Algebra and Logic, Vol. 46, No. 2, 2007

CARTER SUBGROUPS OF FINITE ALMOST SIMPLE GROUPS

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UDC 512.542

Keywords: *finite almost simple group, Carter subgroup.*

In the paper we work to complete the classification of Carter subgroups in finite almost simple groups. In particular, it is proved that Carter subgroups of every finite almost simple group are conjugate. Based on our previous results, together with those obtained by F. Dalla Volta, A. Lucchini, and M. C. Tamburini, as a consequence we derive that Carter subgroups of every finite group are conjugate.

1. PRELIMINARIES

We recall that a subgroup of a finite group is called a *Carter subgroup* if it is nilpotent and self-normalizing. By a well-known result, any finite solvable group contains exactly one conjugacy class of Carter subgroups (cf. [1]). Therefore it seems reasonable to conjecture that a finite group contains at most one conjugacy class of Carter subgroups. In favor of this conjecture is evidence coming from extensive studies of classes of finite groups that are close to simple. In particular, it was shown that the conjecture holds true for symmetric and alternating groups [2]; for any group A such that $SL_n(p^t) \leq A \leq GL_n(p^t)$ [3, 4]; for symplectic groups $Sp_{2n}(p^t)$, full unitary groups $GU_n(p^{2t})$, and full orthogonal groups $GO_n^{\pm}(p^t)$ where p is odd [5] (p^t is a power of a prime p). In [6] the results of [5] were extended to any group G with $O^{p'}(S) \leq G \leq S$, where S is a full classical matrix group. Also some of the sporadic simple groups were investigated (see, e.g., [7]). In the non-solvable case Carter subgroups (if any) always turned out to be normalizers of Sylow 2-subgroups.

In the paper we study into the so-called conjugacy problem worded as follows.

Problem. Are any two Carter subgroups of a finite group conjugate?

In [8] it was proved that the minimal counterexample A to this problem should be almost simple. In [9] a stronger result was obtained.

Definition. A finite group G is said to *satisfy condition* **(C)** if, for every non-Abelian composition factor S of every composition series of G and for every nilpotent subgroup N of G, Carter subgroups of $\langle \mathrm{Aut}_N(S), S\rangle$ are conjugate (for definition of $\mathrm{Aut}_N(S)$, see below).

Since S is simple, it is isomorphic to its inner automorphism group $\mathrm{Inn}(S)$, and we identify S with the subgroup $\mathrm{Inn}(S)$ of $\mathrm{Aut}(S)$.

*Supported by RFBR grant No. 05-01-00797; by the Council for Grants (under RF President) for Support of Young Russian Scientists via projects MK-1455.2005.1 and MK-3036.2007.1; by SB RAS Young Researchers Support grant No. 29; via Integration Project No. 2006.1.2.

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Translated from *Algebra i Logika*, Vol. 46, No. 2, pp. 157-216, March-April, 2007. Original article submitted October 28, 2005; revised February 9, 2007.

THEOREM 1.1 [9]. If a finite group G satisfies condition **(C)**, then Carter subgroups of G are conjugate.

Thus our goal is to prove that for every known finite simple group S and every nilpotent subgroup N of $\mathrm{Aut}(S)$, Carter subgroups of $\langle S, N\rangle$ are conjugate. Some classes of almost simple groups which fail to be minimal counterexamples to the conjugacy problem can be found in [6, 10]. The table of almost simple groups for which the conjugacy problem has an affirmative answer is given in [9].

Our notation is standard. If G is a finite group, then $\mathbf{P}G$ denotes the factor group $G/Z(G) \simeq \mathrm{Inn}(G)$. If π is a set of primes then π' denotes its complement in the set of all primes. For a positive integer n, the set of prime divisors of n is denoted by $\pi(n)$, and the maximal divisor t of n with $\pi(t) \subseteq \pi$ — by n_π. As usual, we write $O_\pi(G)$ for a maximal normal π-subgroup of G and write $O^{\pi'}(G)$ for the subgroup generated by all π-elements of G. If $\pi = \{2\}'$ is a set of all odd primes, then $O_\pi(G) = O_{2'}(G)$ is denoted by $O(G)$. If $g \in G$, then we define g_π to be the π-part of g, that is, $g_\pi = g^{|g|_{\pi'}}$. We denote by $F(G)$ a Fitting subgroup of G, and by $F^*(G)$ the generalized Fitting subgroup of G. A central product of groups G and H is denoted by $G * H$. For a finite group G, $\mathrm{Aut}(G)$ stands for the automorphism group of G. If $\lambda \in \mathrm{Aut}(G)$, then G_λ is defined to be a set of λ-stable points, that is, $G_\lambda = \{g \in G \mid g^\lambda = g\}$. If $Z(G) = \{e\}$, then $G \simeq \mathrm{Inn}(G)$, and we may suppose that $G \leq \mathrm{Aut}(G)$. A finite group G is said to be *almost simple* if there is a simple group S with $S \leq G \leq \mathrm{Aut}(S)$, that is, $F^*(G)$ is a simple group.

If G is a group, A, B, and H are subgroups of G, and B is normal in A ($B \trianglelefteq A$), then $N_H(A/B) = N_H(A) \cap N_H(B)$. If $x \in N_H(A/B)$, then x induces an automorphism $Ba \mapsto Bx^{-1}ax$ of A/B. Thus there is a homomorphism of $N_H(A/B)$ into $\mathrm{Aut}(A/B)$. An image of this homomorphism is denoted by $\mathrm{Aut}_H(A/B)$, and its kernel — by $C_H(A/B)$. In particular, if $S = A/B$ is a composition factor of G, then the group $\mathrm{Aut}_H(S)$ is defined for any subgroup $H \leq G$. If A and H are subgroups of G, then $\mathrm{Aut}_H(A) = \mathrm{Aut}_H(A/\{e\})$ by definition.

LEMMA 1.2. Let G be a finite group, H a normal subgroup of G, $S = (A/H)/(B/H)$ a composition factor of G/H, and L a subgroup of G. Then $\mathrm{Aut}_L(A/B) \simeq \mathrm{Aut}_{LH/H}((A/H)/(B/H))$.

Proof. Since $H \leq B$, $H \leq C_G(A/B)$, and so we may assume that $L = LH$. Also we can suppose that $L \leq N_G(A) \cap N_G(B)$ and $G = LA$. Then the action on A/B given by the rule $x : Ba \mapsto Bx^{-1}ax$ coincides with that on $(A/H)/(B/H)$ given by the rule $xH : BaH \mapsto Bx^{-1}axH$, whence the result. □

Well known is the following:

LEMMA 1.3. Let G be a finite group, H be a normal subgroup of G, and $\overline{N}$ be a nilpotent subgroup of $\overline{G} = G/H$. Then there exists a nilpotent subgroup N of G such that $NH/H = \overline{N}$.

Proof. We may assume that $G/H = \overline{N}$. There exists a subgroup U of G such that $UH = G$. Choose a subgroup of minimal order with this property. Then $U \cap H \leq \Phi(U)$, where $\Phi(U)$ is a Frattini subgroup of U. Indeed, if there exists a maximal subgroup M of U not containing $U \cap H$, then clearly $MH = G$, which contradicts the minimality of U. Thus the group $U/\Phi(U)$ is nilpotent; hence U is nilpotent and $N = U$. □

Lemmas 1.2 and 1.3 imply that if a finite group G satisfies **(C)** then the subgroup HN satisfies **(C)**, for any normal subgroup N and any solvable subgroup H of G.

LEMMA 1.4. Let G be a finite group, K be a Carter subgroup of G, and N be a normal subgroup of G. Assume that KN satisfies **(C)** (this is always true if G satisfies **(C)** or N is solvable), or $KN = G$. Then KN/N is a Carter subgroup of G/N.

Proof. If $KN = G$ then the statement is clear. Suppose $KN \neq G$, that is, KN satisfies **(C)**. Consider $x \in G$, letting $xN \leq N_{G/N}(KN/N)$. It follows that $x \in N_G(KN)$. The group K^x is a Carter subgroup of KN. Since KN satisfies **(C)**, its Carter subgroups are conjugate. Thus there exists $y \in KN$ such that

$K^y = K^x$. Since K is a Carter subgroup of G, we have $xy^{-1} \in N_G(K) = K$ and $x \in KN$. □

LEMMA 1.5 [9, Lemma 5]. Let K be a Carter subgroup of a finite group G, assume that a non-identity element z is in $Z(K)$, and suppose that $C_G(z)$ satisfies **(C)**. Then:

(a) every subgroup Y which contains K and satisfies **(C)** is self-normalizing in G;

(b) no conjugate of z in G except z lies in $Z(K)$;

(c) if H is a Carter subgroup of G which is not conjugate to K, then z is not conjugate to any element in the center of H.

Specifically, the centralizer $C_G(z)$ is self-normalizing in G, and z is not conjugate to any power $z^k \neq z$.

LEMMA 1.6. Let G be a finite group and Q be a Sylow 2-subgroup of G. Then a Carter subgroup K of G containing Q exists if and only if $N_G(Q) = QC_G(Q)$.

Proof. Assume that there exists a Carter subgroup K of G containing Q. Since K is nilpotent, Q is normal in K and $K \leq QC_G(Q) \trianglelefteq N_G(Q)$. By the Feit–Thompson theorem [11], we see that $N_G(Q)$ is solvable. By Lemma 1.5(a), we conclude that $QC_G(Q)$ is self-normalizing in $N_G(Q)$, and so $N_G(Q) = QC_G(Q)$.

Suppose now that $N_G(Q) = QC_G(Q)$, that is, $N_G(Q) = Q \times O(C_G(Q))$. Since $O(C_G(Q))$ is of odd order, it is solvable and, hence, contains a Carter subgroup K_1. Consider a nilpotent subgroup $K = Q \times K_1$ of G. Clearly, $N_G(K) \leq N_G(Q)$. But K is a Carter subgroup of $N_G(Q)$, and so K is one of G. □

Definition. With Lemma 1.6 in mind, we say that a finite group G *satisfies condition* **(ESyl2)** if $N_G(Q) = QC_G(Q)$ for a Sylow 2-subgroup Q of G. In other words, G satisfies **(ESyl2)** if each element of odd order normalizing Q centralizes Q.

LEMMA 1.7. Let G be a finite group, Q be a Sylow 2-subgroup of G, and x be an element of odd order in $N_G(Q)$. Assume that there exist normal subgroups $G_1, \ldots, G_k$ of G such that $G_1 \cap \ldots \cap G_k \cap Q \leq Z(N_G(Q))$ and x centralizes Q modulo G_i for all i. Then x centralizes Q. In particular, if G/G_i satisfies **(ESyl2)** for all i, then G satisfies **(ESyl2)**.

Proof. Consider a normal series $Q \trianglerighteq Q_1 \trianglerighteq \ldots \trianglerighteq Q_k \trianglerighteq Q_{k+1} = \{e\}$, where $Q_i = Q \cap (G_1 \cap \ldots \cap G_i)$. The conditions of the lemma imply that x centralizes each factor Q_{i-1}/Q_i. Since x is of odd order, x centralizes Q. □

LEMMA 1.8 [9, Lemma 3]. Let K be a Carter subgroup of a finite group G. Assume that there exists a normal subgroup $B = T_1 \times \ldots \times T_k$ of G such that $G = KB$, $Z(T_i) = \{e\}$, and T_i does not factor into a direct product of its proper subgroups, for all i. Then $\mathrm{Aut}_K(T_i)$ is a Carter subgroup of $\langle \mathrm{Aut}_K(T_i), T_i \rangle$.

LEMMA 1.9. Let H be a subgroup of a finite group G such that $|G : H| = 2^t$, H satisfies **(ESyl2)**, and each element of odd order in G is in H (this property is obviously equivalent to H being subnormal). Then G satisfies **(ESyl2)**.

Proof. Let Q be a Sylow 2-subgroup of G such that $Q \cap H$ is a Sylow 2-subgroup of H. Consider an element $x \in N_G(Q)$ of odd order. Since $x \in H$, $x \in N_H(Q) \leq N_H(Q \cap H) = (Q \cap H) \times O(N_H(Q \cap H))$, that is, $x \in O(N_H(Q \cap H))$. Thus the set of elements of odd order in $N_G(Q)$ form a subgroup $R = O(N_H(Q \cap H)) \cap N_G(Q)$ of $N_G(Q)$. Clearly, R is normal in $N_G(Q)$; hence $R = O(N_G(Q))$. On the other hand, Q is normal in $N_G(Q)$ by definition, and $Q \cap R = \{e\}$; so $N_G(Q) = Q \times O(N_G(Q))$. □

2. GROUPS OF LIE TYPE

The notation for groups of Lie type are borrowed from [12], and for linear algebraic groups — from [13]. If G is a canonical finite group of Lie type (for definition, see below) with trivial center, then $\widehat{G}$ denotes the inner-diagonal automorphism group of G. (Here, we do not exclude non-simple groups of Lie type such as $A_1(2)$, with all the exceptions given in [12, Thms. 11.1.2 and 14.4.1]). In view of [14, 3.2], $\mathrm{Aut}(G)$ is generated by inner-diagonal, field, and graph automorphisms. Since we are assuming that $Z(G)$ is trivial, $G \simeq \mathrm{Inn}(G)$, and hence we may suppose that $G \leq \widehat{G} \leq \mathrm{Aut}(G)$.

Let $\overline{G}$ be a simple connected linear algebraic group over an algebraically closed field $\overline{\mathbb{F}}_p$ of positive characteristic p. In this instance $Z(\overline{G})$ may be non-trivial. An endomorphism σ of $\overline{G}$ is called a *Frobenius map* if $\overline{G}_\sigma$ is finite and σ is an automorphism of $\overline{G}$ treated as an abstract group. Groups $O^{p'}(\overline{G}_\sigma)$ are called *canonical finite groups of Lie type*, and every group G satisfying $O^{p'}(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma$ is called a *finite group of Lie type*. If $\overline{G}$ is a simple algebraic group of adjoint type then we say that G is also of adjoint type. In [12], note, groups of Lie type referred to $O^{p'}(\overline{G})$ only. In [15], however, every group $\overline{G}_\sigma$ was used to refer to a finite group of Lie type, for an arbitrary connected reductive group $\overline{G}$. Moreover, in [16, 17] every group G with $O^{p'}(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma$ was called, without any explanation or grounding, a finite group of Lie type. We intend to clarify the situation here, in giving definitions of finite groups of Lie type and of canonical finite groups of Lie type. For example, $\mathbf{P}SL_2(3)$ is a canonical finite group of Lie type and $\mathbf{P}GL_2(3)$ is a finite group of Lie type. Note that an element of order 3 is not conjugate to its inverse in $\mathbf{P}SL_2(3)$ but is conjugate to its inverse in $\mathbf{P}GL_2(3)$. Since such information on conjugation is important in many cases (including those in the present paper), we will adhere to the following notation.

Denote by $\Phi(\overline{G})$ the root system of a group $\overline{G}$, and by Φ, or $\Phi(G)$, the root system of a group $O^{p'}(G)$. Write $\Delta(\overline{G})$ for a fundamental group of $\overline{G}$ and write $\Delta(\Phi)$ for a factor group of the lattice generated by fundamental weights in the root system Φ w.r.t. the lattice generated by all roots in Φ. Note that $\Delta(\overline{G})$ is always a factor of $\Delta(\Phi(\overline{G}))$, and for each root system Φ distinct from D_{2n}, the group $\Delta(\Phi)$ is cyclic and $\Delta(D_{2n})$ is elementary Abelian of order 4. The Weyl group of $\overline{G}$ is denoted by $W(\overline{G})$, and the Weyl group of Φ — by $W(\Phi)$. If $W(\Phi)$ is a Weyl group of the root system Φ, then by w_0 we denote the unique element mapping all positive roots to negative ones.

