ON THE EXISTENCE OF CARTER SUBGROUPS

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In the paper we obtain the existence criterion of a Carter subgroup in a finite group in terms of its normal series. An example showing that the criterion cannot be reformulated in terms of composition factors is given.

1 Introduction

Recall that a nilpotent self-normalizing subgroup of a group G is called a *Carter subgroup*. The classical result by Carter [1] states that every finite solvable group contains Carter subgroups and all of them are conjugate. A finite group G is said to satisfy condition **(C)** if, for every its nonabelian composition factor S and for every its nilpotent subgroup N, Carter subgroups (if exist) of $\langle \mathrm{Aut}_N(S), S\rangle$ are conjugate (the definition of $\mathrm{Aut}_N(S)$ is given below). In the paper [2] it is proven that if a finite group satisfies **(C)**, then its Carter subgroups are conjugate. In the recent paper [3, Theorem 10.1] it is proven that in every almost simple group with known simple socle Carter subgroups are conjugate. Thus, modulo the classification of finite simple groups, in every finite group Carter subgroups are conjugate. In the paper by a finite group we almost mean a finite group satisfying **(C)**, thus the results of the paper does not depend on the classification of finite simple groups. There exist finite groups without Carter subgroups, the minimal example is Alt_5. In the paper we give a criterion of existence of Carter subgroups in terms of normal series.

If G is a group, A, B, H are subgroups of G and B is normal in A ($B \trianglelefteq A$), then $N_H(A/B) = N_H(A) \cap N_H(B)$. If $x \in N_H(A/B)$, then x induces an automorphism $Ba \mapsto Bx^{-1}ax$ of A/B. Thus, there is a homomorphism of $N_H(A/B)$ into $\mathrm{Aut}(A/B)$. The image of this homomorphism is denoted by $\mathrm{Aut}_H(A/B)$ while its kernel is denoted by $C_H(A/B)$. In particular, if S is a composition factor of G, then for any $H \leq G$ the group $\mathrm{Aut}_H(S)$ is defined.

Let $G = G_0 \geq G_1 \geq \ldots \geq G_n = \{e\}$ be a chief series of G (recall that G is assumed to satisfy **(C)**). Then $G_i/G_{i+1} = T_{i,1} \times \ldots \times T_{i,k_i}$, where $T_{i,1} \simeq \ldots \simeq T_{i,k_i} \simeq T_i$ and T_i is a simple group. If $i \geqslant 1$, then denote by $\overline{K}_i$ a Carter subgroup of G/G_i (if it exists) and by K_i its complete preimage in G/G_{i+1}. If $i = 0$, then $\overline{K}_0 = \{e\}$ and $K_0 = G/G_1$. We say that a finite group G satisfies condition **(E)**, if, for every i, j, either $\overline{K}_i$ does not exist, or $\mathrm{Aut}_{K_i}(T_{i,j})$ contains a Carter subgroup.

The following lemma shows that the homomorphic image of a Carter subgroup is a Carter subgroup. We shall use this fact substantially.

Lemma 1. [3, Lemma 2.1] *Let G be a finite group, let K be a Carter subgroup of G and assume that N is a normal subgroup of G. Assume that either KN satisfies* **(C)** *(this condition is automatically satisfied if G satisfies* **(C)** *or if N is solvable), or $KN = G$. Then KN/N is a Carter subgroup of G/N.*

2 Criterion

Lemma 2. *Let H be a normal subgroup of a finite group G, $S = (A/H)/(B/H)$ be a composition factor of G/H, and L be a subgroup of G.*

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Then $\mathrm{Aut}_L(A/B) \simeq \mathrm{Aut}_{LH/H}((A/H)/(B/H))$.

Proof. Since $H \leq B$, then $H \leq C_G(A/B)$, hence we may assume that $L = LH$. Moreover. we may assume that $L \leq N_G(A) \cap N_G(B)$ and $G = LA$. Then the action on A/B given by $x : Ba \mapsto Bx^{-1}ax$ coincides with the action on $(A/H)/(B/H)$ given by $xH : BaH \mapsto Bx^{-1}axH$, and the lemma follows. $\square$

Below we shall need to know some additional information about the structure of Carter subgroups in groups of special type. Let A' be a group with a normal subgroup T'. Consider the direct product $A_1 \times \ldots \times A_k$, where $A_1 \simeq \ldots \simeq A_k \simeq A'$ and its normal subgroup $T = T_1 \times \ldots \times T_k$, where $T_1 \simeq \ldots \simeq T_k \simeq T'$. Consider the symmetric group Sym_k, acting on $A_1 \times \ldots \times A_k$ by $A_i^s = A_{i^s}$ for all $s \in S$ and define X to be a semidirect product $(A_1 \times \ldots \times A_k) \leftthreetimes \mathrm{Sym}_k$ (permutation wreath product of A' and Sym_k). Denote by A the direct product $A_1 \times \ldots \times A_k$ and by π_i the projection $\pi_i : A \rightarrow A_i$. In the introduced notations the following lemma holds.

