{7,11,23,29,31,37,43\}$	6	$\{2,23,29,31,37,43\}$
Ru	4	$\{5,7,13,29\}$	2	$\{2,29\}$
He	3	$\{5,7,17\}$	2	$\{2,17\}$
McL	3	$\{5,7,11\}$	2	$\{2,11\}$
HN	3	$\{7,11,19\}$	2	$\{2,19\}$
HiS	3	$\{5,7,11\}$	3	$\{2,7,11\}$
Suz	4	$\{5,7,11,13\}$	3	$\{2,11,13\}$
Co_1	4	$\{7,11,13,23\}$	2	$\{2,23\}$
Co_2	4	$\{5,7,11,23\}$	3	$\{2,11,23\}$
Co_3	4	$\{5,7,11,23\}$	2	$\{2,23\}$
Fi_{22}	4	$\{5,7,11,13\}$	2	$\{2,13\}$
Fi_{23}	5	$\{7,11,13,17,23\}$	3	$\{2,17,23\}$
Fi'_{24}	6	$\{7,11,13,17,23,29\}$	4	$\{2,17,23,29\}$
O'N	5	$\{5,7,11,19,31\}$	4	$\{2,11,19,31\}$
LyS	6	$\{5,7,11,31,37,67\}$	4	$\{2,31,37,67\}$
F_1	11	$\{11,13,17,19,23,29,31,41,47,59,71\}$	5	$\{2,29,41,59,71\}$
F_2	8	$\{7,11,13,17,19,23,31,47\}$	3	$\{2,31,47\}$
F_3	5	$\{5,7,13,19,31\}$	4	$\{2,13,19,31\}$

Table 3. Simple alternating groups

G	Conditions	$t(G)$	$\rho(G)$	$t(2,G)$	$\rho(2,G)$
Alt_n	$n=5,6$	3	$\{2,3,5\}$	3	$\{2,3,5\}$
	$n=8$	3	$\{2,5,7\}$	3	$\{2,5,7\}$
	$n\geqslant 7$, $s'_n+s''_n>n$	$\lvert\tau(n)\rvert+1$	$\tau(n)\cup\{s'_n\}$	$\lvert\tau(2,n)\rvert+1$	$\tau(2,n)\cup\{2\}$
	$n\geqslant 9$, $s'_n+s''_n\leqslant n$	$\lvert\tau(n)\rvert$	$\tau(n)$	$\lvert\tau(2,n)\rvert+1$	$\tau(2,n)\cup\{2\}$