We say that groups G of Lie type, for which $O^{p'}(G)$ is equal to one of the groups ${}^2A_n(q^2)$, ${}^2D_n(q^2)$, or ${}^2E_6(q^2)$, are defined over $GF(q^2)$, groups ${}^3D_4(q^3)$ of Lie type are defined over $GF(q^3)$, and that other groups of Lie type are defined over $GF(q)$. The field $GF(q)$ is called the *base field* in all cases. In view of [18, Lemma 2.5.8], if $\overline{G}$ is of adjoint type then $\overline{G}_\sigma$ is an inner-diagonal automorphism group of $O^{p'}(\overline{G}_\sigma)$. If $\overline{G}$ is simply connected, then $\overline{G}_\sigma = O^{p'}(\overline{G}_\sigma)$ (cf. [19, 12.4]). By [18, Thm. 2.2.6(g)], $\overline{G}_\sigma = \overline{T}_\sigma O^{p'}(\overline{G}_\sigma)$ for each σ-stable maximal torus $\overline{T}$ of $\overline{G}$ in any case. For a given finite group G of Lie type (treated as an abstract group), the corresponding algebraic group is not uniquely determined in general. For example, if $G = \mathbf{P}SL_2(5) \simeq SL_2(4)$, then G derives either as $(SL_2(\overline{\mathbb{F}}_2))_\sigma$ or as $O^{5'}((\mathbf{P}SL_2(\overline{\mathbb{F}}_5))_\sigma)$ (for appropriate σ). For every finite group G of Lie type, therefore, we will (somehow) fix a corresponding algebraic group $\overline{G}$ and a Frobenius map σ such that $O^{p'}(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma$.

Let $U = \langle X_r \mid r \in \Phi(\overline{G})^+ \rangle$ be a maximal unipotent subgroup of G. If we fix an order on $\Phi(\overline{G})$, then every $u \in U$ can be uniquely written in the form

$$u = \prod_{r \in \Phi^+} x_r(t_r), \tag{1}$$

where the roots are taken in the given order and t_r are from the definition field of G. We say that G is *twisted* if $O^{p'}(G)$ coincides with one of the groups ${}^2A_n(q^2)$, ${}^2B_2(2^{2n+1})$, ${}^2D_n(q^2)$, ${}^3D_4(q^3)$, ${}^2E_6(q^2)$, ${}^2G_2(3^{3n+1})$, or ${}^2F_4(2^{n+1})$, and is *split* otherwise. If $O^{p'}(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma$ is a twisted group of Lie type and $r \in \Phi(\overline{G})$, then by $\bar{r}$ we always denote an image of r under the symmetry of a root system corresponding to a graph automorphism used in constructing G. Sometimes we write $\Phi^\varepsilon(q)$, where $\varepsilon \in \{+, -\}$; $\Phi^+(q) = \Phi(q)$ is a split group of Lie type with base field $GF(q)$ and $\Phi^-(q) = {}^2\Phi(q^2)$ is a twisted group of Lie type defined over a field $GF(q^2)$ (with base field $GF(q)$).

Further, let $\overline{R}$ be a closed σ-stable subgroup of $\overline{G}$. Put $R = G \cap \overline{R}$ and $N(G, R) = G \cap N_{\overline{G}}(\overline{R})$. Note that $N(G, R) \neq N_G(R)$ in general, and we call $N(G, R)$ an *algebraic normalizer* of R. For example, if $G = SL_n(2)$, then the diagonal matrix subgroup H of G is trivial; hence $N_G(H) = G$. But $G = (SL_n(\overline{\mathbb{F}}_2))_\sigma$, where σ is the Frobenius map $\sigma : (a_{i,j}) \mapsto (a_{i,j}^2)$. Therefore $H = \overline{H}_\sigma$, where $\overline{H}$ is a subgroup of diagonal matrices in $SL_n(\overline{\mathbb{F}}_2)$. Thus $N(G, H)$ is the monomial matrix group of G. We use the term an "algebraic normalizer" in order to avoid such difficulties and make our proofs universal. A group R is called a *torus* (resp., a *reductive subgroup*, a *parabolic subgroup*, a *maximal torus*, and a *reductive subgroup of maximal rank*) if $\overline{R}$ is a torus (resp., a reductive subgroup, a parabolic subgroup, a maximal torus, and a reductive subgroup of maximal rank) of $\overline{G}$. A maximal σ-stable torus $\overline{T}$ such that $\overline{T}_\sigma$ is a Cartan subgroup of $\overline{G}_\sigma$ is referred to as a *maximal split torus*.

If $\overline{R}$ is a connected reductive subgroup of $\overline{G}$ of maximal rank, then $\overline{R} = \overline{G}_1 * \ldots * \overline{G}_k * \overline{Z}$, where $\overline{G}_i$ are simple connected linear algebraic groups and $\overline{Z} = Z(\overline{R})^0$ (see [13, Thm. 27.5]). Moreover, if $\Phi_1, \ldots, \Phi_k$ are root systems of respective groups $\overline{G}_1, \ldots, \overline{G}_k$, then $\Phi_1 \cup \ldots \cup \Phi_k$ is a subsystem of $\Phi(\overline{G})$. There is a nice algorithm determining subsystems of an arbitrary root system Φ, described by Borel and de Siebental in [20], and independently, by Dynkin in [21]. We need only remove some nodes from the extended Dynkin diagram of Φ and then repeat the procedure for connected components that appear. The so obtained connected components are Dynkin diagrams for indecomposable subsystems, and the Dynkin diagram of every indecomposable subsystem can be arrived at in just this way.

Now we assume that a reductive subgroup $\overline{R}$ is σ-stable. In view of [19, 10.10], there exists a σ-stable maximal torus $\overline{T}$ of $\overline{R}$. Let $\overline{G}_{i_1}, \ldots, \overline{G}_{i_{j_i}}$ be a σ-orbit of $\overline{G}_{i_1}$. Consider an induced action of σ on the factor group

$$(\overline{G}_{i_1} * \ldots * \overline{G}_{i_{j_i}})/Z(\overline{G}_{i_1} * \ldots * \overline{G}_{i_{j_i}}) \simeq \mathbf{P}\overline{G}_{i_1} \times \ldots \times \mathbf{P}\overline{G}_{i_{j_i}}.$$

Since $\mathbf{P}\overline{G}_{i_1} \simeq \ldots \simeq \mathbf{P}\overline{G}_{i_{j_i}}$ are simple (as abstract groups), σ induces a cyclic permutation on $\mathbf{P}\overline{G}_{i_1}, \ldots, \mathbf{P}\overline{G}_{i_{j_i}}$, and we may choose a numbering so that $\mathbf{P}\overline{G}_{i_1}^\sigma = \mathbf{P}\overline{G}_{i_2}, \ldots, \mathbf{P}\overline{G}_{i_{j_i}}^\sigma = \mathbf{P}\overline{G}_{i_1}$. Thus

$$(\mathbf{P}\overline{G}_{i_1} \times \ldots \times \mathbf{P}\overline{G}_{i_{j_i}})_\sigma = \{x \mid x = g \cdot g^\sigma \cdot \ldots \cdot g^{\sigma^{j_i - 1}} \text{ for some } g \in \mathbf{P}\overline{G}_{i_1}\}_\sigma \simeq (\mathbf{P}\overline{G}_{i_1})_{\sigma^{j_i}}.$$

By [19, 10.15], the group $\mathbf{P}\overline{G}_{\sigma^{j_i}}$ is finite; so $O^{p'}((\mathbf{P}\overline{G}_{i_1})_{\sigma^{j_i}})$ is a finite canonical group of Lie type, probably with a larger base field than is one for $O^{p'}(\overline{G}_\sigma)$.

Let $\overline{B}_{i_1}$ be a preimage of a σ^{j_i}-stable Borel subgroup of $\mathbf{P}\overline{G}_{i_1}$ in $\overline{G}_{i_1}$ under the natural epimorphism and let $\overline{T}_{i_1}$ be a σ^{j_i}-stable maximal torus of $\overline{G}_{i_1}$ lying in $\overline{B}_{i_1}$ (such exist by [19, 10.10]). From a remark in [19, Sec. 11], it follows that subgroups $\overline{U}_{i_1}$ and $\overline{U}_{i_1}^-$ generated by $\overline{T}_{i_1}$-stable root subgroups taken over all positive and negative roots are also σ^{j_i}-stable. Since $\overline{G}_{i_1}$ is a simple algebraic group, $\overline{G}_{i_1}$ is generated by the subgroups $\overline{U}_{i_1}$ and $\overline{U}_{i_1}^-$. Now $Z(\overline{G}_{i_1} * \ldots * \overline{G}_{i_{j_i}})$ consists of semisimple elements, and so restrictions of the natural epimorphism $\overline{G}_{i_1} \to \mathbf{P}\overline{G}_{i_1}$ to $\overline{U}_{i_1}$ and to $\overline{U}_{i_1}^-$ are isomorphism. Therefore, for every k, $(\overline{U}_{i_1})^{\sigma^k}$ and $(\overline{U}_{i_1}^-)^{\sigma^k}$ are maximal σ^{j_i}-stable connected unipotent subgroups of $\overline{G}_{i_k}$ which generate $\overline{G}_{i_k}$.

Thus $\overline{U}_{i_1} \times (\overline{U}_{i_1})^\sigma \times \ldots \times (\overline{U}_{i_1})^{\sigma^{j_i - 1}}$ and $\overline{U}_{i_1}^- \times (\overline{U}_{i_1}^-)^\sigma \times \ldots \times (\overline{U}_{i_1}^-)^{\sigma^{j_i - 1}}$ are maximal σ-stable connected unipotent subgroups of $\overline{G}_{i_1} * \ldots * \overline{G}_{i_{j_i}}$ which generate $\overline{G}_{i_1} * \ldots * \overline{G}_{i_{j_i}}$. By [19, Cor. 12.3(a)], we have

$$\begin{aligned} O^{p'}((\overline{G}_{i_1} * \ldots * \overline{G}_{i_{j_i}})_\sigma) &= \langle (\overline{U}_{i_1} \times (\overline{U}_{i_1})^\sigma \times \ldots \times (\overline{U}_{i_1})^{\sigma^{j_i - 1}})_\sigma, \\ &\qquad (\overline{U}_{i_1}^- \times (\overline{U}_{i_1}^-)^\sigma \times \ldots \times (\overline{U}_{i_1}^-)^{\sigma^{j_i - 1}})_\sigma \rangle \\ &\simeq \langle (\overline{U}_{i_1})_{\sigma^{j_i}}, (\overline{U}_{i_1}^-)_{\sigma^{j_i}} \rangle = O^{p'}((\overline{G}_{i_1})_{\sigma^{j_i}}). \end{aligned}$$

In view of [19, 11.6 and Cor. 12.3], $\langle(\overline{U}_{i_1})_{\sigma^{j_i}}, (\overline{U}^-_{i_1})_{\sigma^{j_i}}\rangle$ is a canonical finite group of Lie type. Moreover, the above argument implies that $\langle(\overline{U}_{i_1})_{\sigma^{j_i}}, (\overline{U}^-_{i_1})_{\sigma^{j_i}}\rangle/Z(\langle(\overline{U}_{i_1})_{\sigma^{j_i}}, (\overline{U}^-_{i_1})_{\sigma^{j_i}}\rangle)$ and $O^{p'}((\mathbf{P}\overline{G}_{i_1})_{\sigma^{j_i}})$ are isomorphic. Denote $O^{p'}((\overline{G}_{i_1} * \ldots * \overline{G}_{i_{j_i}})_\sigma)$ by G_i. We see that G_i is a canonical finite group of Lie type, for all i. The subgroups G_i of $O^{p'}(\overline{G}_\sigma)$ appearing in so doing are called *subsystem subgroups* of $O^{p'}(\overline{G}_\sigma)$.

Since $\overline{G}_{i_1} * \ldots * \overline{G}_{j_i}$ is a σ-stable subgroup, $\overline{G}_{i_1} * \ldots * \overline{G}_{j_i} \cap \overline{T}$ is a σ-stable maximal torus of $\overline{G}_{i_1} * \ldots * \overline{G}_{j_i}$. Hence we may assume that $\overline{T} \cap \overline{G}_{i_1} * \ldots * \overline{G}_{i_{j_i}}$ is a maximal σ-stable torus in $\overline{G}_{i_1} * \ldots * \overline{G}_{i_{j_i}}$, for any σ-orbit $\{\overline{G}_{i_1}, \ldots, \overline{G}_{i_{j_i}}\}$. We have $\overline{R}_\sigma = \overline{T}_\sigma(G_1 * \ldots * G_m)$, and $\overline{T}_\sigma$ normalizes each of the subgroups G_i.

For the σ-orbit $\{\overline{G}_{i_1}, \ldots, \overline{G}_{i_{j_i}}\}$ of $\overline{G}_{i_1}$ with $G_i = O^{p'}((\overline{G}_{i_1} * \ldots * \overline{G}_{i_{j_i}})_\sigma)$, we consider $\mathrm{Aut}_{\overline{R}_\sigma}(G_i)$. Since $G_1 * \ldots * G_{i-1} * G_{i+1} * \ldots * G_k * \overline{Z}_\sigma \leq C_{\overline{R}_\sigma}(G_i)$, we have $\mathrm{Aut}_{\overline{R}_\sigma}(G_i) \simeq \left(\overline{T}_\sigma G_i\right)/Z\left(\overline{T}_\sigma G_i\right)$. From [18, Prop. 2.6.2], it follows that the automorphisms induced by $\overline{T}_\sigma$ on G_i are diagonal. Therefore the inclusions $\mathbf{P}G_i \leq \mathrm{Aut}_{\overline{R}_\sigma}(G_i) \leq \widehat{\mathbf{P}G_i}$ hold; specifically, $\mathrm{Aut}_{\overline{R}_\sigma}(G_i)$ is a finite group of Lie type.

Let $\overline{R}$ be a σ-stable connected reductive subgroup of maximal rank (in particular, $\overline{R}$ can be a maximal torus) in G. The groups $N_{\overline{G}}(\overline{R})/\overline{R}$ and $N_W(W_{\overline{R}})/W_{\overline{R}}$ being isomorphic yields an induced action of σ on $N_W(W_{\overline{R}})/W_{\overline{R}}$, and we say that $w_1 \equiv w_2$ for $w_1, w_2 \in N_W(W_{\overline{R}})/W_{\overline{R}}$ if there exists an element $w \in N_W(W_{\overline{R}})/W_{\overline{R}}$ for which $w_1 = w^{-1}w_2w^\sigma$. Let $Cl(\overline{G}_\sigma, \overline{R})$ be a set of $\overline{G}_\sigma$-conjugacy classes of σ-stable subgroups $\overline{R}^g$, where $g \in \overline{G}$. Then $Cl(\overline{G}_\sigma, \overline{R})$ is in 1-1 correspondence with the set of σ-conjugacy classes $Cl(N_W(W_{\overline{R}})/W_{\overline{R}}, \sigma)$, where W is a Weyl group of $\overline{G}$ and $W_{\overline{R}}$ is one of $\overline{R}$ (and is a subgroup of W). If w is an element of $N_W(W_{\overline{R}})/W_{\overline{R}}$, and $(\overline{R}^g)_\sigma$ corresponds to a σ-conjugacy class of w, then we say that $(\overline{R}^g)_\sigma$ is obtained by "twisting" the group $\overline{R}$ with an element $w\sigma$. Moreover, $(\overline{R}^g)_\sigma \simeq \overline{R}_{\sigma w}$. For more details about twisting, we ask the reader to consult [23].

LEMMA 2.1. Let $\overline{G}$ be a simple connected linear algebraic group over a field of characteristic p and $t \in G$ be an element of order r, not divisible by p. Then $C_{\overline{G}}(t)/C_{\overline{G}}(t)^0$ is a $\pi(r)$-group.