Lemma 3. *Let G be a subgroup of X such that $T \leq G$, $G/(G \cap T)$ is nilpotent and $(G \cap A)^{\pi_i} = A_i$. Assume also that A is solvable. Let K be a Carter subgroup of G.*
Then $(K \cap A)^{\pi_i}$ is a Carter subgroup of A_i.

Proof. Assume that the statement is not true and let G be a counterexample of minimal order with minimal k. Then $S = G/(G \cap A)$ is transitive and primitive. Indeed, if S is not transitive, then $S \leq \mathrm{Sym}_{k_1} \times \mathrm{Sym}_{k-k_1}$, hence $G \leq G_1 \times G_2$. If we denote by $\psi_i : G \rightarrow G_i$ the natural homomorphism, then $G^{\psi_i} = G_i$ satisfies conditions of the lemma and $K^{\psi_i} = K_i$ is a Carter subgroup of G_i. Clearly $(G \cap A)^{\pi_j} = (G_i \cap A^{\psi_i})^{\pi_j}$, where $i = 1$ if $j \in \{1, \ldots, k_1\}$ and $i = 2$ if $j \in \{k_1 + 1, \ldots, k\}$, i. e., the following diagrams are commutative:

$$\begin{tikzcd} G \cap A \arrow[rr, "\pi_j"] \arrow[dr, "\psi_1"'] & & A_j, \\ & G_1 \cap A^{\psi_1} \arrow[ur, "\pi_j"'] & \end{tikzcd} \qquad \begin{tikzcd} G \cap A \arrow[rr, "\pi_j"] \arrow[dr, "\psi_2"'] & & A_j. \\ & G_2 \cap A^{\psi_2} \arrow[ur, "\pi_j"'] & \end{tikzcd}$$

Thus we obtain the statement by induction. If S is transitive, but is not primitive, let

$$\Omega_1 = \{T_1, \ldots, T_m\}, \Omega_2 = \{T_{m+1}, \ldots, T_{2m}\}, \ldots, \Omega_l = \{T_{(l-1)m+1}, \ldots, T_{lm}\}$$

be a system of imprimitivity. Then it contains a nontrivial intransitive normal subgroup

$$F' \leq \underbrace{\mathrm{Sym}_m \times \ldots \times \mathrm{Sym}_m}_{l \text{ times}},$$

where $k = m \cdot l$. Consider the complete preimage F of F' in X. Then $G \cap F \leq F_1 \times \ldots \times F_l$. Denote by $\psi_i : F \rightarrow F_i$ the natural projection, then $(G \cap F)^{\psi_i} = F_i$. Note that all of F_i satisfy conditions of the lemma and, if we define $T_i' = T_{(i-1)m+1} \times \ldots \times T_{im}$, then G satisfies conditions of the lemma with $T' = T_1' \times \ldots \times T_l'$ and $A' = F$. By induction we have that $(K \cap F)^{\psi_i}$ is a Carter subgroup of F_i and, if $j \in \{m \cdot (i-1) + 1, \ldots m \cdot i\}$, then $\big((K \cap F)^{\psi_i} \cap A^{\psi_i}\big)^{\pi_j}$ is a Carter subgroup of A_j. Since $(G \cap A)^{\pi_j} = \big((K \cap F)^{\psi_i} \cap A^{\psi_i}\big)^{\pi_j}$ (for suitable i), we obtain the statement by induction.

Let Y' be a minimal normal subgroup of G contained in T (if Y' is trivial, then T is trivial and we have nothing to prove, since G is nilpotent in this case). Thus Y' is a normal elementary

Abelian p-group. Let $Y_i = (Y')^{\pi_i}$, then $Y = Y_1 \times \ldots \times Y_k$ is a nontrivial normal subgroup of G (Y is a subgroup of G since $T \leq G$). Let $\bar{\pi}_i : (G \cap A) \to A_i/Y_i = \overline{A}_i$ be the projection corresponding to π_i. Denote by $\overline{K} = KY/Y$ the corresponding Carter subgroup of $\overline{G} = G/Y$. Then $\overline{G}$ satisfies conditions of the lemma. By induction, $(\overline{K} \cap \overline{A})^{\bar{\pi}_i}$ is a Carter subgroup of $\overline{A}_i$. Let K_1 be a complete preimage of $\overline{K}$ in G and let Q be a Hall p'-subgroup of K_1. Then $(Q \cap A)^{\pi_i}$ is a Hall p'-subgroup of $(K_1 \cap A)^{\pi_i}$. In view of the proof of [4, Theorem 20.1.4], we obtain that $K = N_{K_1}(Q)$ is a Carter subgroup of G and $(N_{K_1\cap A}(Q \cap A))^{\pi_i}$ is a Carter subgroup of A_i. Thus we need to show that $(N_{K_1\cap A}(Q\cap A))^{\pi_i} = (N_{K_1\cap S}(Q))^{\pi_i}$. By induction, the equality $(N_{\overline{K}\cap\overline{A}}(\overline{A}\cap\overline{Q}))^{\bar{\pi}_i} = (N_{\overline{K}\cap\overline{G}}(\overline{Q}))^{\bar{\pi}_i}$ holds. Thus we need to prove that $(N_Y(Q\cap A))^{\pi_i} = (N_Y(Q))^{\pi_i}$. Note also that $(N_Y(Q \cap A))^{\pi_i} \leq N_{Y_i}((Q \cap A)^{\pi_i})$.