Table 4. p-independence numbers for finite simple classical groups

G	Conditions	$t(p,G)$	$\rho(p,G)$
$A_{n-1}(q)$	$n=2$, $q>3$	3	$\{p,r_1,r_2\}$
	$n=3$, $(q-1)_3=3$ and $q+1\neq 2^k$	4	$\{p,3,r_2,r_3\}$
	$n=3$, $(q-1)_3\neq 3$ and $q+1\neq 2^k$	3	$\{p,r_2,r_3\}$
	$n=3$, $(q-1)_3=3$ and $q+1=2^k$	3	$\{p,3,r_3\}$
	$n=3$, $(q-1)_3\neq 3$ and $q+1=2^k$	2	$\{p,r_3\}$
	$n=6$, $q=2$	2	$\{2,31\}$
	$n=7$, $q=2$	2	$\{2,127\}$
	$n>3$ $(n,q)\neq(6,2),(7,2)$	3	$\{p,r_{n-1},r_n\}$
${}^2A_{n-1}(q)$	$n=3$, $q\neq 2$, $(q+1)_3=3$ and $q-1\neq 2^k$	4	$\{p,3,r_1,r_6\}$
	$n=3$, $(q+1)_3\neq 3$ and $q-1\neq 2^k$	3	$\{p,r_1,r_6\}$
	$n=3$, $(q+1)_3=3$ and $q-1=2^k$	3	$\{p,3,r_6\}$
	$n=3$, $(q+1)_3\neq 3$ and $q-1=2^k$	2	$\{p,r_6\}$
	$n=4$, $q=2$	2	$\{2,5\}$
	$n\equiv 0(\mathrm{mod}\ 4)$, $(n,q)\neq(4,2)$	3	$\{p,r_{2n-2},r_n\}$
	$n\equiv 1(\mathrm{mod}\ 4)$	3	$\{p,r_{n-1},r_{2n}\}$
	$n\equiv 2(\mathrm{mod}\ 4)$, $n\neq 2$	3	$\{p,r_{2n-2},r_{n/2}\}$
	$n\equiv 3(\mathrm{mod}\ 4)$, $n\neq 3$	3	$\{p,r_{(n-1)/2},r_{2n}\}$
$B_n(q)$ or $C_n(q)$	$n=3$, $q=2$	2	$\{2,7\}$
	n is even	2	$\{p,r_{2n}\}$
	$n>1$ is odd, $(n,q)\neq(3,2)$	3	$\{p,r_n,r_{2n}\}$
$D_n(q)$	$n=4$, $q=2$	2	$\{2,7\}$
	$n\equiv 0(\mathrm{mod}\ 2)$, $n\geqslant 4$, $(n,q)\neq(4,2)$	3	$\{p,r_{n-1},r_{2n-2}\}$
	$n\equiv 1(\mathrm{mod}\ 1)$, $n>4$	3	$\{p,r_n,r_{2n-2}\}$
${}^2D_n(q)$	$n=4$, $q=2$	3	$\{2,7,17\}$
	$n\equiv 0(\mathrm{mod}\ 2)$, $n\geqslant 4$, $(n,q)\neq(4,2)$	4	$\{p,r_{n-1},r_{2n-2},r_{2n}\}$
	$n\equiv 1(\mathrm{mod}\ 2)$, $n>4$	3	$\{p,r_{2n-2},r_{2n}\}$

Table 5. p-independence numbers for finite simple exceptional groups of Lie type

G	Conditions	$t(p,G)$	$\rho(p,G)$
$G_2(q)$	$q>2$	3	$\{p, r_3, r_6\}$
$F_4(q)$	none	3	$\{p, r_8, r_{12}\}$
$E_6(q)$	none	4	$\{p, r_8, r_9, r_{12}\}$
${}^2E_6(q)$	none	4	$\{p, r_8, r_{12}, r_{18}\}$
$E_7(q)$	none	5	$\{p, r_7, r_9, r_{14}, r_{18}\}$
$E_8(q)$	none	5	$\{p, r_{15}, r_{20}, r_{24}, r_{30}\}$
${}^3D_4(q)$	none	2	$\{p, r_{12}\}$
${}^2B_2(2^{2n+1})$	$n \geqslant 1$	4	$\{2, s_1, s_2, s_3\}$, where s_1 divides $2^{2n+1}-1$, s_2 divides $2^{2n+1}-2^{n+1}+1$ s_3 divides $2^{2n+1}+2^{n+1}+1$
${}^2G_2(3^{2n+1})$	$n \geqslant 1$	5	$\{3, s_1, s_2, s_3, s_4\}$, where $s_1 \neq 2$ divides $3^{2n+1}-1$ $s_2 \neq 2$ divides $3^{2n+1}+1$ s_3 divides $3^{2n+1}-3^{n+1}+1$ s_4 divides $3^{2n+1}+3^{n+1}+1$
${}^2F_4(2^{2n+1})$	$n \geqslant 1$	4	$\{2, s_1, s_2, s_3\}$, where $s_1 \neq 3$ and divides $2^{4n+2}-2^{2n+1}+1$ s_2 divides $2^{4n+2}+2^{3n+2}+2^{2n+1}+2^{n+1}+1$ s_3 divides $2^{4n+2}-2^{3n+2}+2^{2n+1}-2^{n+1}+1$
${}^2F_4(2)'$	none	2	$\{2, 13\}$