Proof. Since p does not divide r, t is semisimple. Hence $C_{\overline{G}}(t)^0$ is a connected reductive subgroup of maximal rank in $\overline{G}$, and every p-element of $C_{\overline{G}}(t)$ is contained in $C_{\overline{G}}(t)^0$ (see [22, Thm. 2.2]). Assume that there exists a prime $s \notin \pi(r)$ dividing the order $|C_{\overline{G}}(t)/C_{\overline{G}}(t)^0|$. Then $s \neq p$, and for some natural $k > 0$, the centralizer $C_{\overline{G}}(t)$ contains an element x of order s^k such that $x \notin C_{\overline{G}}(t)^0$. Since x and t commute, $x \cdot t$ is a semisimple element of $\overline{G}$ of order rs^k. Therefore there exists a maximal torus $\overline{T}$ of $\overline{G}$ for which $x \cdot t \in \overline{T}$. It follows that $(xt)^r = x^r \in \overline{T}$. Since $(s, r) = 1$, there is m such that $rm \equiv 1 \pmod{s^k}$, whence $(x^r)^m = x \in \overline{T}$. The fact that $xt, x \in \overline{T}$ implies $t \in \overline{T}$, so $\overline{T} \leq C_{\overline{G}}(t)^0$, and hence $x \in C_{\overline{G}}(t)^0$, a contradiction. □

Recall that an element x of a linear algebraic group $\overline{G}$ is said to be *regular* if its centralizer has the minimal possible dimension. In particular, if x is semisimple and $\overline{G}$ is connected and reductive, then x is *regular* if the connected component of its centralizer is a maximal torus in $\overline{G}$.

Suppose now that $\overline{R}$ is a σ-stable parabolic subgroup of $\overline{G}$ and $\overline{U}$ is its unipotent radical. Then $\overline{R}$ contains a connected reductive subgroup $\overline{L}$ such that $\overline{R}/\overline{U} \simeq \overline{L}$. The subgroup $\overline{L}$ is called a *Levi factor* of $\overline{R}$. Moreover, if $\overline{Z} = Z(\overline{L})^0$, then $\overline{L} = C_{\overline{G}}(\overline{Z})$ (see [13, 30.2]). Let $Rad(\overline{R})$ be the radical of $\overline{R}$. Then $Rad(\overline{R})$ is a σ-stable connected solvable subgroup, and it contains a σ-stable torus $\overline{Z}$ by [19, 10.10]. Now $C_{\overline{G}}(\overline{Z}) = C_{\overline{R}}(\overline{Z})$ is a σ-stable Levi factor of $\overline{R}$, that is, every σ-stable parabolic subgroup of $\overline{G}$ contains a σ-stable Levi factor $\overline{L}$ and $\overline{L}$ is a connected reductive subgroup of maximal rank in $\overline{G}$.

LEMMA 2.2 (Hartley–Shute lemma) [24, Lemma 2.2]. Let $G = O^{p'}(\overline{G}_\sigma)$ be a finite canonical adjoint group of Lie type with definition field $GF(q)$, H be a Cartan subgroup of G, and $s \in GF(q)$. If $r = \bar{r}$ and the group G is twisted, then we also assume that s is contained in the base field of G. Then there exists an element $h(\chi) \in H$ such that $\chi(r) = s$ except for the cases below where $h(\chi)$ is chosen so that $\chi(r)$ takes up the following values:

(a) $G = A_1(q)$, with $\chi(r) = s^2$;

(b) $G = C_n(q)$ and r is a long root, with $\chi(r) = s^2$;

(c) $G = {}^2A_2(q^2)$ and $r \neq \bar{r}$, with $\chi(r) = s^3$;

(d) $G = {}^2A_3(q^2)$ and $r \neq \bar{r}$, with $\chi(r) = s^2$;

(e) $G = {}^2D_n(q^2)$ and $r \neq \bar{r}$, with $\chi(r) = s^2$;

(f) $G = {}^2G_2(3^{2n+1})$ and $r = a$ or $r = 3a + b$, where a is a short fundamental root and b is a long one, with $\chi(r) = s^2$.

LEMMA 2.3. Let $O^{p'}(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma$ be a finite adjoint group of Lie type over a field of odd characteristic p and let the root system Φ of $\overline{G}$ be one of the following: A_n $(n \geqslant 2)$, D_n $(n \geqslant 4)$, B_n $(n \geqslant 3)$, G_2, F_4, E_6, E_7, or E_8. Assume $G \not\simeq {}^2G_2(3^{2n+1})$. Suppose U is a maximal unipotent subgroup of G, H is a Cartan subgroup of G which normalizes U, and Q is a Sylow 2-subgroup of H. Then $C_U(Q) = \{e\}$.

Proof. Clearly, we need only prove the lemma for the case $G = O^{p'}(\overline{G}_\sigma) = O^{p'}(G)$, that is, we may assume that G is a canonical adjoint group of Lie type. If G is split or $G \simeq {}^2D_n(q^2)$, then the lemma follows from [10, Lemma 2.8]. Suppose that $G \simeq {}^2A_n(q^2)$ or $G \simeq {}^2E_6(q^2)$; then $\Phi(\overline{G})$ is equal to A_n and E_6, respectively. Denote by $\bar{r}$ the image of a root r in Φ under a suitable symmetry. In terms of [12], the root system $\Phi(\overline{G})$ is a union of equivalence classes Ψ_i, where each Ψ_i has either type A_1, or $A_1 \times A_1$, or A_2.

By [12, Prop. 13.6.1], $U = \prod\limits_i X_{\Psi_i}$, where

$$X_{\Psi_i} = \{x_r(t) \mid t \in GF(q)\}$$

if $\Psi_i = \{r\}$ has type A_1 (in which case $r = \bar{r}$);

$$X_{\Psi_i} = \{x_r(t)x_{\bar{r}}(t^q) \mid t \in GF(q^2)\}$$

if $\Psi_i = \{r, \bar{r}\}$ has type $A_1 \times A_1$ (in which case $r \neq \bar{r}$ and $r + \bar{r} \notin \Phi(\overline{G})$);

$$X_{\Psi_i} = \{x_r(t)x_{\bar{r}}(t^q)x_{r+\bar{r}}(u) \mid t \in GF(q^2),\ u + u^q = -N_{r,\bar{r}}tt^q\}$$

if $\Psi_i = \{r, \bar{r}, r + \bar{r}\}$ has type A_2 (in which case $r \neq \bar{r}$ and $r + \bar{r} \in \Phi(\overline{G})$).

Now if $h(\chi)$ is an element of H, then the following equalities hold (see [12, item (b) in the proof of Thm. 14.4.1]):

$$h(\chi)x_r(t)h(\chi)^{-1} = x_r(\chi(r)t)$$

if $r = \bar{r}$ and $\Psi_i = \{r\}$ is of type A_1;

$$h(\chi)x_r(t)x_{\bar{r}}(t^q)h(\chi)^{-1} = x_r(\chi(r)t)x_{\bar{r}}(\chi(\bar{r})t^q)$$

if $r \neq \bar{r}$, $r + \bar{r} \notin \Phi(\overline{G})$, and $\Psi_i = \{r, \bar{r}\}$ is of type $A_1 \times A_1$;

$$h(\chi)x_r(t)x_{\bar{r}}(t^q)x_{r+\bar{r}}(u)h(\chi)^{-1} = x_r(\chi(r)t)x_{\bar{r}}(\chi(\bar{r})t^q)x_{r+\bar{r}}(\chi(r+\bar{r})u)$$

if $r \neq \bar{r}$, $r + \bar{r} \in \Phi(\overline{G})$, and $\Psi_i = \{r, \bar{r}, r + \bar{r}\}$ is of type A_2.

Let u be a non-trivial element of $C_U(Q)$. Then u contains a non-trivial multiplier of X_{Ψ_i}, for some i. We may assume that $u \in X_\Psi$ since factoring into $\prod\limits_i X_{\Psi_i}$ is unique (see [12, Prop. 13.6.1]).

Suppose that Ψ has type A_1, that is, $u = x_r(t)$, $t \in GF(q)$, and $r = \bar{r}$. By Lemma 2.2, for every $s \in GF(q)$, there exists $h(\chi) \in H$ such that $\chi(r) = s$. Take $s = -1$. Then there is $h(\chi) \in H$ for which $\chi(r) = -1$. Since $h(\chi)^2 = h(\chi^2)$ (see a formula at p. 98 in [12]), $\chi^2(r) = 1$, that is, $|h(\chi)^2| < |h(\chi)|$.

Hence the order $|h(\chi)|$ is even, and we may factor $h(\chi)$ into a product of its 2- and 2′-parts and write $h(\chi) = h_2 \cdot h_{2'} = h(\chi_1) \cdot h(\chi_2)$. Now $\chi(r) = \chi_1(r) \cdot \chi_2(r)$; consequently $\chi_1(r) = -1$ and $\chi_2(r) = 1$. Therefore $h(\chi_1)x_r(t)h(\chi_1)^{-1} = x_r(-t) \neq x_r(t)$. Hence the case where $u = x_r(t)$ and $\Psi = \{r\}$ has type A_1 is impossible.

Assume that $\Psi = \{r, \bar{r}\}$ is of type $A_1 \times A_1$. By Lemma 2.2, for every $s \in GF(q^2)$, there is $h(\chi) \in H$ such that $\chi(r) = s^2$. There exists $s \in GF(q^2)$ for which $s^2 = -1$, and so there exists $h(\chi) \in H$ such that $\chi(r) = -1$. As above, $h(\chi)$ can be represented as $h(\chi_1) \cdot h(\chi_2)$, a product of its 2- and 2′-parts. Consequently $\chi_1(r) = -1$, and so

$$h(\chi_1)x_r(t)x_{\bar{r}}(t^q)h(\chi_1)^{-1} = x_r(-t)x_{\bar{r}}(-t^q) \neq x_r(t)x_{\bar{r}}(t^q).$$

Thus the case where $u = x_r(t)x_{\bar{r}}(t^q)$ and $\Psi = \{r, \bar{r}\}$ has type $A_1 \times A_1$ is impossible.

Lastly suppose that $\Psi = \{r, \bar{r}, r + \bar{r}\}$ is of type A_2. By Lemma 2.2, for every $s \in GF(q^2)$, there exists $h(\chi) \in H$ such that $\chi(r) = s^3$. Choose $s = -1$; then there is $h(\chi) \in H$ for which $\chi(r) = -1$. Again $h(\chi) = h(\chi_1) \cdot h(\chi_2)$ can be represented as a product of its 2- and 2′-parts, and $\chi_1(r) \neq 1$. It follows that

$$\begin{aligned} h(\chi_1)x_r(t)x_{\bar{r}}(t^q)x_{r+\bar{r}}(u)h(\chi_1)^{-1} &= x_r(-t)x_{\bar{r}}(\chi_1(-t^q)x_{r+\bar{r}}(\chi_1(r+\bar{r})u) \\ &\neq x_r(t)x_{\bar{r}}(t^q)x_{r+\bar{r}}(u) \end{aligned}$$

for $t \neq 0$. If $t = 0$, then we choose s so that $s^2 = -1$. Consequently $\chi_1(r + \bar{r}) = -1$, and we arrive at an inequality as above. Hence this last case is also impossible.

In view of Lemma 2.2, we can use a similar argument to prove the lemma for the remaining cases — $G \simeq {}^3D_4(q^3)$, $G \simeq G_2(q)$, and $G \simeq F_4(q)$. Our further reasoning does not make use of Lemma 2.3 for the groups in hand, and so we do not give a detailed proof for these. □

LEMMA 2.4. Let $O^{p'}(\overline{G}_\sigma) = G$ be a canonical finite adjoint group of Lie type over a field of odd characteristic p and let -1 not be a square in the base field of G. Assume that the root system Φ of $\overline{G}$ is equal to C_n. Suppose U is a maximal unipotent subgroup of G, H is a Cartan subgroup of G which normalizes U, and Q is a Sylow 2-subgroup of H. Then $C_U(Q) = \langle X_r \mid r \text{ is a long root}\rangle$.

Proof. If r is a short root, then there exists a root s with $\langle s, r\rangle = 1$. Thus $x_r(t)^{h_s(-1)} = x_r((-1)^{\langle s,r\rangle}t) = x_r(-t)$ (cf. [12, Prop. 6.4.1]). Therefore if $x \in C_U(Q)$ and $x_r(t)$ is a non-trivial multiplier in the representation (1) for x, then r is a long root. Now if r is long, then either $|\langle s, r\rangle| = 2$ or $\langle s, r\rangle = 0$ for any root s, that is, $x_r(t)^{h_s(-1)} = x_r(t)$. Since -1 is not a square in the base field of G (i.e., in $GF(q)$), we have $q \equiv -1 \pmod 4$, whence $\langle h_s(-1) \mid s \in \Phi\rangle = Q$. □

LEMMA 2.5. Let $G = PSp_{2n}(q)$ be a simple canonical group of Lie type, J be a subset of the set of fundamental roots containing r_n as a long fundamental root, P_J be a parabolic subgroup generated by the Borel subgroup B and by groups X_r with $-r \in J$, and L be a Levi factor of P_J. Denote by S a quasisimple normal subgroup of L isomorphic to $Sp_{2k}(q)$ (such always exists since $r_n \in J$). Then $\mathrm{Aut}_L(S/Z(S)) = S/Z(S)$.

Proof. This statement is known; it was proved in an unpublished paper by N. A. Vavilov. We give a proof here for completeness. As noted, L is a reductive subgroup of maximal rank in G; so $S/Z(S) \leq \mathrm{Aut}_L(S/Z(S)) \leq \widehat{S/Z(S)}$. For q even, the statement is obvious, since $|\widehat{C_n(q)} : C_n(q)| = (2, q-1)$. If q is odd, then there are only two possibilities for $\mathrm{Aut}_L(S/Z(S))$: either $\mathrm{Aut}_L(S/Z(S)) = S/Z(S)$, or $\mathrm{Aut}_L(S/Z(S)) = \widehat{S/Z(S)}$. We claim that the second equality is impossible.

In our notation, fundamental roots in the root system of S are $r_{n-k+1}, \dots, r_n$. If $\mathrm{Aut}_L(S/Z(S)) = \widehat{S/Z(S)}$, then there exist elements $s_1, \dots, s_k$ of $\mathbb{Z}\Phi = \mathbb{Z}C_n$ such that

$$\langle s_i, r_{n-k+j}\rangle = \frac{(s_i, r_{n-k+j})}{(s_i, s_i)} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

(These generate a lattice of fundamental weights, and so allow us to obtain all diagonal automorphisms of S.) For each root r of C_n, either $\langle r, r_n\rangle = 0$, or $\langle r, r_n\rangle = \pm 2$, that is, for each element $s \in \mathbb{Z}\Phi$, the number $\langle s, r_n\rangle$ is even and, in particular, is distinct from 1. Therefore such a set of elements $s_1, \dots, s_k$ does not exist. □

LEMMA 2.6. Let G be a finite group of Lie type over a field of odd characteristic and let $\overline{G}$ and σ be chosen so that $O^{p'}(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma$. If G satisfies **(ESyl2)**, then every group L with $G \leq L \leq \overline{G}_\sigma$ will satisfy **(ESyl2)**.

Proof. Let Q be a Sylow 2-subgroup of $\overline{G}_\sigma$ and $Q^0 = O^{p'}(\overline{G}_\sigma) \cap Q$ be one of $O^{p'}(\overline{G}_\sigma)$. If $N_{\overline{G}_\sigma}(Q^0) = QC_{\overline{G}_\sigma}(Q)$, then the statement of the lemma is clearly true. In view of [25, Thm. 1], for a classical group $\overline{G}_\sigma$, the equality $N_{\overline{G}_\sigma}(Q^0) = QC_{\overline{G}_\sigma}(Q)$ may fail only if the root system of $\overline{G}$ has type A_1, or C_n. If the root system of $\overline{G}$ is of type A_1 or C_n, then $|\overline{G}_\sigma : O^{p'}(\overline{G}_\sigma)| = 2$ and the lemma follows from Lemma 1.9.