Since S is a transitive and primitive nilpotent subgroup of Sym_k, then $k = r$ is prime and $S = \langle s\rangle$ is cyclic. If $r = p$, then $Q \cap A = Q$ and we have nothing to prove. Otherwise let h be an r-element of K, generating S modulo $K \cap A$. Clearly $Q = (Q \cap A)\langle h\rangle$. Let $t \in Y_i$ be an element of $N_{Y_i}((Q \cap A)^{\pi_i})$. Then $(t \cdot t^h \cdot \ldots \cdot t^{h^{r-1}}) \in N_Y(Q)$ and $t^{\pi_i} = (t \cdot t^h \cdot \ldots \cdot t^{h^{r-1}})^{\pi_i}$, hence $(N_Y(Q \cap A))^{\pi_i} \leq N_{Y_i}((Q \cap A)^{\pi_i}) \leq (N_Y(Q))^{\pi_i} \leq (N_Y(Q \cap A))^{\pi_i}$. □

Theorem 1. *Let G be a finite group. Then G contains a Carter subgroup if and only if G satisfies* **(E)**.

Proof. We prove first the part "only if". Let H be a minimal normal subgroup of G. Then $H = T_1 \times \ldots \times T_k$, where $T_1 \simeq \ldots \simeq T_k \simeq T$ are simple groups.

If H is elementary Abelian (i. e., T is cyclic of prime order), then $\mathrm{Aut}(T)$ is solvable and contains a Carter subgroup. Assume that T is a nonabelian simple group. Clearly K is a Carter subgroup of KH. By [2, Lemma 3] we obtain that $\mathrm{Aut}_{KH}(T_i)$ contains a Carter subgroup for all i.

Now we prove the part "if". Again assume by contradiction that G is a counterexample of minimal order, i. e., that G does not contain a Carter subgroup, but, G satisfies **(E)**. Let H be a minimal normal subgroup of G. Then $H = T_1 \times \ldots \times T_k$, where $T_1 \simeq \ldots \simeq T_k \simeq T$, and T is a finite simple group.

By definition G/H satisfies **(E)**, thus, by induction, there exists a Carter subgroup $\overline{K}$ of $\overline{G} = G/H$. Let K be a complete preimage of $\overline{K}$, then K satisfies **(E)**. If $K \neq G$, then, by induction K contains a Carter subgroup K'. Note that K' is a Carter subgroup of G. Indeed, assume that $x \in N_G(K') \setminus K'$. Since $K'H/H = \overline{K}$ is a Carter subgroup of $\overline{G}$, we have that $x \in K$. But K' is a Carter subgroup of K, thus $x \in K'$. Hence $G = K$, i. e. G/H is nilpotent.

If H is Abelian, then G is solvable, therefore G contains a Carter subgroup. So assume that T is a nonabelian finite simple group. We first show that $C_G(H)$ is trivial. Assume that $C_G(H) = M$ is nontrivial. Since T is a nonabelian simple group, it follows that $M \cap H = \{e\}$, so M is nilpotent. By Lemma 2 we have that G/M satisfy **(E)**. By induction we obtain that G/M contains a Carter subgroup $\overline{K}$. Let K' be a complete preimage of $\overline{K}$ in G. Then K' is solvable, hence contains a Carter subgroup K. Like above we obtain that K is a Carter subgroup of G, a contradiction. Hence $C_G(H) = \{e\}$.

Since H is a minimal normal subgroup of G, we obtain that $\mathrm{Aut}_G(T_1) \simeq \mathrm{Aut}_G(T_2) \simeq \ldots \simeq \mathrm{Aut}_G(T_k)$. Thus there exists a monomorphism

$$\varphi : G \to (\mathrm{Aut}_G(T_1) \times \ldots \times \mathrm{Aut}_G(T_k)) \leftthreetimes \mathrm{Sym}_k = G_1$$

and we identify G with G^φ. Denote by K_i a Carter subgroup of $\mathrm{Aut}_G(T_i)$ and by A the subgroup $\mathrm{Aut}_G(T_1) \times \ldots \times \mathrm{Aut}_G(T_k)$. Since G/H is nilpotent, then $K_iT_i = \mathrm{Aut}_G(T_i)$ and

$G_1 = (K_1T_1 \times \ldots \times K_kT_k) \leftthreetimes \mathrm{Sym}_k$. Let $\pi_i : G \cap A \to (G \cap A)/C_{(G\cap A)}(T_i)$ be the canonical projections. Since $G/(G \cap A)$ is transitive, we obtain that $(G \cap A)^{\pi_i} = K_iT_i$.