Table 6. 2-independence numbers for finite simple classical groups of characteristic $p \neq 2$

G	Conditions	$t(2,G)$	$\rho(2,G)$
$A_{n-1}(q)$	$n=2$, $q \equiv 1 (\mathrm{mod}\ 4)$	3	$\{2, r_2, p\}$
	$n=2$, $q \equiv 3 (\mathrm{mod}\ 4)$, $q \neq 3$	3	$\{2, r_1, p\}$
	$n \geqslant 3$ and $n_2 < (q-1)_2$	2	$\{2, r_n\}$
	$n \geqslant 3$ and either $n_2 > (q-1)_2$, or $n_2 = (q-1)_2 = 2$	2	$\{2, r_{n-1}\}$
	$2 < n_2 = (q-1)_2$	3	$\{2, r_{n-1}, r_n\}$
${}^2A_{n-1}(q)$	$n_2 > (q+1)_2$	2	$\{2, r_{2n-2}\}$
	$n_2 = 1$	2	$\{2, r_{2n}\}$
	$2 < n_2 < (q+1)_2$	2	$\{2, r_n\}$
	$n \geqslant 3$, $2 = n_2 \leqslant (q+1)_2$	2	$\{2, r_{n/2}\}$
	$2 < n_2 = (q+1)_2$	3	$\{2, r_{2n-2}, r_n\}$
$B_n(q)$ or $C_n(q)$	$n > 1$ is odd and $(q-1)_2 = 2$	2	$\{2, r_n\}$
	n is even or $(q-1)_2 > 2$	2	$\{2, r_{2n}\}$
$D_n(q)$	$n \equiv 0 (\mathrm{mod}\ 2)$, $n \geqslant 4$, $q \equiv 3 (\mathrm{mod}\ 4)$	2	$\{2, r_{n-1}\}$
	$n \equiv 0 (\mathrm{mod}\ 2)$, $n \geqslant 4$, $q \equiv 1 (\mathrm{mod}\ 4)$	2	$\{2, r_{2n-2}\}$
	$n \equiv 1 (\mathrm{mod}\ 2)$, $n > 4$, $q \equiv 3 (\mathrm{mod}\ 4)$	2	$\{2, r_n\}$
	$n \equiv 1 (\mathrm{mod}\ 2)$, $n > 4$, $q \equiv 1 (\mathrm{mod}\ 8)$	2	$\{2, r_{2n-2}\}$
	$n \equiv 1 (\mathrm{mod}\ 2)$, $n > 4$, $q \equiv 5 (\mathrm{mod}\ 8)$	3	$\{2, r_n, r_{2n-2}\}$
${}^2D_n(q)$	$n \equiv 0 (\mathrm{mod}\ 2)$, $n \geqslant 4$	2	$\{2, r_{2n}\}$
	$n \equiv 1 (\mathrm{mod}\ 2)$, $n > 4$, $q \equiv 1 (\mathrm{mod}\ 4)$	2	$\{2, r_{2n}\}$
	$n \equiv 1 (\mathrm{mod}\ 2)$, $n > 4$, $q \equiv 7 (\mathrm{mod}\ 8)$	2	$\{2, r_{2n-2}\}$
	$n \equiv 1 (\mathrm{mod}\ 2)$, $n > 4$, $q \equiv 3 (\mathrm{mod}\ 8)$	3	$\{2, r_{2n-2}, r_{2n}\}$

Table 7. 2-independence numbers for finite simple exceptional groups of Lie type of characteristic $p \neq 2$