Assume now that G is a group of exceptional type. If $\overline{G}_\sigma = O^{p'}(\overline{G}_\sigma)$, then the statement is obviously true. The equality $N_{\overline{G}_\sigma}(Q^0) = QC_{\overline{G}_\sigma}(Q)$ might fail only if the root system of $\overline{G}$ has type E_6, or E_7. If the root system of $\overline{G}$ is of type E_7, then $|\overline{G}_\sigma : O^{p'}(\overline{G}_\sigma)| = 2$ and the lemma follows from Lemma 1.9.

Suppose that the root system of $\overline{G}$ has type E_6. Then either $\overline{G}_\sigma = O^{p'}(\overline{G}_\sigma)$ or $|\overline{G}_\sigma : O^{p'}(\overline{G}_\sigma)| = 3$. In the former case there is nothing to prove, and so we let $|\overline{G}_\sigma : O^{p'}(\overline{G}_\sigma)| = 3$. Since the group G coincides with $\overline{G}_\sigma$ or with $O^{p'}(\overline{G}_\sigma)$, and the case where $G = \overline{G}_\sigma$ is trivial, we may suppose that $G = O^{p'}(\overline{G}_\sigma)$. By [18, Thm. 4.10.2], there exists a maximal torus T of $\overline{G}_\sigma$ such that Q is contained in $N(\overline{G}_\sigma, T)$. Since $|\overline{G}_\sigma : G| = 3$, we have $Q = Q^0 \leq N(G, T \cap G)$. By [26, Thm. 6], $N_G(Q) = Q \times R^0$, where $R^0 \leq T$ is a cyclic group of odd order.

Now since $\overline{G}_\sigma = TG$, we have $N_{\overline{G}_\sigma}(Q) = \langle N_T(Q), N_G(Q)\rangle$. Indeed, $N(G, T\cap G)/(T\cap G) \simeq N(G,T)/T$. Hence a Sylow 2-subgroup QT/T of $N(G,T)/T$ coincides with its normalizer. The factor group $\overline{G}_\sigma/G$ is cyclic of order 3; so $N_{\overline{G}_\sigma}(Q) = \langle tg, N_G(Q)\rangle$, where $t \in T$ and $g \in G$. Moreover, $|\overline{G}_\sigma : G| = 3$, and we can therefore assume that tg is an element of order 3^k, for some $k > 0$. Since $t \in T \leq N(\overline{G}_\sigma, T)$, $Q^t \leq N(G, T \cap G)$. Hence there exists an element $g_1 \in N(G, T\cap G)$ such that $Q^t = Q^{g_1^{-1}}$. Therefore we may suppose that $tg = tg_1 \in N(\overline{G}_\sigma, T)$. Under the natural epimorphism $\pi : N(\overline{G}_\sigma, T) \to N(\overline{G}_\sigma, T)/T$, the image of $N_{N(\overline{G}_\sigma,T)}(Q)$ coincides with Q. Hence $(tg)^\pi = e$, and so $tg \in T$. Thus each element of odd order in $\overline{G}_\sigma$ which normalizes Q lies in T. Since T is a torus, T is Abelian; hence the set of elements of $N_{\overline{G}_\sigma}(Q)$ of odd order form a normal subgroup R of $N_{\overline{G}_\sigma}(Q)$. Therefore $N_{\overline{G}_\sigma}(Q) = Q \times R$, that is, $\overline{G}_\sigma$ satisfies **(ESyl2)**. □

The following lemma follows immediately from [25, Thm. 1].

LEMMA 2.7. Let $O^{p'}(\overline{G}_\sigma) = G$ be a canonical finite group of Lie type, $\overline{G}$ be either of type A_1 or of type C_n, p be odd, and $q = p^\alpha$ be the order of the base field of G. Then G satisfies **(ESyl2)** if and only if $q \equiv \pm 1 \pmod 8$.

Note that Lemma 2.6, together with [25, Thm. 1] and [26, Thm. 6], implies that every group of Lie type over a field of odd characteristic, which is distinct from a Ree group and the groups specified in Lemma 2.7, satisfies **(ESyl2)**.

LEMMA 2.8. Let $O^{p'}(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma$ be a finite adjoint group of Lie type with base field of characteristic p and order q. Assume also that $O^{p'}(G)$ is not isomorphic to ${}^2D_{2n}(q^2)$, ${}^3D_4(q^3)$, ${}^2B_2(2^{2n+1})$, ${}^2G_2(3^{2n+1})$, and ${}^2F_4(2^{2n+1})$. Then there exists a maximal σ-stable torus $\overline{T}$ of $\overline{G}$ such that:

(a) $(N_{\overline{G}}(\overline{T})/\overline{T})_\sigma \simeq (N_{\overline{G}}(\overline{T}))_\sigma/(\overline{T}_\sigma) = N(\overline{G}_\sigma, \overline{T}_\sigma)/\overline{T}_\sigma \simeq W$, where W is the Weyl group of $\overline{G}$;

(b) if r is an odd prime divisor of $q - (\varepsilon 1)$, where $\varepsilon = +$ whenever G is split, and $\varepsilon = -$ whenever G is twisted, then $N(\overline{G}_\sigma, \overline{T}_\sigma)$ contains a Sylow r-subgroup of $\overline{G}_\sigma$;

(c) if r is a prime divisor of $q - (\varepsilon 1)$, and s is an element of order r in G such that $C_{\overline{G}}(s)$ is connected, then s is, up to conjugation by an element of G, contained in $T = \overline{T}_\sigma \cap G$.

The torus $\overline{T}$ is unique up to conjugation in $O^{p'}(\overline{G}_\sigma)$, and $|\overline{T}_\sigma| = (q - \varepsilon 1)^n$, where n is the rank of $\overline{G}$.

Proof. Since $\overline{G}_\sigma = TO^{p'}(\overline{G}_\sigma)$ holds for every maximal torus T of $\overline{G}_\sigma$, there is no loss of generality in assuming that $G = \overline{G}_\sigma$. If G is split then the lemma can be proved readily. In this instance $\overline{T}$ is a maximal torus such that $\overline{T}_\sigma$ is a Cartan subgroup of $\overline{G}_\sigma$ (i.e., $\overline{T}$ is a maximal split torus), and so (a) is clear. Item (b) follows from [29, (10.1)]. Moreover, [29, (10.2)] implies that the order of $\overline{T}_\sigma$ is determined uniquely and is equal to $(q-1)^n$, where n is the rank of $\overline{G}$. By [27, F, Sec. 6], every element of order r in $\overline{T}$ is contained in $\overline{G}_\sigma$. Now there exists $g \in \overline{G}$ such that $s^g \in \overline{T}$, and hence $s^g \in G$. Since the centralizer of s is connected, elements s and s^g are conjugate in $\overline{G}$ iff they are conjugate in G; so s and s^g are conjugate in G, which yields (c). The data on classes of maximal tori given in [27, G; 28] implies that, up to conjugation by an element of G, there exists a unique torus $\overline{T}$ such that $|\overline{T}_\sigma| = (q-1)^n$.

Assume that $O^{p'}(G) \simeq {}^2A_n(q^2)$. Then $\overline{T}$ is a maximal torus for which $|\overline{T}_\sigma| = (q+1)^n$. Note that $\overline{T}_\sigma$ can be obtained from a maximal split torus by twisting with $w_0\sigma$. Using [15, Prop. 3.3.6], by direct calculations, we can show that $N(\overline{G}_\sigma, \overline{T}_\sigma)/\overline{T}_\sigma$ is isomorphic to $W(\overline{G})$ which is in turn isomorphic to Sym_{n+1}. The uniqueness follows from [16, Prop. 8]. To prove (b), we need to appeal to [29, (10.1)].

In order to verify (c), we first show that every element of order r in $\overline{T}$ is contained in G. Assume that t is an element of order r in $\overline{T}$. (Recall that r divides $q+1$ in this instance.) Let $\overline{H}$ be a σ-stable maximal split torus of $\overline{G}$. The torus $\overline{T}_\sigma$ is obtained from $\overline{H}$ by twisting with an element w_0, where $w_0 \in W(\overline{G})$ is a unique element mapping all positive roots to negative, and $\overline{T}_\sigma \simeq \overline{H}_{\sigma w_0}$. Let $r_1, \ldots, r_n$ be a set of fundamental roots in A_n. Then t, being an element of $\overline{H}$, can be written in the form $h_{r_1}(\zeta_1) \cdot \ldots \cdot h_{r_n}(\zeta_n)$. Now, for every i we have $\sigma w_0 : h_{r_i}(\lambda) \mapsto h_{-r_i}(\lambda^q) = h_{r_i}(\lambda^{-q})$, that is, $t^{\sigma w_0} = t^{-q}$. Since r divides $q+1$, we obtain $t^{q+1} = e$, that is, $t = t^{-q}$. Hence $t^{\sigma w_0} = t$ and $t \in \overline{T}_\sigma$. As in the untwisted case, there exists an element $g \in \overline{G}$ such that $s^g \in \overline{T}$; hence $s^g \in \overline{T}_\sigma$. Since $C_{\overline{G}}(s)$ is connected, elements s and s^g are conjugate in G.

For $O^{p'}(G) = {}^2D_{2n+1}(q^2)$, we take $\overline{T}$ to be a unique (up to conjugation in G) maximal torus of order $|\overline{T}_\sigma| = (q+1)^{2n+1}$ (whose uniqueness follows from [16, Prop. 10]). For $O^{p'}(G) = {}^2E_6(q^2)$, we choose $\overline{T}$ to be a unique (again up to conjugation in G) maximal torus of order $|\overline{T}_\sigma| = (q+1)^6$ (whose uniqueness follows from [17, Table 1, p. 128]). As with $G = {}^2A_n(q^2)$, it is easy to show that $\overline{T}$ satisfies items (a), (b), and (c) of the lemma. □

LEMMA 2.9. Let G be a finite group of Lie type and $\overline{G}$ and σ be chosen so that $O^{p'}(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma$. If s is a regular semisimple element of odd prime order r in G, then $N_G(C_{\overline{G}}(s)) \neq C_G(s)$.

Proof. In view of [22, Prop. 2.10], the factor group $C_{\overline{G}}(s)/C_{\overline{G}}(s)^0$ is isomorphic to a subgroup of $\Delta(\overline{G})$. Now, if the root system Φ of $\overline{G}$ is not equal to A_n, or E_6, then $|\Delta(\Phi)|$ is a power of 2. Since $\Delta(\overline{G})$ is a factor of $\Delta(\Phi(\overline{G}))$, Lemma 2.1 implies that $C_{\overline{G}}(s) = C_{\overline{G}}(s)^0 = \overline{T}$ is a maximal torus and $C_G(s) = C_{\overline{G}}(s) \cap G = T$. We have $N_G(T) \geq N(G, T) \neq T$, which yields the statement of the lemma for the present case. We may therefore assume that either $\Phi = A_n$ or $\Phi = E_6$.

Suppose first that $\Phi = A_n$, that is, $O^{p'}(G) = A_n^{\varepsilon}(q)$, where $\varepsilon \in \{+,-\}$. Clearly, $T = C_{\overline{G}}(s)^0 \cap G$ is a normal subgroup of $C_G(s)$, and hence $C_G(s) \leq N(G,T)$. Assume that $N_G(C_{\overline{G}}(s)) = C_G(s)$. Then $C_G(s) = N_{N(G,T)}(C_{\overline{G}}(s))$ and $C_G(s)/T$ is a self-normalizing subgroup of $N(G,T)/T$. As noted, $C_G(s)/T$ is isomorphic to a subgroup of $\Delta(A_n)$; that is, it is cyclic. Lemma 2.1 also implies that $C_G(s)/T$ is an r-group; in other words, $C_G(s)/T = \langle x\rangle$ for some r-element $x \in N(G,T)/T$. Thus $\langle x\rangle$ is a Carter subgroup of $N(G,T)/T$.

In view of [15, Prop. 3.3.6], we have $N(G,T)/T \simeq C_{W(\overline{G})}(y)$ for some $y \in W(\overline{G}) \simeq \mathrm{Sym}_{n+1}$. Clearly, $C_{C_{W(\overline{G})}(y)}(x)$ contains y. Therefore y must be an r-element, since otherwise $N_{C_{W(\overline{G})}(y)}(\langle x\rangle)$ would contain an element y of order coprime to r; that is, $N_{C_{W(\overline{G})}(y)}(\langle x\rangle) \neq \langle x\rangle$. We have arrived at a contradiction with $\langle x\rangle$ being a Carter subgroup of $C_{W(\overline{G})}(y)$.

Now let $y = \tau_1 \cdot \ldots$ be a factorization of y into a product of independent cycles and $l_1, \ldots$ be lengths of the respective cycles $\tau_1, \ldots$. Assume first that m_1 cycles have equal length l_1, m_2 cycles have length l_2, etc. Let $m_0 = n+1-(l_1m_1 + \ldots + l_km_k)$. Then

$$C_{W(\overline{G})}(y) \simeq \left(Z_{l_1} \wr \mathrm{Sym}_{m_1}\right) \times \ldots \times \left(Z_{l_k} \wr \mathrm{Sym}_{m_k}\right) \times \mathrm{Sym}_{m_0},$$

where Z_{l_i} is a cyclic group of order l_i. If $m_j > 1$ for some $j \geqslant 0$, then there exists a normal subgroup N of $C_{W(\overline{G})}(y)$ such that $C_{W(\overline{G})}(y)/N \simeq \mathrm{Sym}_{m_j} \neq \{e\}$. In view of [9, Table; 10, Table], we see that Carter subgroups of S satisfying $\mathrm{Alt}_\ell \leq S \leq \mathrm{Aut}(\mathrm{Alt}_\ell)$ are conjugate, for all $\ell \geqslant 5$. Thus $C_{W(\overline{G})}(y)$ and N satisfy **(C)**, and $\langle x\rangle$ is a unique (up to conjugation) Carter subgroup of $C_{W(\overline{G})}(y)$. By Lemma 1.4, $\langle x\rangle$ maps onto a Carter subgroup of $C_{W(\overline{G})}(y)/N \simeq \mathrm{Sym}_{m_j}$. In view of [2], only a Sylow 2-subgroup of Sym_{m_j} can be a Carter subgroup of Sym_{m_j}. We have arrived at a contradiction with the fact that x is an r-element and r is odd.

Thus we may assume that $C_{W(\overline{G})}(y) \simeq (Z_{l_1} \times \ldots \times Z_{l_k})$ and $l_i \neq l_j$ if $i \neq j$. We know how maximal tori and their normalizers in $A_n^{\varepsilon}(q)$ are structured (see, e.g., [16, Props. 7, 8]). With this in mind, we look into the structure of T and $N(G,T)$, of which we gain an idea by using matrices. Below, $GL_n^{\varepsilon}(q)$ is a group isomorphic to $GL_n(q)$ if $\varepsilon = +$, and to $GU_n(q)$ if $\varepsilon = -$. Given the decomposition $l_1 + l_2 + \ldots + l_k = n+1$ in $GL_{n+1}^{\varepsilon}(q)$, we consider a group L consisting of block-diagonal matrices of the form

$$\begin{pmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & A_k \end{pmatrix},$$

where $A_i \in GL_{l_i}^{\varepsilon}(q)$. Then $L \simeq GL_{l_1}^{\varepsilon}(q) \times \ldots \times GL_{l_k}^{\varepsilon}(q)$. Denote $GL_{l_i}^{\varepsilon}(q)$ by G_i for brevity. In every group G_i, consider a Singer cycle T_i. We know that $N_{G_i}(T_i)/T_i$ is a cyclic group of order l_i and $N(G_i,T_i) = N_{G_i}(T_i)$. There exists a subgroup Z of $Z(SL_{n+1}^{\varepsilon}(q))$ such that $O^{p'}(G) \simeq SL_{n+1}^{\varepsilon}(q)/Z$. Consequently $T \simeq \left((T_1 \times \ldots \times T_k) \cap SL_{n+1}^{\varepsilon}(q)\right)/Z$ and $N(G,T) \simeq \left((N(G_1,T_1) \times \ldots \times N(G_k,T_k)) \cap SL_{n+1}^{\varepsilon}(q)\right)/Z$. Since the group $N(G_i,T_i)/T_i$ is cyclic for every Singer cycle T_i, we may assume that $N(G,T) = C_G(s)$ and T is a Singer cycle, that is, a cyclic group of order $\frac{q^{n+1}-(\varepsilon 1)^{n+1}}{q-(\varepsilon 1)}$, where $n+1 = r^k$ for some $k \geqslant 1$. (The last-mentioned equality holds since $N(G,T)/T$ is an r-group.) But $q^{r^k} \equiv q \pmod r$, and hence r divides $q-(\varepsilon 1)$. By Lemma 2.8, s is in $N(G,H)$, where H is a maximal torus such that the factor group $N(G,H)/H$ is isomorphic to Sym_{n+1}, and $|H| = (q-\varepsilon 1)^n$. In particular, H is not a Singer cycle.