Since $\mathrm{Aut}_{G\cap A}(T_i) = K_iT_i$, hence $G \cap A$ satisfies **(E)**. By induction it contains a Carter subgroup M. By [2, Lemma 3] we obtain that M^{π_i} is a Carter subgroup of K_iT_i, therefore we may assume $M^{\pi_i} = K_i$. In particular, if $R = (K_1 \cap T_1) \times \ldots \times (K_k \cap T_k)$, then $M \leq N_G(R)$. In view of [3], Carter subgroups in every finite group are conjugate. Since $(G \cap A)/H$ is nilpotent we obtain that $G \cap A = MH$, hence $G = N_G(M)H$. Moreover $N_G(M) \cap A = M$, so $N_G(M)$ is solvable. Since M normalizes R, $M^{\pi_i} = K_i$, we obtain that $N_G(M)$ normalizes R, hence $N_G(M)R$ is solvable. Therefore it contains a Carter subgroup K. By Lemma 3, $(K \cap A)^{\pi_i}$ is a Carter subgroup of $(N_G(M)R \cap A)^{\pi_i}$ (R plays the role of T from Lemma 3 in this case), so $(K \cap A)^{\pi_i} = K_i$. Assume that $x \in N_G(K) \setminus K$. Since $G/H = N_G(M)H/H = KH/H$ it follows that $x \in H$. Therefore $x^{\pi_i} \in (N_G(K) \cap A)^{\pi_i} \leq N_{T_i}((K \cap A)^{\pi_i}) = K_i$. Since $\bigcap_i \mathrm{Ker}(\pi_i) = \{e\}$, it follows that $x \in R \leq N_G(M)R$. But K is a Carter subgroup of $N_G(M)R$, hence $x \in K$. This contradiction completes the proof. $\square$

3 Example

In this section we construct an example showing, that we can not substitute condition **(E)** by a weaker condition: for every composition factor S of G, $\mathrm{Aut}_G(S)$ contains a Carter subgroup. This example also shows that an extension of a group containing a Carter subgroup by a group containing a Carter subgroup may fail to contain a Carter subgroup.

Consider $L = \Gamma SL_2(3^3) = PSL_2(3^3) \leftthreetimes \langle\varphi\rangle$, where φ is a field automorphism of $PSL_2(3^3)$. Let $X = (L_1 \times L_2) \leftthreetimes \mathrm{Sym}_2$, where $L_1 \simeq L_2 \simeq L$ and if $\sigma = (1, 2) \in \mathrm{Sym}_2 \setminus \{e\}$, $(x, y) \in L_1 \times L_2$, then $\sigma(x, y)\sigma = (y, x)$ (permutation wreath product of L and Sym_2). Denote by $H = PSL_2(3^3) \times PSL_2(3^3)$ the minimal normal subgroup of X and by $M = L_1 \times L_2$. Let $G = (H \leftthreetimes \langle(\varphi, \varphi^{-1})\rangle) \leftthreetimes \mathrm{Sym}_2$ be a subgroup of X. Then the following statements hold:

1. For every composition factor S of G, $\mathrm{Aut}_G(S)$ contains a Carter subgroup.

2. $G \cap M \trianglelefteq G$ contains a Carter subgroup.

3. $G/(G \cap M)$ is nilpotent.

4. G does not contain a Carter subgroup.

1. Clearly we need to verify the statement for nonabelian composition factors only. Every nonabelian composition factor S of G is isomorphic to $PSL_2(3^3)$ and $\mathrm{Aut}_G(S) = L$. In view of [3, Theorem 7.1] (Theorem 3 below) we obtain that L contains a Carter subgroup (coinciding with a Sylow 3-subgroup).

2. Since $(G \cap M)/H$ is nilpotent and from the previous statement we obtain that $G \cap M$ satisfies **(E)**, hence contains a Carter subgroup (it is easy to see that a Sylow 3-subgroup of $G \cap M$ is a Carter subgroup of $G \cap M$).

3. Evident.

4. Assume that K is a Carter subgroup of G. Then KH/H is a Carter subgroup of G/H. But G/H is a nonabelian group of order 6, hence $G/H \simeq \mathrm{Sym}_3$ and KH/H is a Sylow 2-subgroup of G/H. In view of [2, Lemma 3] it follows that $\mathrm{Aut}_K(PSL_2(3^3))$ is a Carter subgroup of $\mathrm{Aut}_{KH}(PSL_2(3^3)) = PSL_2(3^3)$. But $PSL_2(3^3)$ does not contain Carter subgroups in view of [3, Theorem 7.1] (Theorem 3 below).

4 The classification of Carter subgroups

It is proven in [3, Lemma 2.3] that a Carter subgroup of a finite group G contains a Sylow 2-subgroup S of G if and only if $N_G(S) = SC_G(S)$. A finite group G is said to satisfy **(ESyl2)**, if for its Sylow 2-subgroup S the condition $N_G(S) = SC_G(S)$ holds. We give main classification theorems from [3] here for convenience.