G	Conditions	$t(2,G)$	$\rho(2,G)$
$G_2(q)$	none	3	$\{2, r_3, r_6\}$
$F_4(q)$	none	2	$\{2, r_{12}\}$
$E_6(q)$	none	3	$\{2, r_9, r_{12}\}$
${}^2E_6(q)$	none	3	$\{2, r_{12}, r_{18}\}$
$E_7(q)$	$q \equiv 1 (\mathrm{mod}\ 4)$	3	$\{2, r_{14}, r_{18}\}$
	$q \equiv 3 (\mathrm{mod}\ 4)$	3	$\{2, r_7, r_9\}$
$E_8(q)$	none	5	$\{2, r_{15}, r_{20}, r_{24}, r_{30}\}$
${}^3D_4(q)$	none	2	$\{2, r_{12}\}$
${}^2G_2(3^{2n+1})$	none	3	$\{2, s_1, s_2\}$, where s_1 divides $3^{2n+1} - 3^{n+1} + 1$, s_2 divides $3^{2n+1} + 3^{n+1} + 1$.

Table 8. Independence numbers for finite simple classical groups

G	Conditions	$t(G)$	$\rho(G)$
$A_{n-1}(q)$	$n=2,\ q>3$	3	$\{p, r_1, r_2\}$
	$n=3,\ (q-1)_3=3$ and $q+1\neq 2^k$	4	$\{p, 3, r_2, r_3\}$
	$n=3,\ (q-1)_3\neq 3$ and $q+1\neq 2^k$	3	$\{p, r_2, r_3\}$
	$n=3,\ (q-1)_3=3$ and $q+1=2^k$	3	$\{p, 3, r_3\}$
	$n=3,\ (q-1)_3\neq 3$ and $q+1=2^k$	2	$\{p, r_3\}$
	$n=4$	3	$\{p, r_{n-1}, r_n\}$
	$n=5,6,\ q=2$	3	$\{5, 7, 31\}$
	$7\leqslant n\leqslant 11,\ q=2$	$\left[\frac{n-1}{2}\right]$	$\{r_i \mid i\neq 6, \left[\frac{n}{2}\right]<i\leqslant n\}$
	$n\geqslant 5$ and $q>2$ or $n\geqslant 12$ and $q=2$	$\left[\frac{n+1}{2}\right]$	$\{r_i \mid \left[\frac{n}{2}\right]<i\leqslant n\}$
${}^2A_{n-1}(q)$	$n=3, q\neq 2, (q+1)_3=3,$ and $q-1\neq 2^k$	4	$\{p, 3, r_1, r_6\}$
	$n=3,\ (q+1)_3\neq 3$ and $q-1\neq 2^k$	3	$\{p, r_1, r_6\}$
	$n=3,\ (q+1)_3=3$ and $q-1=2^k$	3	$\{p, 3, r_6\}$
	$n=3,\ (q+1)_3\neq 3$ and $q-1=2^k$	2	$\{p, r_6\}$
	$n=4,\ q=2$	2	$\{2, 5\}$
	$n=4,\ q>2$	3	$\{p, r_4, r_6\}$
	$n=5,\ q=2$	3	$\{2, 5, 11\}$
	$n\geqslant 5$ and $(n,q)\neq(5,2)$	$\left[\frac{n+1}{2}\right]$	$\{r_{i/2} \mid \left[\frac{n}{2}\right]<i\leqslant n,$ $i\equiv 2(\mathrm{mod}\ 4)\}\cup$ $\cup\{r_{2i} \mid \left[\frac{n}{2}\right]<i\leqslant n,$ $i\equiv 1(\mathrm{mod}\ 2)\}$ $\cup\{r_i \mid \left[\frac{n}{2}\right]<i\leqslant n,$ $i\equiv 0(\mathrm{mod}\ 4)\}$
$B_n(q)$ or $C_n(q)$	$n=2,\ q>2$	2	$\{p, r_4\}$
	$n=3$ and $q=2$	2	$\{5, 7\}$
	$n=4$ and $q=2$	3	$\{5, 7, 17\}$
	$n=5$ and $q=2$	4	$\{7, 11, 17, 31\}$
	$n=6$ and $q=2$	5	$\{7, 11, 13, 17, 31\}$
	$n>2,\ (n,q)\neq(3,2),(4,2),(5,2),(6,2)$	$\left[\frac{3n+5}{4}\right]$	$\{r_{2i} \mid \left[\frac{n+1}{2}\right]\leqslant i\leqslant n\}\cup$ $\cup\{r_i \mid \left[\frac{n}{2}\right]<i\leqslant n,$ $i\equiv 1(\mathrm{mod}\ 2)\}$
$D_n(q)$	$n=4$ and $q=2$	2	$\{5, 7\}$
	$n=5$ and $q=2$	4	$\{5, 7, 17, 31\}$
	$n=6$ and $q=2$	4	$\{7, 11, 17, 31\}$
	$n\geqslant 4,\ (n,q)\neq(4,2),(5,2),(6,2)$	$\left[\frac{3n+1}{4}\right]$	$\{r_{2i} \mid \left[\frac{n+1}{2}\right]\leqslant i<n\}\cup$ $\cup\{r_i \mid \left[\frac{n}{2}\right]<i\leqslant n,$ $i\equiv 1(\mathrm{mod}\ 2)\}$
${}^2D_n(q)$	$n=4$ and $q=2$	3	$\{5, 7, 17\}$
	$n=5$ and $q=2$	3	$\{7, 11, 17\}$
	$n=6$ and $q=2$	5	$\{7, 11, 13, 17, 31\}$
	$n=7$ and $q=2$	5	$\{11, 13, 17, 31, 43\}$
	$n\geqslant 4,\ n\not\equiv 1(\mathrm{mod}\ 4),$ $(n,q)\neq(4,2),(6,2),(7,2)$	$\left[\frac{3n+4}{4}\right]$	$\{r_{2i} \mid \left[\frac{n}{2}\right]\leqslant i\leqslant n\}\cup$ $\cup\{r_i \mid \left[\frac{n}{2}\right]<i\leqslant n,$ $i\equiv 1(\mathrm{mod}\ 2)\}$
	$n>4,\ n\equiv 1(\mathrm{mod}\ 4),\ (n,q)\neq(5,2)$	$\left[\frac{3n+4}{4}\right]$	$\{r_{2i} \mid \left[\frac{n}{2}\right]<i\leqslant n\}\cup$ $\cup\{r_i \mid \left[\frac{n}{2}\right]<i<n,$ $i\equiv 1(\mathrm{mod}\ 2)\}$