For $s \in H$, this immediately yields a contradiction with the choice of s. If $s \notin H$, then the intersection $\langle s\rangle \cap H$ is trivial, since the order of s is prime. Hence, under the natural homomorphism $N(G,H) \to$

$N(G,H)/H \simeq \mathrm{Sym}_{n+1}$, the element s maps onto an element of order r. In Sym_{n+1}, however, every element of odd order is conjugate to its inverse. Therefore there exists a 2-element z of G normalizing but not centralizing $\langle s\rangle$. Therefore $z \leq N_{\overline{G}}(C_{\overline{G}}(s)) \leq N_{\overline{G}}(C_{\overline{G}}(s)^0)$ and $|N(G,T)/T|$ is divisible by 2, which clashes with the above statement that $N(G,T)/T$ is an r-group. The case $\Phi(\overline{G}) = A_n$ is described out.

For the case $\Phi = E_6$, it is easy to verify that for every $y \in W(E_6)$, the group $C_{W(E_6)}(y)$ does not contain Carter subgroups of order 3. Indeed, if $C_{W(E_6)}(y)$ has a Carter subgroup of order 3, then it is generated by y. However it is well known (and can be easily checked against [28, Table 9]) that $W(E_6)$ lacks in elements of order 3 whose centralizer, too, has order 3. Since $|C_G(s)/T|$ divides 3 and $C_G(s)/T$ is a Carter subgroup of $C_{W(E_6)}(y)$ for some y, we are led to a contradiction. □

3. SEMILINEAR GROUPS OF LIE TYPE

Now we define some overgroups of finite groups of Lie type. First we give a detailed description of a Frobenius map σ. Note that all maps in this section are automorphisms, if $\overline{G}$ is treated as an abstract group, and are endomorphisms if $\overline{G}$ is treated as an algebraic group. By reason of the fact that our maps are used to construct respective automorphisms of finite groups and of groups over an algebraically closed field, we will call all maps automorphisms.

Let $\overline{G}$ be a simple connected linear algebraic group over an algebraically closed field $\overline{\mathbb{F}}_p$ of positive characteristic p. Below, unless otherwise stated, we consider groups of adjoint type. Choose a Borel subgroup $\overline{B}$ of $\overline{G}$, letting $\overline{U} = R_u(\overline{B})$ be the unipotent radical of $\overline{B}$. There exists a Borel subgroup $\overline{B}^-$ with $\overline{B} \cap \overline{B}^- = \overline{T}$, where $\overline{T}$ is a maximal torus of $\overline{B}$ (and hence of $\overline{G}$).

Let Φ be the root system of $\overline{G}$ and $\{X_r \mid r \in \Phi^+\}$ be the set of $\overline{T}$-stable 1-dimensional root subgroups of $\overline{U}$. Every X_r is isomorphic to the additive group of $\overline{\mathbb{F}}_p$; so every element of X_r can be written in the form $x_r(t)$, where t is an image of $x_r(t)$ under this isomorphism. Denote by $\overline{U}^- = R_u(\overline{B}^-)$ the unipotent radical of $\overline{B}^-$. As above, define $\overline{T}$-stable 1-dimensional subgroups $\{\overline{X}_r \mid r \in \Phi^-\}$ of $\overline{U}^-$. Then $\overline{G} = \langle \overline{U}, \overline{U}^-\rangle$. Let $\bar{\varphi}$ be a field automorphism of $\overline{G}$ (treated as an abstract group) and $\bar{\gamma}$ be a graph automorphism of $\overline{G}$. It is known that $\bar{\varphi}$ can be chosen so as to act by the rule $x_r(t)^{\bar{\varphi}} = x_r(t^p)$ (see, e.g., [12, 12.2; 15, 1.7]). In view of [12, Props. 12.2.3 and 12.3.3], we can choose $\bar{\gamma}$ so that it acts by the rule $x_r(t)^{\bar{\gamma}} = x_{\bar{r}}(t)$ if Φ has no roots of distinct lengths, or by the rule $x_r(t)^{\bar{\gamma}} = x_{\bar{r}}(t^{\lambda_r})$, for appropriate $\lambda_r \in \{1,2,3\}$, if Φ has roots of distinct lengths. Here, $\bar{r}$ is an image of r under the symmetry ρ (corresponding to $\bar{\gamma}$) of the root system Φ. In both cases $x_r(t)^{\bar{\gamma}} = x_{\bar{r}}(t^{\lambda_r})$, where $\lambda_r \in \{1,2,3\}$. Obviously, $\bar{\varphi} \cdot \bar{\gamma} = \bar{\gamma} \cdot \bar{\varphi}$. Let $n_r(t) = x_r(t)x_{-r}(-t^{-1})x_r(t)$, $\overline{N} = \langle n_r(t) \mid r \in \Phi, t \in \overline{\mathbb{F}}_p^*\rangle$, $h_r(t) = n_r(t)n_r(-1)$, and $\overline{H} = \langle h_r(t) \mid r \in \Phi, t \in \overline{\mathbb{F}}_p^*\rangle$. In view of [12, Chaps. 6, 7], $\overline{H}$ is a maximal torus of $\overline{G}$ and $\overline{N} = N_{\overline{G}}(\overline{H})$ and $\overline{X}_r$ are root subgroups w.r.t. $\overline{H}$. Therefore we can substitute $\overline{H}$ for $\overline{T}$ and suppose that $\overline{T}$, by our choice, is $\bar{\varphi}$- and $\bar{\gamma}$-stable. Moreover, $\bar{\varphi}$ induces a trivial automorphism of $\overline{N}/\overline{H}$.

An automorphism $\bar{\varphi}^k$, $k \in \mathbb{N}$, is called a *classical Frobenius automorphism*. We refer to σ as a *Frobenius automorphism* if σ is conjugate in $\overline{G}$ to $\bar{\gamma}^\epsilon\bar{\varphi}^k$, with $\epsilon \in \{0,1\}$ and $k \in \mathbb{N}$. It follows from the Lang–Steinberg theorem [19, Thm. 10.1] that for any $\bar{g} \in \overline{G}$, elements σ and $\sigma\bar{g}$ are conjugate w.r.t. $\overline{G}$. In view of [19, 11.6], therefore, we see that a Frobenius map defined as in the previous section coincides with a Frobenius automorphism as defined here.

Now we fix $\overline{G}$, $\bar{\varphi}$, $\bar{\gamma}$, and $\sigma = \bar{\gamma}^\epsilon\bar{\varphi}^k$, assuming that $|\bar{\gamma}| \leqslant 2$, that is, we do not consider the triality automorphism of $\overline{G}$ with the root system $\Phi(\overline{G}) = D_4$. Put $B = \overline{B}_\sigma$, $H = \overline{H}_\sigma$, and $U = \overline{U}_\sigma$. Since $\overline{B}$, $\overline{H}$, and $\overline{U}$ are $\bar{\varphi}$- and $\bar{\gamma}$-stable, they give us a Borel subgroup, a Cartan subgroup, and a maximal unipotent subgroup (a Sylow p-subgroup) of $\overline{G}_\sigma$ (for details, see [15, 1.7-1.9; 18, Chap. 2]).

Assume first that $\epsilon = 0$, that is, $O^{p'}(\overline{G}_\sigma)$ is not twisted (it is split). Then $U = \langle X_r \mid r \in \Phi^+\rangle$, where X_r is isomorphic to the additive group of $GF(p^k) = GF(q)$ and every element of X_r can be written in the form $x_r(t)$, $t \in GF(q)$. Also let $U^- = \overline{U}_\sigma^-$. As for U, we put $U^- = \langle X_r \mid r \in \Phi^-\rangle$ and write every element of X_r in the form $x_r(t)$, $t \in GF(q)$. Now we can define an automorphism φ to be a restriction of $\bar{\varphi}$ to $\overline{G}_\sigma$ and define an automorphism γ to be a restriction of $\bar{\gamma}$ to $\overline{G}_\sigma$. By definition, $x_r(t)^\varphi = x_r(t^p)$ and $x_r(t)^\gamma = x_{\bar{r}}(t^{\lambda_r})$ for all $r \in \Phi$ (see definition of $\bar{\gamma}$ above).

We define $\zeta = \gamma^\varepsilon\varphi^\ell$, $\varphi^\ell \neq e$, $\varepsilon \in \{0,1\}$, to be an automorphism of $\overline{G}_\sigma$ and define $\bar{\zeta} = \bar{\gamma}^\varepsilon \cdot \bar{\varphi}^\ell$ to be an automorphism of $\overline{G}$. Choose a ζ-stable subgroup G with $O^{p'}(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma$. Note that if the root system Φ of $\overline{G}$ is not D_{2n}, then $\overline{G}_\sigma/(O^{p'}(\overline{G}_\sigma))$ is cyclic. Thus for most of the groups and automorphisms except groups of type D_{2n} over a field of odd characteristic, any subgroup G of $\overline{G}_\sigma$ satisfying $O^{p'}(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma$ is γ- and φ-stable. Define ΓG to be a set of subgroups of the form $\langle G, \zeta g\rangle \leq \overline{G}_\sigma \leftthreetimes \langle\zeta\rangle$, where $g \in \overline{G}_\sigma$, $\langle\zeta g\rangle \cap \overline{G}_\sigma \leq G$, and define $\Gamma\overline{G}$ to be a set of subgroups like $\overline{G} \leftthreetimes \langle\bar{\zeta}\rangle$. Following [18, Def. 2.5.13], we call ζ a *field* automorphism if $\varepsilon = 0$, that is, $\zeta = \varphi^\ell$, and call it a *graph-field* automorphism in all other cases (under the assumption that $\varphi^\ell \neq e$).

Suppose now that $\epsilon = 1$, that is, $O^{p'}(\overline{G}_\sigma)$ is twisted. Then $U = \overline{U}_\sigma$ and $U^- = \overline{U}_\sigma^-$. Define φ on $U^\pm$ to be a restriction of $\bar{\varphi}$ to $U^\pm$. Since $O^{p'}(\overline{G}_\sigma) = \langle U^+, U^-\rangle$, we obtain an automorphism φ of the group $O^{p'}(\overline{G}_\sigma)$. Consider $\zeta = \varphi^\ell \neq e$ and let G be a ζ-stable group with $O^{p'}(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma$. Then $\bar{\zeta} = \bar{\varphi}^\ell$ is an automorphism of $\overline{G}$. Define ΓG as a set of subgroups of the form $\langle G, \zeta g\rangle \leq \overline{G}_\sigma \leftthreetimes \langle\zeta\rangle$, where $g \in \overline{G}_\sigma$, $\langle\zeta g\rangle \cap \overline{G}_\sigma \leq G$, and define $\Gamma\overline{G}$ as a set of subgroups like $\overline{G} \leftthreetimes \langle\bar{\zeta}\rangle$. Following [18, Def. 2.5.13], we call ζ a *field* automorphism if $|\zeta|$ is not divisible by $|\gamma|$ (this definition will also be used in the case where $|\gamma| = 3$ and $\overline{G}_\sigma \simeq {}^3D_4(q^3)$), and call it a *graph* automorphism in all other cases.

Groups in the set ΓG defined above are called *semilinear finite groups of Lie type* (or else *semilinear canonical finite groups of Lie type* if $G = O^{p'}(\overline{G}_\sigma)$), and those in the set $\Gamma\overline{G}$ are referred to as *semilinear algebraic groups*. Note that $\Gamma\overline{G}$ cannot be defined without ΓG, since we need to know that $\varphi^\ell \neq e$. If G is written in the notation of [12], that is, $O^{p'}(G) = G = A_n(q)$ or $O^{p'}(G) = G = {}^2A_n(q^2)$, etc., then we denote ΓG by $\Gamma A_n(q)$, $\Gamma^2A_n(q^2)$, etc.

Consider $A \in \Gamma G$ and $x \in A \setminus G$. Then $x = \zeta^k y$ for some $k \in \mathbb{N}$ and $y \in \overline{G}_\sigma$. Set $\bar{x}$ equal to $\bar{\zeta}^k y$. Conversely, if $\bar{x} = \bar{\zeta}^k y$ for some $y \in \overline{G}_\sigma$, $\zeta^k \neq e$, and $\langle\zeta^k y\rangle \cap \overline{G}_\sigma \leq G$, then we set x equal to $\zeta^k y$. Note that we do not need to suppose that $\bar{x} \notin \overline{G}$ since $|\bar{\zeta}| = \infty$. If $x \in G$ then we put $\bar{x} = x$.

LEMMA 3.1. Let X be a subgroup of G. Then x normalizes X if and only if $\bar{x}$ normalizes X as a subgroup of $\overline{G}$.

Proof. Since ζ is the restriction of $\bar{\zeta}$ to G, our statement is trivial. □

Let X_1 be a subgroup of $A \in \Gamma G$. Then X_1 is generated by a normal subgroup $X = X_1 \cap G$ and by an element $x = \zeta^k y$. In view of Lemma 3.1, we can consider the subgroup $\overline{X}_1 = \langle\bar{x}, X\rangle$ of $\overline{G} \leftthreetimes \langle\bar{\zeta}\rangle$. Now we explain why we use such complicated notation and complex definitions. The order of ζ is always finite, whereas $\bar{\zeta}$ is invariably of infinite order. Even if $Z(G)$ is trivial, therefore, we cannot conceive of $G \leftthreetimes \langle\bar{\zeta}\rangle$ as a subgroup of $\mathrm{Aut}(G)$. Hence, in order to use the machinery of linear algebraic groups, we have to somehow define (one possible way has been just given) a connection between elements of $\mathrm{Aut}(G)$ and those of $\mathrm{Aut}(\overline{G})$.

Let $\overline{R}$ be a σ-stable maximal torus (a reductive subgroup of maximal rank, a parabolic subgroup) of $\overline{G}$ and $y \in N_{\overline{G}\leftthreetimes\langle\bar{\zeta}\rangle}(\overline{R})$ be chosen so that there exists $x \in \langle G, \zeta g\rangle$ with $y = \bar{x}$. Then $R_1 = \langle x, \overline{R} \cap G\rangle$ is called a *maximal torus* (a *reductive subgroup of maximal rank*, a *parabolic subgroup*) of $\langle G, \zeta g\rangle$.

LEMMA 3.2. Let $M = \langle x, X\rangle$, where $X = M \cap G \trianglelefteq M$, be a subgroup of $\langle G, \zeta g\rangle$ such that $O_p(X)$

is non-trivial. Then there exists a σ- and $\bar{x}$-stable parabolic subgroup $\overline{P}$ of $\overline{G}$ for which $X \leq \overline{P}$ and $O_p(X) \leq R_u(\overline{P})$.