Theorem 2. [3, Theorem 6.1] *Let G be a group of Lie type over a field of characteristic p with trivial center and $\overline{G}$, σ are chosen so that $O^{p'}(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma$, and $O^{p'}(\overline{G}_\sigma)$ is isomorphic to either $D_4(q)$, or ${}^3D_4(q^3)$. Assume that τ is a graph automorphism of order 3 of $O^{p'}(G)$ if $O^{p'}(G) \simeq D_4(q)$ and is a field automorphism of order 3 which has the set of stable points isomorphic to $G_2(q)$ if $G \simeq {}^3D_4(q^3)$. Denote by A_1 the subgroup of $\mathrm{Aut}(D_4(q))$ generated by inner-diagonal and field automorphisms, and also by a graph automorphism of order 2. Let $A \leq \mathrm{Aut}(G)$ be such that $A \not\leq A_1$ (if $O^{p'}(G) \simeq D_4(q)$), and K be a Carter subgroup of A. Assume also that $|O^{p'}(G)| \leqslant$ **Cmin**, $G = A \cap \overline{G}_\sigma$ and $A = KG$. Then one of the following statements holds:*

(a) *$G \simeq {}^3D_4(q^3)$, $(|A : G|, 3) = 1$, q is odd and K contains a Sylow 2-subgroup of A;*

(b) *$(|A : G|, 3) > 1$, q is odd, $\tau \in A$ and, up to conjugation by an element of G, the subgroup K contains a Sylow 2-subgroup of $C_A(\tau) \simeq \Gamma G_2(q)$ and $\tau \in K$;*

(c) *$(|A : G|, 3) > 1$, $q = 2^t$, $\tau \in a$ and, up to conjugation by an element of G, the subgroup K contains a Sylow 2-subgroup of $C_G(\tau) \simeq G_2(2^{t_2'})$ and $\tau \in K$;*

(d) *$O^{p'}(G) \simeq D_4(p^{3t})$, p is odd, the quotient A/G is cyclic, $O^{p'}(G) \leftthreetimes \langle\tau\rangle \not\leq A$, $A = G \leftthreetimes \langle\zeta\rangle$, where $g \in \overline{G}_\sigma$, and $\zeta = \tau\varphi^m$ is a graph-field automorphism, and , up to conjugation by an element of G, $K = Q \leftthreetimes \langle\zeta\rangle$, where Q is a Sylow 2-subgroup of $C_G(\zeta_{2'}) \simeq {}^3D_4(p^{3t/|\zeta_{2'}|})$.*

Theorem 3. [3, Theorem 7.1] *Let G be a finite group of Lie type (G is not necessary simple) with trivial center over a field of characteristic p and $\overline{G}$, σ are chosen so that $O^{p'}(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma$. Assume also that $G \not\simeq {}^3D_4(q^3)$. Choose a subgroup A of $\mathrm{Aut}(O^{p'}(\overline{G}_\sigma))$ with $A \cap \overline{G}_\sigma = G$ and assume that A is contained in the subgroup A_1 defined in Theorem 2, if $O^{p'}(G) = D_4(q)$. Let K be a Carter subgroup of A and assume that $A = KG$.*

Then exactly one of the following statements holds:

(a) *$A = \Gamma G$ and either $\Gamma G = \langle {}^2A_2(2^{2t}), \zeta g\rangle$, or $\Gamma G = \widehat{{}^2A_2(2^{2t})} \leftthreetimes \langle\zeta\rangle$, where the order $|\zeta| = t$ is odd, $C_G(\zeta) \simeq \widehat{{}^2A_2(2)}$ ($C_G(\zeta) \simeq \widehat{{}^2A_2(2)}$ if $G = \widehat{{}^2A_2(2^{2t})}$ or t is divisible by 3), the subgroup $K \cap G$ is Abelian and has order $2 \cdot 3^k$, where $3^{k-1} = t_3$;*

(b) *G is defined over $GF(2^t)$, a field automorphism ζ is in A, $|\zeta| = t$, and, up to conjugation in G, the equality $K = Q \leftthreetimes \langle\zeta, \tau\rangle$ holds, where Q is a Sylow 2-subgroup of $G_{\zeta_{2'}}$ and τ is a graph automorphism of order $\leqslant 2$ of $O^{p'}(G)$ contained in A;*

(c) *$G \simeq \mathbf{P}SL_2(3^t)$, a field automorphism ζ is in A, $|\zeta| = t$ is odd, and, up to conjugation in G, the equality $K = Q \leftthreetimes \langle\zeta\rangle$ holds, where Q is a Sylow 3-subgroup of $G_{\zeta_{3'}}$;*

(d) *$A = \Gamma G = {}^2G_2(3^{2n+1}) \leftthreetimes \langle\zeta\rangle$, $|\zeta| = 2n+1$, and, up to conjugation in G the equality $K \cap {}^2G_2(3^{2n+1}) = S \times P$ holds, where Q is of order 2 and $|P| = 3^{|\zeta|_3}$.*

(e) *p does not divide $|K\cap G|$ and K contains a Sylow 2-subgroup of A, the group A satisfies* **(ESyl2)** *if and only if G satisfies*(**ESyl2**).