Table 9. Independence numbers for finite simple exceptional groups of Lie type

G	Conditions	$t(G)$	$\rho(G)$
$G_2(q)$	$q > 2$	3	$\{p, r_3, r_6\}$
$F_4(q)$	$q = 2$	4	$\{5, 7, 13, 17\}$
	$q > 2$	5	$\{r_3, r_4, r_6, r_8, r_{12}\}$
$E_6(q)$	$q = 2$	5	$\{5, 13, 17, 19, 31\}$
	$q > 2$	6	$\{r_4, r_5, r_6, r_8, r_9, r_{12}\}$
${}^2E_6(q)$		5	$\{r_4, r_8, r_{10}, r_{12}, r_{18}\}$
$E_7(q)$		7	$\{r_7, r_8, r_9, r_{10}, r_{12}, r_{14}, r_{18}\}$
$E_8(q)$		11	$\{r_7, r_8, r_9, r_{10}, r_{12}, r_{14}, r_{15}, r_{18}, r_{20}, r_{24}, r_{30}\}$
${}^3D_4(q)$	$q = 2$	2	$\{2, 13\}$
	$q > 2$	3	$\{r_3, r_6, r_{12}\}$
${}^2B_2(2^{2n+1})$	$n \geqslant 1$	4	$\{2, s_1, s_2, s_3\}$, where s_1 divides $2^{2n+1} - 1$, s_2 divides $2^{2n+1} - 2^{n+1} + 1$ s_3 divides $2^{2n+1} + 2^{n+1} + 1$
${}^2G_2(3^{2n+1})$	$n \geqslant 1$	5	$\{3, s_1, s_2, s_3, s_4\}$, where $s_1 \neq 2$ divides $3^{2n+1} - 1$ $s_2 \neq 2$ divides $3^{2n+1} + 1$ s_3 divides $3^{2n+1} - 3^{n+1} + 1$ s_4 divides $3^{2n+1} + 3^{n+1} + 1$
${}^2F_4(2^{2n+1})$	$n \geqslant 2$,	5	$\{s_1, s_2, s_3, s_4, s_5\}$, where s_1 and divides $2^{2n+1} + 1$, s_2 divides $2^{4n+2} + 1$, $s_3 \neq 3$ and divides $2^{4n+2} - 2^{2n+1} + 1$, s_4 divides $2^{4n+2} - 2^{3n+2} + 2^{2n+1} - 2^{n+1} + 1$, s_5 divides $2^{4n+2} + 2^{3n+2} + 2^{2n+1} + 2^{n+1} + 1$,
${}^2F_4(2)'$	none	3	$\{3, 5, 13\}$
${}^2F_4(8)$	none	4	$\{7, 19, 37, 109\}$