Proof. Let $U_0 = O_p(X)$ and $N_0 = N_{\overline{G}}(U_0)$. Then $U_i = U_0 R_u(N_{i-1})$ and $N_i = N_{\overline{G}}(U_i)$. Clearly, U_i and N_i are $\bar{x}$- and σ-stable subgroups, for all i. In view of [13, Prop. 30.3], the chain $N_0 \leq N_1 \leq \ldots \leq N_k \leq \ldots$ is finite and $\overline{P} = \bigcup_i N_i$ is a proper parabolic subgroup of $\overline{G}$. Obviously, $\overline{P}$ is σ- and $\bar{x}$-stable, $X \leq \overline{P}$, and $O_p(X) \leq R_u(\overline{P})$. □

LEMMA 3.3. Let G be a finite group of Lie type over a field of odd characteristic p. Assume that $\overline{G}$ and σ are chosen so that $O^{p'}(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma$. Suppose that ψ is a field automorphism of $O^{p'}(\overline{G}_\sigma)$ of odd order. Then ψ centralizes a Sylow 2-subgroup of G, and there exists a ψ-stable Cartan subgroup H such that ψ centralizes a Sylow 2-subgroup of H. Moreover, if $G \not\simeq {}^2G_2(3^{2n+1})$, ${}^3D_4(q^3)$, or ${}^2D_{2n}(q^2)$, then there exists a ψ-stable torus T of G such that ψ centralizes a Sylow 2-subgroup of T and the factor group $N(G,T)/T$ is isomorphic to $N_{\overline{G}}(\overline{T})/\overline{T}$.

Proof. Obviously, we need to prove the lemma only for the case $G = \overline{G}_\sigma$. Let $|\psi| = k$ and $GF(q)$ be the base field of G. Then $q = p^\alpha$ and $\alpha = k \cdot m$. Now $|G|$ can be written in the form $|G| = q^N(q^{m_1} + \varepsilon_1 1) \cdot \ldots \cdot (q^{m_n} + \varepsilon_n 1)$ for some N, where n is the rank of $\overline{G}$, and $\varepsilon_i = \pm$ (cf. [12, Thms. 9.4.10 and 14.3.1]). Similarly, $|G_\psi| = (p^m)^N((p^m)^{m_1} + \varepsilon_1 1) \cdot \ldots \cdot ((p^m)^{m_n} + \varepsilon_n 1)$. Since k is odd, $((p^{km})^{m_i} + \varepsilon_i 1)_2 = ((p^m)^{m_i} + \varepsilon_i 1)_2$ for all i, that is, $|G|_2 = |G_\psi|_2$ and a Sylow 2-subgroup of G_ψ is a Sylow 2-subgroup of G. By [18, Prop. 2.5.17], there exists an automorphism ψ_1 of $\overline{G}$ such that $\sigma = \psi_1^k$ and ψ coincides with the restriction of ψ_1 to $\overline{G}_\sigma$. Note that ψ_1, in general, is not equal to the automorphism $\bar{\psi}$ defined above. Consider a maximal split torus $\overline{H}_{\psi_1}$ of $\overline{G}_{\psi_1}$. Then $H = \overline{H}_\sigma$ is a ψ-stable Cartan subgroup of G. Since $|H| = (q^{k_1} + \varepsilon_1 1) \cdot \ldots \cdot (q^{k_l} + \varepsilon_l 1)$, where $\varepsilon_i = \pm$, likewise we can prove that $|H|_2 = |H_\psi|_2$.

Now assume that $G \not\simeq {}^2G_2(3^{2n+1})$, ${}^3D_4(q^3)$, or ${}^2D_{2n}(q^2)$. By Lemma 2.8, there exists a maximal torus T of G_ψ such that $N(G_\psi, T)/T \simeq N_{\overline{G}}(\overline{T})/\overline{T}$ and $|T_\psi| = (p^m - \varepsilon 1)^n$. Since $|\psi|$ is odd and $\overline{T}_{\psi_1}$ is obtained by twisting a maximal split torus $\overline{H}$ with an element w_0, $\overline{T}_\sigma$, too, is obtained by twisting $\overline{H}$ with w_0 (see proof of Lemma 2.6). Therefore $|\overline{T}_\sigma| = (q - \varepsilon 1)^n$ and $|\overline{T}_{\psi_1}| = (p^m - \varepsilon 1)^n$; hence $|\overline{T}_\sigma|_2 = |T|_2 = |T_\psi|_2$. □

LEMMA 3.4 [29, (7-2)]. Let $\overline{G}$ be a connected simple linear algebraic group of adjoint type over a field of characteristic p, σ be a Frobenius map of $\overline{G}$, and $G = \overline{G}_\sigma$ be a finite group of Lie type. Suppose φ is a field or graph-field automorphism of G and φ' is an element of $(G \leftthreetimes \langle\varphi\rangle) \setminus G$ such that $|\varphi'| = |\varphi|$ and $G \leftthreetimes \langle\varphi\rangle = \langle G, \varphi'\rangle = G \leftthreetimes \langle\varphi'\rangle$. Then there exists an element $g \in G$ for which $\langle\varphi\rangle^g = \langle\varphi'\rangle$. In particular, if $G/O^{p'}(G)$ is a 2-group and φ is of odd order, then such g can be chosen in $O^{p'}(G)$.

LEMMA 3.5. Let G be a finite adjoint group of Lie type over a field of odd characteristic, $G \not\simeq {}^3D_4(q^3)$, and $\overline{G}$ and σ be chosen so that $O^{p'}(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma$. Assume A is a subgroup of $\mathrm{Aut}(O^{p'}(\overline{G}_\sigma))$ such that $A \cap \overline{G}_\sigma = G$. If $O^{p'}(G) \simeq D_4(q)$, then we also suppose that A is contained in a group generated by inner-diagonal and field automorphisms and by a graph automorphism of order 2. Then A satisfies **(ESyl2)** if and only if G satisfies **(ESyl2)**.

Proof. Suppose G satisfies **(ESyl2)**. In the conditions of the lemma, the factor group A/G is Abelian; so $A/G = \overline{A}_1 \times \overline{A}_2$, where $\overline{A}_1$ is a Hall $2'$-subgroup of A/G and $\overline{A}_2$ is a Sylow 2-subgroup of A/G. Denote by A_1 a complete preimage of $\overline{A}_1$ in A. If A_1 satisfies **(ESyl2)**, then so does A by Lemma 1.9. Thus we can think of the order $|A/G|$ as odd. By assumption, an order 3 graph automorphism is not contained in A, so A/G is cyclic, and hence $A = \langle G, \psi g\rangle$, where ψ is a field automorphism of odd order and $g \in \overline{G}_\sigma$. Since $|A : G| = |\psi|$ is odd, we may assume that $|\psi g|$ likewise is odd. By Lemma 3.3, ψ centralizes a Sylow 2-subgroup of $\overline{G}_\sigma$; therefore g is of odd order.

Now the factor $\overline{G}_\sigma/G$ is Abelian and is representable as $\overline{L} \times \overline{Q}$, where $\overline{L}$ is a Hall $2'$-subgroup of $\overline{G}_\sigma/G$ and $\overline{Q}$ is a Sylow 2-subgroup of $\overline{G}_\sigma/G$. Let L be a complete preimage of $\overline{L}$ in $\overline{G}_\sigma$ under the natural homomorphism. Then $g \in L$. Consider a group $L \leftthreetimes \langle\psi\rangle \geq A$. By construction, the index $|L \leftthreetimes \langle\psi\rangle : A| = |L : G|$ is odd. By Lemma 2.6, the group L satisfies **(ESyl2)**. In view of Lemma 3.3, the field automorphism ψ centralizes a Sylow 2-subgroup Q of L. Thus

$$N_{L\leftthreetimes\langle\psi\rangle}(Q) = N_L(Q) \times \langle\psi\rangle = QC_L(Q) \times \langle\psi\rangle = QC_{L\leftthreetimes\langle\psi\rangle}(Q),$$

that is, the group $L \leftthreetimes \langle\psi\rangle$ satisfies **(ESyl2)**. Since $|L \leftthreetimes \langle\psi\rangle : A|$ is odd, A also satisfies **(ESyl2)**.

Now assume that A satisfies **(ESyl2)**. If G does not satisfy **(ESyl2)**, then [25, Thm. 1] and [26, Thm. 6] imply that the root system of $\overline{G}$ is of type A_1, or C_n. In both cases the factor group $\mathrm{Aut}(O^{p'}(\overline{G}_\sigma))/\overline{G}_\sigma$ is cyclic and is generated by a field automorphism φ. Furthermore, from [25, Thm. 1] it follows that the order of the base field (which coincides with the definition field in this instance, for G is not twisted) is equal to $q = p^t$ and $q \equiv \pm 3 \pmod 8$. Hence t is odd and so therefore is $|\mathrm{Aut}(O^{p'}(\overline{G}_\sigma))/\overline{G}_\sigma|$. Thus $|A : G|$ is odd and G satisfies **(ESyl2)**. □

The conclusion of the next lemma is known to hold for classical groups (see, e.g., [30]).

LEMMA 3.6. Let G be a finite adjoint split group of Lie type and $\overline{G}$ and σ be chosen so that $O^{p'}(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma$. Assume that τ is a graph automorphism of $O^{p'}(G)$ of order 2. Then every semisimple element $s \in G$ is conjugate to its inverse in $\langle O^{p'}(\overline{G}_\sigma), \tau a\rangle$, where a is an element of $\overline{G}_\sigma$.

Proof. If $\Phi(\overline{G})$ is not of types A_n, D_{2n+1}, and E_6, then the lemma follows from [10, Lemma 2.2], and so we need only consider groups of these types. Denote by $\bar\tau$ a graph automorphism of $\overline{G}$ for which $\bar\tau|_G = \tau$. Let $\overline{T}$ be a maximal σ-stable torus of $\overline{G}$ such that $\overline{T}_\sigma \cap G$ is a Cartan subgroup of G. Let $r_1, \ldots, r_n$ be fundamental roots in $\Phi(\overline{G})$ and ρ be the symmetry corresponding to $\bar\tau$. Denote r_i^ρ by $\bar r_i$. Then $\overline{T} = \langle h_{r_i}(t_i) \mid 1 \leqslant i \leqslant n,\ t_i \neq 0\rangle$ and $h_{r_i}(t_i)^{\bar\tau} = h_{\bar r_i}(t_i)$. Denote by W the Weyl group of $\overline{G}$. Let w_0 be a unique element of W mapping all positive roots to negative and n_0 be its preimage in $N_{\overline{G}}(\overline{T})$ under the natural homomorphism $N_{\overline{G}}(\overline{T}) \to N_{\overline{G}}(\overline{T})/\overline{T} \simeq W$. We can choose $n_0 \in G$, that is, $n_0^\sigma = n_0$, since σ acts trivially on $W = N(G,T)/T$ (see Lemma 2.8 above). For all r_i and t, we then have

$$h_{r_i}(t)^{n_0\bar\tau} = h_{r_i^{w_0\rho}}(t) = h_{-r_i}(t) = h_{r_i}(t^{-1}).$$

Thus $x^{n_0\bar\tau} = x^{-1}$ for all $x \in \overline{T}$.

Now let s be a semisimple element of G. Then there exists a maximal σ-stable torus $\overline{S}$ of $\overline{G}$ containing s. Since maximal tori of $\overline{G}$ are all conjugate, there exists $g \in \overline{G}$ such that $\overline{S}^g = \overline{T}$. With $\overline{G}_\sigma = O^{p'}(\overline{G}_\sigma)\overline{T}_\sigma$ in mind, we may assume that $a \in \overline{T}_\sigma$. Therefore $s^{gn_0\bar\tau ag^{-1}} = s^{-1}$. The equalities $n_0^\sigma = n_0$ and $\bar\tau^\sigma = \bar\tau$ yield $(gn_0\bar\tau ag^{-1})^\sigma = g^\sigma n_0\bar\tau a(g^{-1})^\sigma$. Moreover, since $\overline{S}$ is σ-stable, $x^{gn_0\bar\tau ag^{-1}} = x^{g^\sigma n_0\bar\tau a(g^{-1})^\sigma} = x^{-1}$ for every $x \in \overline{S}$; in other words, $gn_0\bar\tau ag^{-1}\overline{S} = g^\sigma n_0\bar\tau a(g^{-1})^\sigma\overline{S}$. In particular, there exists $t \in \overline{S}$ such that $gn_0\bar\tau ag^{-1}t = g^\sigma n_0\bar\tau a(g^{-1})^\sigma$. On the Lang–Steinberg theorem, there is $y \in \overline{S}$ for which $t = y\cdot(y^{-1})^\sigma$ (see [19, Thm. 10.1]). Therefore $gn_0\bar\tau ag^{-1}y = (gn_0\bar\tau ag^{-1}y)^\sigma$, that is, $gn_0\tau ag^{-1}y \in \overline{G}_\sigma \leftthreetimes \langle\tau\rangle$, and $s^{gn_0\tau ag^{-1}y} = s^{-1}$. Since $O^{p'}(\overline{G}_\sigma)\overline{S}_\sigma = \overline{G}_\sigma$ and $\overline{S}_\sigma$ is Abelian, we may find an element $z \in \overline{S}_\sigma$ so as to satisfy $gn_0\tau ag^{-1}yz \in \langle O^{p'}(\overline{G}_\sigma), \tau a\rangle$. □

LEMMA 3.7. Let $\langle G, \zeta g\rangle$ be a finite semilinear group of Lie type and $\overline{G}$ and σ be chosen so that $O^{p'}(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma$. Suppose s is a regular semisimple element of odd order in G. Then $N_{\langle G,\zeta g\rangle}(C_{\langle G,\zeta g\rangle}(s)) \neq C_{\langle G,\zeta g\rangle}(s)$.

Proof. Since s is semisimple, there exists a σ-stable maximal torus $\overline{S}$ of $\overline{G}$ containing s. By virtue of $\overline{G}_\sigma = O^{p'}(\overline{G}_\sigma)\overline{S}_\sigma$, we may assume that $g \in \overline{S}_\sigma$, that is, the element g commutes with the element s. If

$C_{\langle G,\zeta g\rangle}(s)G \neq \langle G, \zeta g\rangle$, then we can replace $\langle G, \zeta g\rangle$ by $C_{\langle G,\zeta g\rangle}(s)G$ and prove the statement for this group. Moreover, if $C_{\langle G,\zeta g\rangle}(s) = C_G(s)$, then the lemma follows from Lemma 2.9, and so we can suppose that ζ centralizes s. If either G is not twisted or $|\zeta|$ is odd, then we may put $\sigma = \bar{\zeta}^k$ for some $k > 0$, as follows from [18, Prop. 2.5.17]. By Lemma 2.9, there exists an element of $N_{G_{\zeta g}}(C_{\overline{G}}(s))$ not in $C_{G_{\zeta g}}(s)$, whence the result.