We give also two technical lemmas that will be used in the classification of Carter subgroups in almost simple groups.

Lemma 4. [2, Lemma 5] *Assume that G is a finite gorup. Let K be a Carter subgroup of G with center $Z(K)$. Assume also that $e\neq z\in Z(K)$ and $C_G(z)$ satisfies* **(C)**.

(1) *Every subgroup Y containing K and satisfying* **(C)** *is self-normalized in G.*

(2) *No conjugate of z in G, except z, lies in $Z(G)$.*

(3) *If H is a Carter subgroup of G, non-conjugate to K, then z is not conjugate to any element in the center of H.*

In particular the centralizer $C_G(z)$ is self-normalizing in G, and z is not conjugate to any power $z^k\neq z$.

Lemma 5. [3, Lemma 2.6] *Let G be a finite group, let H be a normal subgroup of G such that $|G:H|=2^t$. Let S,T be Sylow 2-subgroups of G,H respectively and $N_H(T)=TC_H(T)$. Then $N_G(S)=SC_G(S)$.*

In particular, G,H contain Carter subgroups K,L respectively, satisfying $S\leq K$ and $T\leq L$.

In view of condition **(E)** and Theorem 1 the investigation of Carter sbugroups in finite groups is reduced to the classification of Carter subgroups in almost simple groups A, satisfying the additional condition: $A/F^*(A)$ is nilpotent. The classification of Carter subgroups in almost simple groups satisfying this condition is obtained by several authors and we give it here in the form that is convinient to use.

We prove first the following theorem showing that if for some subgroup S of $\mathrm{Aut}(G)$ there exists a Carter subgroup, then it exists in every larger group $S\leq A\leq \mathrm{Aut}(G)$ (here G is a known finite simple group).

Theorem 4. *Let G be a finite simple group and $G\leq A\leq \mathrm{Aut}(G)$ be an almost simple group with simple socle G. Assume that A contains a subgroup S such that $G\leq S$ and S contains a Carter subgroup.*

Then A contains a Carter subgroup.

Proof. Let K be a Carter subgroup of S. Clearly we may assume that $S=KG$.

Assume that either $G\simeq \mathrm{Alt}_n$ for some $n\geqslant 5$, or G is sporadic. Since by [5, Lemma 2.10] every element of odd order of G is conjugate to its nontrivial power, and since $|\mathrm{Aut}(G):G|$ is a power of 2, Lemmas 4 and 5 imply that, if some $G\leq S\leq \mathrm{Aut}(G)$ contains a Carter subroup K, then K coincides with a Sylow 2-subgroup of S. Since $|A:S|$ is a 2-power, the statement of the theorem in this case follows from Lemma 5.

Assume that $G={}^3D_4(q^3)$. By [6, Theorem 1.2(vi)] every element of G is conjugate to its inverse. If q is odd, then [3, Lemma 4.3] implies that K is a Sylow 2-subgroup of S. Thus by Lemma 5 and [3, Lemma 4.3] it follows that A satisfies **(ESyl2)**, i. e. contains a Carter subgroup. If $q=2^t$ is even, then by Theorems 2 and 3 it follows that either $|\mathrm{Aut}(G):S|=3$ and a Carter subgroup of S is a Carter subgroup of A, or $|\mathrm{Aut}(G):S|=2$ and, if $A\neq S$, then A contains a subgroup M of index 3 such that M satisfies Theorem 3(2) and a Carter subgroup of M is a Carter subgroup of A.

Assume that G is a group of Lie type, $G \not\simeq {}^3D_4(q^3)$ and, if $G \simeq D_4(q)$, then $S \leq A_1$, where $A_1 \leq \mathrm{Aut}(D_4(q))$ is defined in Theorem 2. Then S satisfies one of the statements (1)–(5) of Theorem 3. Consider these cases separately.

Assume that S satisfies (1). There can be two cases: either 3 divides $|\zeta|$, or 3 divides $|\zeta|$. In the first case we have that $|\mathrm{Aut}(G) : S| = 2$ thus either $A = S$, or $A = \mathrm{Aut}(G)$. If $A = S$ then there is nothing to prove, if $A = \mathrm{Aut}(G)$, then A satisfies (2) of Theorem 3, thus contains a Carter subgroup. In the second case (when 3 divides $|\zeta|$) we have that A is equal to either ${}^2A_2(2^{2t}) \leftthreetimes \langle\zeta\rangle$, or $\widehat{{}^2A_2(2^{2t})} \leftthreetimes \langle\zeta\rangle$, or ${}^2A_2(2^{2t}) \leftthreetimes \langle\zeta\tau\rangle$, or $\mathrm{Aut}(G)$ (here τ is a field automorphism of order 2, commuting with ζ). First two groups contain a Carter subgroup in view of Theorem 3(1), the least two groups contain a Carter subgroup in view of Theorem 3(2).

Assume that S satisfies (2) of Theorem 3. Then $|\mathrm{Aut}(G) : S| \leqslant 2$, hence either $A = S$, or $A = \mathrm{Aut}(G)$. In the first case A clearly contains a Carter subgroup. In the second case $|A : S| = 2$ and A contains a Carter subgroup in view of Theorem 3(2).