References

[1] J. S. Williams, "Prime graph components of finite groups", J. Algebra, **69**, No. 2 (1981), 487–513.

[2] A. S. Kondrat'ev, "On prime graph components for finite simple groups", Mat. Sb., **180**, No. 6 (1989), 787–797.

[3] V. D. Mazurov, "Characterization of groups by arithmetic properties", Algebra Colloc., **11**, No. 1 (2004), 129–140.

[4] A. V. Vasil'ev, "On connection between the structure of finite group and properties of its prime graph", Sib. Math. J., **46**, No. 3 (2005).

[5] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, Atlas of finite groups, Clarendon Press, Oxford (1985).

[6] The GAP Group, GAP — Groups, Algorithms, and Programming, Version 4.4 (2004); (http://www.gap-system.org).

[7] R. W. Carter, Simple groups of Lie type, John Wiley and Sons, London, 1972.

[8] J. E. Humphreys, Linear algebraic groups, Springer-Verlag, New York, 1972.

[9] *R. Steinberg*, Endomorphisms of algebraic groups, Mem. AMS, **80**, 1968.

[10] A. Borel and J. de Siebental, "Les-sous-groupes fermés de rang maximum des groupes de Lie clos", Comment.Math.Helv., **23**, (1949) 200-221.

[11] E. B. Dynkin, "Semisimple subalgebras of semisimple Lie algebras", Mat.sb., **30**, No 2 (1952), 349–462. (In russian)

[12] R. W. Carter, Finite groups of Lie type, conjugacy classes and complex characters, John Wiley and Sons, 1985.

[13] R. W. Carter, "Centralizers of semisimple elements in the finite classical groups", Proc. London Math. Soc., (3), **42**, No. 1 (1981), 1-41.

[14] R. W. Carter, "Conjugacy classes in the Weyl group", Compositio Mathematica, **25** No. 1 (1972), 1-59.

[15] D. Deriziotis, "Conjugacy classes and centralizers of semisimple elements in finite groups of Lie type", Vorlesungen aus dem Fachbereich Mathematic der Universität Essen, 1984, Heft 11.

[16] K. Zsigmondy, "Zur Theorie der Potenzreste", Monatsh. für Math. und Phys., **3**, (1892), 265-284.

[17] D. Deriziotis, "The centralizers of semisimple elements of the Chevalley groups E_7 and E_8", Tokyo J. Math., **6**, No. 1 (1983), 191-216.

[18] M. S. Lucido, A. R. Moghaddamfar , "Groups with complete prime graph connected components", J. Group Theory, **7**, No. 3 (2004), 373–384.