Assume that G is twisted and $|\zeta|$ is even. Then $\sigma = \bar{\gamma}\bar{\varphi}^k$, where $\bar{\zeta} = \bar{\varphi}^\ell$ and k divides ℓ. Therefore s is in $\overline{G}_{\bar{\gamma}}$. Depending on which root system $\Phi(\overline{G})$ is chosen, we see that $\overline{G}_{\bar{\gamma}}$ is isomorphic to a simple algebraic group with a root system equal to B_m (for some $m > 1$), to C_m (for some $m > 2$), or to F_4. By [10, Lemma 2.2], the element s is conjugate to its inverse w.r.t. $O^{p'}((\overline{G}_{\bar{\gamma}})_{\sigma\bar{\zeta}g}) \leq G_{\zeta g}$; so $N_{\langle G,\zeta g\rangle}(C_{\langle G,\zeta g\rangle}(s)) \neq C_{\langle G,\zeta g\rangle}(s)$. □

LEMMA 3.8. Let $\langle G, \zeta g\rangle$ be a finite semilinear group of Lie type over a field of characteristic p (we do not exclude the case $\langle G, \zeta g\rangle = G$), and G be of adjoint type (recall that $g \in \overline{G}_\sigma$, but not necessarily $g \in G$). Assume that $B = U \leftthreetimes H$, where H is a Cartan subgroup of G, is a ζg-stable Borel subgroup of G and $\langle B, \zeta g\rangle$ contains K as a Carter subgroup of $\langle G, \zeta g\rangle$. Suppose $K \cap U \neq \{e\}$. Then one of the following statements holds:

(a) either $\langle G, \zeta g\rangle = \langle {}^2A_2(2^{2t}), \zeta g\rangle$ or $\langle G, \zeta g\rangle = {}^2\widehat{A_2(2^{2t})} \leftthreetimes \langle\zeta\rangle$; the order $|\zeta| = t$ is odd and is not divisible by 3, $C_G(\zeta) \simeq {}^2\widehat{A_2(2^2)}$, and the subgroup $K \cap G$ is Abelian and has order $2 \cdot 3$;

(b) $\langle G, \zeta g\rangle = \langle {}^2A_2(2^{2t}), \zeta g\rangle$, $C_G(\zeta) \simeq {}^2A_2(2^2)$, and $K \cap G$ is a Sylow 2-subgroup of G_ζ;

(c) either $\langle G, \zeta g\rangle = \langle A_2(2^{2t}), \zeta g\rangle$ or $\langle G, \zeta g\rangle = \widehat{A_2(2^{2t})} \leftthreetimes \langle\zeta\rangle$; ζ is a graph-field automorphism of order $2t$, t is not divisible by 3, $C_G(\zeta) \simeq {}^2\widehat{A_2(2^2)}$, and $K \cap G$ is Abelian and has order $2^{|\zeta_{2'}|} \cdot 3$;

(d) $\langle G, \zeta g\rangle = \langle A_2(2^{2t}), \zeta g\rangle$, ζ is a graph-field automorphism, $C_G(\zeta) \simeq {}^2A_2(2^2)$, and $K \cap G$ is a Sylow 2-subgroup of $G_{\zeta_{2'}}$;

(e) G is defined over $GF(2^t)$, $\langle G, \zeta g\rangle = G \leftthreetimes \langle\zeta g\rangle$, ζ either is a field automorphism of order t in $O^{2'}(G)$, if $O^{2'}(G)$ is split, or is a graph automorphism of order t, if $O^{2'}(G)$ is twisted, and, up to conjugation in G, $K = Q \leftthreetimes \langle\zeta g\rangle$, where Q is a Sylow 2-subgroup of $G_{(\zeta g)_{2'}}$;

(f) G is defined over $GF(2^t)$, $\langle G, \zeta g\rangle = G \leftthreetimes \langle\zeta g\rangle$, ζ is a product of a field automorphism of odd order t in $O^{2'}(G)$ and a graph automorphism of order 2, ζ and ζg are conjugate w.r.t. $\overline{G}_\sigma$, and, up to conjugation in G, $K = Q \leftthreetimes \langle\zeta g\rangle$, where Q is a Sylow 2-subgroup of $G_{(\zeta g)_{2'}}$;

(g) $G/Z(G) \simeq \mathbf{P}SL_2(3^t)$, the order $|\zeta| = t$ is odd (in particular, $\zeta \in \langle G, \zeta g\rangle$), and K contains a Sylow 3-subgroup of $G_{\zeta_{3'}}$;

(h) $\langle G, \zeta g\rangle = {}^2G_2(3^{2n+1}) \leftthreetimes \langle\zeta\rangle$, $|\zeta| = 2n+1$, $K \cap {}^2G_2(3^{2n+1}) = Q \times P$, where Q is of order 2, and $|P| = 3^{|\zeta|_3}$.

Note that every item through and under (a)-(h) insists on there being Carter subgroups of specified forms. Indeed, the existence of Carter subgroups in (a) and (c) follows from there being a Carter subgroup of order 6 in $\mathbf{P}GU_3(2)$ (see [5]). Throughout (b), (d)-(f), such groups exist by reason of the fact that a Sylow 2-subgroup in a group of Lie type defined over a field of order 2 coincides with its normalizer. The existence of Carter subgroups in (g) follows from the fact that a Sylow 3-subgroup of $\mathbf{P}SL_2(3)$ coincides with its normalizer. A Carter subgroup satisfying (h) exists due to there being a Carter subgroup K of order 6 in a (non-simple) group ${}^2G_2(3)$. Lastly, the existence of a Carter subgroup K of order 6 in ${}^2G_2(3)$ follows from structure results in [31, 32].

Proof. If G is one of the groups $A_1(q)$, $G_2(q)$, $F_4(q)$, ${}^2B_2(2^{2n+1})$, or ${}^2F_4(2^{2n+1})$, then we need only appeal to [9, Table; 10, Table]. If $\langle G, \zeta g\rangle = G$, then the lemma follows from [6, 9, 10]. Therefore we may suppose that $\langle G, \zeta g\rangle \neq G$, that is, ζ is a non-trivial field, graph-field, or graph automorphism. If

$\Phi(\overline{G}) = C_n$, then the lemma follows from Theorem 4.1 (see below), in which is Lemma 3.8 not used, and so we assume that $\Phi(\overline{G}) \neq C_n$. If $\Phi(\overline{G}) = D_4$, and either a graph-field automorphism ζ is a product of a field automorphism and a graph automorphism of order 3, or $G \simeq {}^3D_4(q^3)$, then the lemma follows from Theorem 5.1 (see below), in which Lemma 3.8 is not used again, and so we suppose that $\langle G, \zeta g\rangle$ is contained in the group A_1 defined in Theorem 5.1, and $G \not\simeq {}^3D_4(q^3)$. Since Lemma 3.8 will be involved in proving Theorem 6.1, after Theorems 4.1 and 5.1, we can make in advance the following extra assumptions.

Suppose that q is odd and $\Phi(\overline{G})$ is one of the following types: A_n $(n \geqslant 2)$, D_n $(n \geqslant 4)$, B_n $(n \geqslant 3)$, E_6, E_7, or E_8. By Lemma 1.4, KU/U is a Carter subgroup of $\langle B, \zeta g\rangle/U \simeq \langle H, \zeta g\rangle$. Since $\overline{G}_\sigma = G\overline{H}_\sigma$, where $\overline{H}$ is a maximal split torus of $\overline{G}$ and $\overline{H}_\sigma \cap G = H$, we may assume that $g \in \overline{H}_\sigma$ and, in particular, g centralizes H. Therefore $H_\zeta \leq Z(\langle H, \zeta g\rangle)$, and up to conjugation in B, $H_\zeta \leq K$. By Lemma 3.3, the automorphism $\zeta_{2'}$ centralizes a Sylow 2-subgroup Q of H. Thus each element of odd order in $\langle H, \zeta g\rangle$ centralizes Q, and Lemma 1.6 implies that $Q \leq K$ up to conjugation in B. By Lemma 2.3 it follows that $C_U(Q) = \{e\}$, which is a contradiction with $K \cap U$ being non-trivial.

Assume that $G \simeq {}^2G_2(3^{2n+1})$ and $\langle G, \zeta g\rangle = G \leftthreetimes \langle\zeta\rangle$ (in which case $O^{p'}(\overline{G}_\sigma) = \overline{G}_\sigma$). Again by Lemma 1.4, KU/U is a Carter subgroup of $(B \leftthreetimes \langle\zeta\rangle)/U \simeq H \leftthreetimes \langle\zeta\rangle$. In view of [10, Lemma 2.2], every semisimple element of G is conjugate to its inverse. Since non-Abelian composition factors of every semisimple element of G can be isomorphic only to groups $A_1(q)$, it follows from [9, Table] that centralizer of every semisimple element of G satisfies condition **(C)**. By Lemma 1.5, $KU/U \cap B/U$ is, therefore, a 2-group. On the other hand, $|H|_2 = 2$ and $KU/U \geq Z(B/U) \geq H_\zeta$, whence $|H_\zeta| = 2$ and $|\zeta| = 2n+1$. Thus $K \cap G = (K \cap U) \times \langle t\rangle$, where t is an involution. Consequently $K \cap U = C_G(t) \cap G_{\zeta_{3'}}$. Item (h) of the lemma now follows from structure results in [31; 32, Thm. 1].

Let $q = 2^t$. Suppose first that $\Phi(\overline{G})$ has one of the types A_n $(n \geqslant 2)$, D_n $(n \geqslant 4)$, B_n $(n \geqslant 3)$, E_6, E_7, or E_8, G is split, and ζ is a field automorphism. As above, we obtain $H_\zeta \leq K$, and $O^{2'}(G_\zeta)$ is a split group of Lie type with definition field of order $q = 2^{t/|\zeta|}$. By Lemma 2.2, for every $r \in \Phi(\overline{G})$ and every $s \in GF(2^{t/|\zeta|})^*$, there exists $h(\chi) \in H_\zeta \cap O^{2'}(G_\zeta)$ such that $\chi(r) = s$. Arguing as in Lemma 2.3, we see that $K \cap U \leq C_U(H_\zeta) = \{e\}$ for $\frac{t}{|\zeta|} \neq 1$, a contradiction. Hence $|\zeta| = t$ and $H_\zeta = \{e\}$. Since g can be chosen in $\overline{H}_\sigma$, and $\langle\zeta g\rangle \cap \overline{G}_\sigma \leq \langle\zeta g\rangle \cap \overline{H}_\sigma \leq H_\zeta = \{e\}$, we have $\langle\zeta g\rangle \cap \overline{G}_\sigma = \{e\}$. By Lemma 3.4, elements ζg and ζ are conjugate in $\overline{G}_\sigma$, which yields (e).

Next suppose that $\Phi(\overline{G})$ is of one of the types A_n $(n \geqslant 3)$, D_n $(n \geqslant 4)$, or E_6, and either ζ is a graph-field automorphism and G is split, or G is twisted. Let ρ be the symmetry of the Dynkin diagram of $\Phi(\overline{G})$ corresponding to γ (recall that $\zeta = \gamma^\varepsilon\varphi^\ell$ by definition); $\bar{r}$ denotes r^ρ for $r \in \Phi(\overline{G})$. As above, we can prove that up to conjugation, $H_\zeta \leq K$. If $H_\zeta \neq \{e\}$, then $C_U(H_\zeta) = \{e\}$ by Lemma 2.2, which contradicts the condition that $K \cap U \neq \{e\}$. If $H_\zeta = \{e\}$, then either G is twisted and $|\zeta| = t$, (which yields statement (e) of the lemma), or G is split, $|\zeta| = 2t$, and, in particular, t is odd (which yields (f)).

Assume that $O^{2'}(G) \simeq A_2(2^t)$, ζ is a graph-field automorphism, and t is odd. If $|\zeta| \neq 2t$, then arguing similarly to how we did in proving Lemma 1.5 and making use of Lemma 2.2, we see that $C_U(H_\zeta) = \{e\}$, which is a contradiction with $K \cap U \neq \{e\}$. If $|\zeta| = 2t$, then we arrive at (f).

Suppose that $O^{2'}(G) \simeq A_2(2^{2t})$ and ζ is a graph-field automorphism. Again for $|\zeta| \neq 2t$, Lemma 2.2 implies that $C_U(H_\zeta) = \{e\}$, which is a contradiction with $K \cap U \neq \{e\}$. If $|\zeta| = 2t$, then either $G_\zeta \simeq {}^2A_2(2^2)$, or $G_\zeta \simeq {}^2\widehat{A_2(2^2)}$. If $G_\zeta \simeq {}^2A_2(2^2)$, then $H_\zeta = \{e\}$, which yields (d). If $G_\zeta \simeq {}^2\widehat{A_2(2^2)}$, then $|H_\zeta| = 3$, and so $KU/U \cap HU/U$ is a cyclic group $\langle y\rangle$ of order $(2^{t_3}+1)_3 = 3^k$, where $3^{k-1} = t_3$. If $k > 1$, then Lemma 2.2 implies that $C_U(y) = \{e\}$, which is impossible. Thus t is not divisible by 3 and $K \cap U$ is contained in the centralizer of an element x generating H_ζ. Consider a homomorphism $GL_3(2^{2t}) \to \mathbf{P}GL_3(2^{2t})$. Then some preimage of x is similar to the matrix

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda \end{pmatrix},$$

where λ is a generating element of the multiplicative group of $GF(2^2)$. The preimage of U is conjugate to a set of upper triangular matrices with same elements on a diagonal. Direct calculations show that $C_U(x)$ is isomorphic to the additive group of $GF(2^{2t})$. Since K is nilpotent, $K \cap U = (C_U(x))_{\zeta_{2'}}$, which yields (c).

Assume now that $O^{2'}(G) \simeq {}^2A_2(2^{2t})$. By Lemma 1.4, KU/U is a Carter subgroup of $\langle B, \zeta g\rangle/U \simeq \langle H, \zeta g\rangle$, and, as above, we may put $H_\zeta \leq K$. If $|\zeta| = 2t$, then $G_\zeta \simeq SL_2(2)$ and $H_\zeta = \{e\}$, which proves (e). Suppose that t is even. Then either $O^{2'}(G_\zeta) \simeq SL_2(2^{2t/|\zeta|})$ (if the order $|\zeta|$ is even), or $O^{2'}(G_\zeta) \simeq {}^2A_2(2^{2t/|\zeta|})$ (if the order $|\zeta|$ is odd, whence $|\zeta| < t$).

Clearly, H_ζ contains an element x such that $K \cap U \leq C_U(H_\zeta) = \{e\}$, which is a contradiction with $K \cap U \neq \{e\}$. If t is odd and $t \neq |\zeta|$, then $O^{2'}(G_\zeta) \simeq {}^2A_2(2^{2t/|\zeta|})$, from which it follows that H_ζ contains an element x such that $C_U(x) = \{e\}$. If $|\zeta| = t$ and t is odd, then the order $|KU/U \cap B/U|$ is divisible only by 3 (otherwise Lemma 2.2 implies again that $C_U(H_\zeta) = \{e\}$). If $G_\zeta \simeq {}^2A_2(2^{2t/|\zeta|})$, then $H_\zeta = \{e\}$, and we now arrive at (b). If $G_\zeta \simeq {}^2\widehat{A_2(2^{2t/|\zeta|})}$, then $KU/U \cap HU/U$ is a cyclic group $\langle y\rangle$ of order $(2^{t_3}+1)_3 = 3^k$, where $3^{k-1} = t_3$. If $k > 1$, then $C_U(y) = \{e\}$ by Lemma 2.2, which is impossible. Thus t is not divisible by 3 and $K \cap U$ is contained in the centralizer of an element x generating H_ζ. As in the non-twisted case above, we see that $C_U(x)$ is isomorphic to the additive subgroup of $GF(2^t)$. The nilpotency of K implies that $K \cap U = (C_U(x))_{\zeta_{2'}}$, and we are so faced with (a). □

4. CARTER SUBGROUPS IN SYMPLECTIC GROUPS

Consider a set $\mathcal{A}$ of almost simple groups A such that a unique non-Abelian composition factor $S = F^*(A)$ is a canonical simple group of Lie type and A contains non-conjugate Carter subgroups. If the set $\mathcal{A}$ is not empty, then by **Cmin** we denote the minimal possible order of $F^*(A)$, with $A \in \mathcal{A}$. If $\mathcal{A}$ is empty, then we let **Cmin** $= \infty$. We shall prove that **Cmin** $= \infty$, that is, $\mathcal{A} = \varnothing$. Note that if $A \in \mathcal{A}$ and $G = F^*(A)$, then there exists a subgroup A_1 of A such that $A_1 \in \mathcal{A}$ and $A_1 = KG$ for a Carter subgroup K of A. Indeed, if Carter subgroups of NG are conjugate for every nilpotent subgroup N of A, then A satisfies **(C)**, and hence the Carter subgroups of A are conjugate, which contradicts the choice of A. Therefore there exists a nilpotent subgroup N of A such that the Carter subgroups of NG are not conjugate. Let K be a Carter subgroup of NG. Then KG/G is obviously a Carter subgroup of NG/G; that is, it coincides with NG/G. Hence the Carter subgroups of KG are not conjugate and $KG = A_1 \in \mathcal{A}$. This means that the condition $A = KG$ in Theorems 4.1, 5.1, and 6.1 is not a restriction and is used only for ease of the argument.