Assume that S satisfies either (3), or (4) of Theorem 3. Then $S = \mathrm{Aut}(G) = A$ and there is nothing to prove.

Assume that S satisfies (5) of Theorem 3. Let Q be a Sylow 2-subgroup of $S \cap \widehat{G}$. As it is noted in Theorem 3(5), then the eqality $N_{S\cap\widehat{G}}(Q) = QC_{S\cap\widehat{G}}(Q)$ holds. Since $A\cap\widehat{G} \geq S\cap\widehat{G}$, then by [7] we have that $N_{A\cap\widehat{G}}(Q_1) = Q_1C_{A\cap\widehat{G}}(Q_1)$, where Q_1 is a Sylow 2-subgroup of $A\cap\widehat{G}$. By [3, Lemma 4.3] it follows that for a Sylow 2-subgroup Q_2 of A the equality $N_A(Q_2) = Q_2C_A(Q_2)$ holds. Therefore, [3, Lemma 2.3] implies that A contains a Carter subgroup.

Assume now that $G = D_4(q)$ and S satisfies Theorem 2. Since graph automorphisms of order 2 and 3 does not commute, then only one of them can be contained in a nilpotent subgroup, Thus we may assume that only one of them is contained in A. Then every subgroup A containing S either satisfies Theorem 2, or satisfies 3 condition (2), if q is even, or condition (5), if q is odd, i. e., contains a Carter subgroup. $\square$

The tables given below are arranged in the following order. In the first column is given a simple group S such that Carter subgroups of $\mathrm{Aut}(S)$ are classified. In the second column we give conditions for a subgroup A of its group of automorphisms for A to contain a Carter subgroup. In the third column we give the structure of a Carter subgroup K. In every subgroup of $\mathrm{Aut}(S)$ lying between S and A Carter subgroups does non exist. By $P_r(G)$ a Sylow r-subgroup of G is denoted. By φ we denote a field automorphism of a group of Lie type S, by τ we denote a graph automorphism of a group of Lie type S contained in K (since graph automorphisms of order 2 and 3 of $D_4(q)$ does not commute, only one of them can be in K). By λ we denote a graph-field automorphism of $D_4(q^3)$, with the set of stable points isomorphic to ${}^3D_4(q^{3/k})$ for some k. If A does not contains a graph automorphism, then we suppose $\tau = e$. By ψ we denote a field automorphism of S of maximal order contained in A (it is a power of φ, but $\langle\psi\rangle$ can be different from $\langle\varphi\rangle$). If G is solvable, then by $K(G)$ we denote a Carter subgroup of G. In Table 3 by χ the $2'$-part of a field automorphism φ of ${}^2A_2(2^{2t})$ is denoted. To check the condition **(ESyl2)** in an almost simple group we use results from [7] and [8].

Table 1. Groups of automorphisms of alternating groups containing Carter subgroups.

Group S	Conditions for A	Structure of K
Alt_5	$A = \mathrm{Sym}_5$	$K = P_2(\mathrm{Sym}_5)$
Alt_n, $n \geqslant 6$	none	$K = N_A(P_2(S))$

Table 2. Groups of automorphisms of cporadic groups containing Carter subgroups.

Group S	Coditions for A	Structure of K
J_2, J_3, Suz, HN	$A = \mathrm{Aut}(S)$	$K = P_2(A)$
$\not\simeq J_1, J_2, J_3, Suz, HN$	none	$K = P_2(A)$

Table 3. Groups of automorphisms of classical groups containing Carter subgroups.

Group S	Conditions for A	Structure of K
$A_1(q)$, $q \equiv \pm 1 \pmod 8$	none	$K = N_A(P_2(S))$
$A_1(q)$, $q \equiv \pm 3 \pmod 8$	$\widehat{S} \leq A$	$K = N_A(P_2(\widehat{S}))$
$A_n(2^t)$, $t \geqslant 2$, if $n = 1$	$\varphi \in A$	$K = \langle \varphi, \tau \rangle \leftthreetimes S_{\varphi_{2'}}$
$A_n(q)$, q odd, $n \geqslant 2$	none	$K = P_2(A) \times K(O(N_A(P_2(A))))$
${}^2A_2(2^{2t})$, t odd	$3 \mid t$, $\langle \chi \rangle \leftthreetimes S \leq A \leq \langle \chi \rangle \leftthreetimes \widehat{S}$ $3 \nmid t$, $A = \langle \chi \rangle \leftthreetimes \widehat{S}$	$K = \langle \chi \rangle \times \langle x \rangle$ where $\lvert x \rvert = 2 \cdot 3 \cdot t_3$
${}^2A_2(2^{2t})$	$A = \mathrm{Aut}(S)$	$K = \langle \varphi \rangle \leftthreetimes P_2(S_{\varphi_{2'}})$
${}^2A_n(q^2)$, q odd	none	$K = P_2(A) \times K(O(N_A(P_2(A))))$
${}^2A_n(2^{2t})$, $n \geqslant 3$	$A = \mathrm{Aut}(S)$	$K = \langle \varphi \rangle \leftthreetimes P_2(S_{\varphi_{2'}})$
$B_2(q)$, $q \equiv \pm 1 \pmod 8$	none	$K = P_2(A) \times K(O(N_A(P_2(A))))$
$B_2(2^t)$, $t \geqslant 2$	$\varphi \in A$	$K = \langle \varphi, \tau \rangle \leftthreetimes P_2((S_\tau)_\varphi)$
$B_2(q)$, $q \equiv \pm 3 \pmod 8$	$\widehat{S} \leq A$	$K = P_2(A) \times K(O(N_A(P_2(A))))$
$B_n(q)$, q odd, $n \geqslant 3$	none	$K = P_2(A) \times K(O(N_A(P_2(A))))$
$C_n(q)$, $q \equiv \pm 1 \pmod 8$	none	$K = P_2(A) \times K(O(N_A(P_2(A))))$
$C_n(q)$, $q \equiv \pm 3 \pmod 8$	$\widehat{S} \leq A$	$K = P_2(A) \times K(O(N_A(P_2(A))))$
$C_n(2^t)$, $n \geqslant 3$	$A = \mathrm{Aut}(S)$	$K = \langle \varphi \rangle \times P_2(S_{\varphi_{2'}})$

$D_4(q)$, q odd	none	if $\lvert \tau \rvert \leqslant 2$ and $\lambda \notin A$, then
		$K = P_2(A) \times K(O(N_A(P_2(A))))$;
		if $A = G \rightthreetimes \langle \lambda \rangle$, then
		$K = \langle \lambda \rangle \leftthreetimes P_2(S_{\lambda_{2'}})$;
		if $\lvert \tau \rvert = 3$, then
		$K = \langle \tau, \psi \rangle \leftthreetimes P_2(S_\tau)$

$D_4(2^t)$	$\varphi \in A$	if $\lvert \tau \rvert \leqslant 2$, then
		$K = \langle \tau, \varphi \rangle \leftthreetimes P_2(S_{\varphi_{2'}})$;
		if $\lvert \tau \rvert = 3$, then
		$K = \langle \tau, \varphi \rangle \leftthreetimes P_2((S_\tau)_{\varphi_{2'}})$

$D_n(q)$, q odd, $n \geqslant 5$	none	$K = P_2(A) \times K(O(N_A(P_2(A))))$
$D_n(2^t)$, $n \geqslant 5$	$\varphi \in A$	$K = \langle \tau, \varphi \rangle \leftthreetimes P_2(S_{\varphi_{2'}})$
${}^2D_n(q^2)$, q odd	none	$K = P_2(A) \times K(O(N_A(P_2(A))))$
${}^2D_n(2^{2t})$	$A = \mathrm{Aut}(S)$	$K = \langle \varphi \rangle \leftthreetimes P_2(S_{\varphi_{2'}})$

Table 4. Groups of automorphisms of exceptional groups of Lie type containing Carter subgroups.

Group S	Conditions for A	Structure of K
${}^2B_2(2^{2n+1})$, $n \geqslant 1$	$A = \mathrm{Aut}(S)$	$K = \langle \varphi \rangle \times P_2({}^2B_2(2))$
$({}^2F_4(2))'$	none	$K = P_2(A)$
${}^2F_4(2^{2n+1})$, $n \geqslant 1$	$A = \mathrm{Aut}(S)$	$K = \langle \varphi \rangle \times P_2({}^2F_4(2))$
${}^2G_3(3^{2n+1})$	$A = \mathrm{Aut}(G)$	$\langle \varphi \rangle \leftthreetimes (2 \times P)$,

		where $\lvert P\rvert = 3^{\lvert\varphi\rvert_3}$
others, q is odd	none	$K = P_2(A) \times K(O(N_A(P_2(S))))$
others, $q = 2^t$	$\varphi \in A$	$\langle \tau, \varphi\rangle \leftthreetimes P_2(S_{\varphi_{2'}})$

As a corollary note the following interesting result.

Lemma 6. *Let S be a known finite simple group, $S \not\simeq J_1$ and $G = \mathrm{Aut}(S)$. Then G contains a Carter subgroup.*

Proof. By [7, Theorems 2 и 3], if S is not a group of Lie type and distinct from J_1, then the group of its automorphisms $\mathrm{Aut}(S)$ satisfies **(ESyl2)** and contains a Carter subgroup. Now if S is a group of Lie type in even characteristic, then $\mathrm{Aut}(S)$ contains a Carter subgroup in view of 3(2). If S is a group of Lie type in odd characteristic and $S \not\simeq {}^2G_2(3^{2n+1})$, then $\widehat{S}$ satisfies **(ESyl2)**, hence contains a Carter subgroup. By Theorem 4, $\mathrm{Aut}(S)$ contains a Carter subgroup. Now, if $S \simeq {}^2G_2(3^{2n+1})$, then $\mathrm{Aut}(S)$ contains a Carter subgroup in view of 3(4). □

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