In this section we consider Carter subgroups in an almost simple group A with simple socle $S = F^*(A) \simeq \mathbf{P}Sp_{2n}(q)$. Such are taken up in a separate section, since Lemma 2.3 fails for groups of type $\mathbf{P}Sp_{2n}(q)$, and so our present argument will be slightly different from that in Theorem 6.1.

THEOREM 4.1. Let G be a finite adjoint group of Lie type over a field of characteristic p and let $\overline{G}$ and σ satisfy $\mathbf{P}Sp_{2n}(p^t) \simeq O^{p'}(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma$. Choose a subgroup A of $\mathrm{Aut}(\mathbf{P}Sp_{2n}(p^t))$ so that $A \cap \overline{G}_\sigma = G$. Let K be a Carter subgroup of A. Assume also that $|\mathbf{P}Sp_{2n}(p^t)| \leqslant$ **Cmin** and $A = KG$. Then exactly one of the following statements holds:

(a) G is defined over $GF(2^t)$, a field automorphism ζ is in A, $|\zeta| = t$, and, up to conjugation in G, $K = Q \leftthreetimes \langle\zeta\rangle$, where Q is a Sylow 2-subgroup of $G_{\zeta_{2'}}$;

(b) $G \simeq \mathbf{P}SL_2(3^t) \simeq \mathbf{P}Sp_2(3^t)$, a field automorphism ζ is in A, $|\zeta| = t$ is odd, and, up to conjugation in G, $K = Q \leftthreetimes \langle\zeta\rangle$, where Q is a Sylow 3-subgroup of $G_{\zeta_{3'}}$;

(c) p does not divide $|K \cap G|$ and K contains a Sylow 2-subgroup of A.

Specifically, Carter subgroups of A are conjugate, that is, if $A_1 \in \mathcal{A}$ and $|F^*(A_1)| = \mathbf{Cmin}$, then $F^*(A_1) \not\simeq \mathbf{P}Sp_{2n}(p^t)$.

Proof. Assume that the theorem is untrue and A is a counterexample such that $|F^*(A)|$ is minimal. Note that not more than one of the statements of the theorem can be fulfilled. Indeed, if (b) holds, then it follows by Lemmas 2.7 and 3.5 that for a Sylow 2-subgroup Q of A, the condition $N_G(Q) = QC_G(Q)$ is invalid, that is, statement (c) fails. Furthermore, if A_1 is an almost simple group for which $F^*(A_1)$ is a simple group of Lie type of order less than $|F^*(A)|$, then Carter subgroups of A_1 are conjugate. In view of the main theorem in [6], we may suppose that $A \neq G$. Moreover, by [10, Thm. 3.5], we can think of q as odd, that is, $\mathrm{Aut}(\mathbf{P}Sp_{2n}(q))$ does not contain a graph automorphism. Thus we may assume that $A = \langle G, \zeta g\rangle$.

Suppose that K is a Carter subgroup of $\langle G, \zeta g\rangle$ and K does not satisfy the statement of the theorem. Write $K = \langle x, K \cap G\rangle$. If $p \neq 3$ or t is even, then the theorem follows from [10, Thm. 3.5]. We may therefore assume that $q = 3^t$ and t is odd. Since $|\overline{G}_\sigma : O^{p'}(\overline{G}_\sigma)| = 2$ and the order $|\zeta|$ is odd, we can think that the order $|\zeta g|$ likewise is odd, and so $\zeta \in \langle G, \zeta g\rangle$, that is, $\langle G, \zeta g\rangle = G \leftthreetimes \langle\zeta\rangle$. By [10, Lemma 2.2], every semisimple element of odd order is conjugate to its inverse in G. Now, for every semisimple element $t \in G$, each non-Abelian composition factor of $C_G(t)$ is a simple group of Lie type of order less than **Cmin** (cf. [23]). Therefore, for every non-Abelian composition factor S of $C_G(t)$ and every nilpotent subgroup $N \leq C_G(t)$, Carter subgroups of $\langle \mathrm{Aut}_N(S), S\rangle$ are conjugate. Hence $C_G(t)$ satisfies **(C)**. By Lemma 1.5, $|K \cap G| = 2^\alpha \cdot 3^\beta$ for some $\alpha, \beta \geqslant 0$.

If $G = \mathbf{P}\widehat{Sp_{2n}(q)}$, then [34, Thm. 2] implies that every unipotent element is conjugate to its inverse. Since 3 is a good prime for G, it follows by [35, Thms. 1.2, 1.4] that for any element $u \in G$ of order 3, all non-Abelian composition factors in $C_G(u)$ are simple groups of Lie type of order less than **Cmin**. Thus $C_G(u)$ satisfies **(C)**, and hence $K \cap G$ is a 2-group by Lemma 1.5. By Lemmas 3.3 and 3.4, every element $x \in A \setminus G$ of odd order with $\langle x\rangle \cap G = \{e\}$ centralizes some Sylow 2-subgroup of G. Therefore K contains a Sylow 2-subgroup of G and hence of A, that is, K satisfies statement (c) of the theorem.

Thus we may assume that $G = \mathbf{P}Sp_{2n}(q)$ and $\beta \geqslant 1$, that is, a Sylow 3-subgroup $O_3(K \cap G)$ of $K \cap G$ is non-trivial. By Lemma 3.2, $K \cap G$ is contained in some K-stable parabolic subgroup P of G with Levi factor L, and up to conjugation in P, a Sylow 2-subgroup $O_2(K \cap G)$ of $K \cap G$ is contained in L. Note that all non-Abelian composition factors of P are simple groups of Lie type of order less than **Cmin**; hence P satisfies **(C)**, and so does each homomorphic image of P. The group $\widetilde{K} = KO_3(P)/O_3(P)$ is isomorphic to $K/O_3(K \cap G)$, and by Lemma 1.4, $\widetilde{K}$ is a Carter subgroup of $\langle \widetilde{K}, P/O_3(P)\rangle$. Now $\widetilde{K} \cap P/O_3(P) \simeq O_2(K \cap G)$ is a 2-group and every element $x \in \langle \widetilde{K}, P/O_3(P)\rangle \setminus P/O_3(P)$ of odd order with $\langle x\rangle \cap P/O_3(P) = \{e\}$ centralizes a Sylow 2-subgroup of $P/O_3(P) \simeq L$ (see Lemmas 3.3 and 3.4). Therefore $O_2(K \cap G)$ contains a Sylow 2-subgroup of L and, in particular, the Sylow 2-subgroup H_2 of H. Since K is nilpotent, Lemma 2.4 implies that $O_3(K \cap G) \leq C_U(H_2) = \langle X_r \mid r \text{ is a long root in } \Phi(G)^+\rangle$. For every two long positive roots r and s in $\Phi(G)^+$, $r + s \notin \Phi(G)$; so the Chevalley commutator formula in [12, Thm. 5.2.2] implies that the subgroup $\langle X_r \mid r \text{ is a long root in } \Phi(G)^+\rangle$ is Abelian.

Since ζ is a field automorphism, it normalizes each parabolic subgroup of G containing a ζ-stable Borel subgroup. Thus, for every subset J of the set $\Pi = \{r_1, \dots, r_n\}$ of fundamental roots in $\Phi = \Phi(G)$, the parabolic subgroup P_J is ζ-stable. Therefore we may suppose that $P = P_J$, where J is a proper subset of the set Π of fundamental roots in Φ. Choose a numbering of fundamental roots so that r_n is a long fundamental root, while other fundamental roots r_i are short. If $r_n \in J$, then one of the components of the Levi factor L, for example, G_1, is isomorphic to $Sp_{2k}(q)$ for some $k < n$ (note that $q \neq 3$ since $A \neq G$). By Lemma 2.5, $L/C_L(G_1) = \mathrm{Aut}_L(G_1/Z(G_1)) = G_1/Z(G_1)$. By Lemma 1.4, $K_1 = KC_L(G_1)O_3(P)/C_L(G_1)O_3(P)$ is a Carter subgroup of $(P \leftthreetimes \langle\zeta\rangle)/C_L(G_1)O_3(P)$. Since $|K_1 \cap P/C_L(G_1)O_3(P)|$ is not divisible by 3, and ζ centralizes a Sylow 2-subgroup of $G_1/Z(G_1)$ (cf. Lemma 3.3 above), K_1 contains a Sylow 2-subgroup of $P/C_L(G_1)O_3(P) \simeq G_1/Z(G_1) \simeq \mathbf{P}Sp_{2k}(q)$. Moreover, by Lemma 3.3, a Sylow 2-subgroup of $(P/C_L(G_1)O_3(P))_\zeta$ is a Sylow 2-subgroup of $P/C_L(G_1)O_3(P)$. Thus $K_1 \cap P/C_L(G_1)O_3(P)$ is a Sylow 2-subgroup in $(P/C_L(G_1)O_3(P))_\zeta \simeq \mathbf{P}Sp_{2k}(3)$. In view of Lemma 2.7, there exists an element x of odd order in $\mathbf{P}Sp_{2k}(3)$ normalizing but not centralizing a Sylow 2-subgroup, which is a contradiction with K_1 being a Carter subgroup of $(P \leftthreetimes \langle\zeta\rangle)/C_L(G_1)O_3(P)$. Thus we may assume that $r_n \notin J$.

Consider a set $J_n = \Pi \setminus \{r_n\}$ and a parabolic subgroup P_{J_n}. By the above argument, $K \leq P_J \leftthreetimes \langle\zeta\rangle \leq P_{J_n} \leftthreetimes \langle\zeta\rangle$. Now the subgroup $\langle X_r \mid r \text{ is a long root of } \Phi(G)^+\rangle$ is contained in $O_3(P_{J_n})$ and $O_3(K \cap G)$ is contained in $\langle X_r \mid r \text{ is a long root of } \Phi(G)^+\rangle$; so $O_3(K \cap G) \leq O_3(P_{J_n})$ and we can suppose that $P = P_{J_n}$. By Lemma 1.4, $\widetilde{K} = KO_3(P)/O_3(P)$ is a Carter subgroup of $(P \leftthreetimes \langle\zeta\rangle)/O_3(P)$. Note that a unique non-Abelian composition factor of $P \leftthreetimes \langle\zeta\rangle$ is isomorphic to $A_{n-1}(q) \simeq \mathbf{P}SL_n(q)$. By [25, Thm. 1; 26, Thm. 4], we obtain $\widetilde{K} = R \times \langle\zeta\rangle$, where R is a Sylow 2-subgroup of P centralized by ζ. Thus $O_3(K \cap G) \leq C_P(R)$. Consider $Q = O_3(K \cap G) \cap P_\zeta$. Since $O_3(K \cap G)$ is non-trivial and K is nilpotent, $Q = O_3(K \cap G) \cap P_\zeta = Z(K) \cap O_3(K \cap G)$ is non-trivial. Therefore $N_G(Q)$ is a proper subgroup of G, and by Lemma 3.2, $N_G(Q)$ is contained in a proper parabolic subgroup of G. On the other hand, $K \leq N_G(Q)$ and $P = P_{J_n}$ is a maximal proper parabolic subgroup of G. If $N_G(Q)$ is not contained in P, then $N_G(Q)$ and K are contained in a parabolic subgroup P_J with $r_n \in J$. We have proved above that $r_n \notin J$; so $N_G(Q)$ is contained in P.

We claim that $R \times Q$ is a Carter subgroup of G_ζ. Indeed, assume that an element $x \in G_\zeta$ normalizes $R \times Q$. Then x normalizes Q; hence x is in P and normalizes $O_3(P)$. On the other hand, x normalizes R and hence $C_P(R)$; so x normalizes $C_{O_3(P)}(R)$. Moreover, it is evident that x and ζ commute. Thus x normalizes $(R \times C_{O_3(P)}(R)) \leftthreetimes \langle\zeta\rangle$. As noted above, $K \leq (R \times C_{O_3(P)}(R)) \leftthreetimes \langle\zeta\rangle$ and $(R \times C_{O_3(P)}(R)) \leftthreetimes \langle\zeta\rangle$ is solvable. Lemma 1.5(a) implies that $(R \times C_{O_3(P)}(R)) \leftthreetimes \langle\zeta\rangle$ coincides with its normalizer in $G \leftthreetimes \langle\zeta\rangle$, and so $x \in R \times C_{O_3(P)}(R)$. The group $C_{O_3(P)}(R) \leq \langle X_r \mid r \text{ is a long root}\rangle$ is Abelian, and hence every element of $R \times C_{O_3(P)}(R)$ centralizes $C_{O_3(P)}(R) \geq O_3(K \cap G)$. Therefore x normalizes $(R \times O_3(K \cap G)) \leftthreetimes \langle\zeta\rangle = K$, that is, $x \in K$. By construction, $R \times Q = K \cap G_\zeta$; so $x \in R \times Q$ and $R \times Q$ is a Carter subgroup of G_ζ. On the other hand, $O^{3'}(G_\zeta) \simeq \mathbf{P}Sp_{2n}(3^{t/|\zeta|})$ and it follows by induction that groups $\mathbf{P}Sp_{2n}(3^{t/|\zeta|})$ and $\mathbf{P}\widehat{Sp_{2n}(3^{t/|\zeta|})}$ do not contain Carter subgroups whose order is divisible by 3. This final contradiction completes the proof. □

5. CARTER SUBGROUPS IN GROUPS WITH TRIALITY AUTOMORPHISM

THEOREM 5.1. Let G be a finite adjoint group of Lie type over a field of characteristic p and $\overline{G}$ and σ be chosen so that $O^{p'}(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma$ and $O^{p'}(G)$ is isomorphic to $D_4(q)$, or ${}^3D_4(q^3)$. Assume that τ is a graph automorphism of $O^{p'}(G)$ of order 3. (Recall that for $G \simeq {}^3D_4(q^3)$, τ is an automorphism such that the set of its stable points is isomorphic to $G_2(q)$.) Denote by A_1 a subgroup of $\mathrm{Aut}(D_4(q))$ generated by inner-diagonal and field automorphisms, and also by a graph automorphism of order 2. Let $A \leq \mathrm{Aut}(O^{p'}(G))$ be such that $A \not\leq A_1$ (if $O^{p'}(G) \simeq D_4(q)$) and K be a Carter subgroup of A. Assume also that $|O^{p'}(G)| \leqslant \mathbf{Cmin}$, $G = A \cap \overline{G}_\sigma$, and $A = KG$. Then one of the following statements holds:

(a) $G \simeq {}^3D_4(q^3)$, $(|A:G|,3)=1$, q is odd, and K contains a Sylow 2-subgroup of A;

(b) $(|A:G|,3)=3$, q is odd, $\tau \in A$, up to conjugation by an element of G, the subgroup K contains a Sylow 2-subgroup of $C_A(\tau) \in \Gamma G_2(q)$, and $\tau \in K$;

(c) $(|A:G|,3)=3$, $q=2^t$, $|A:G|=3t$, $A = G \leftthreetimes \langle \tau, \varphi \rangle$, where φ is a field automorphism of order t commuting with τ, up to conjugation by an element of G, the subgroup K contains a Sylow 2-subgroup of $C_G(\langle \tau, \varphi \rangle_{2'}) \simeq G_2(2^{t_{2'}})$, and $\tau \in K$;

(d) $O^{p'}(G) \simeq D_4(p^{3t})$, p is odd, the factor group A/G is cyclic, $\tau \notin A$, $A = G \leftthreetimes \langle \zeta \rangle$, where for some natural m, $\zeta = \tau\varphi^m$ is a graph-field automorphism, and, up to conjugation by an element of G, $K = Q \leftthreetimes \langle \zeta \rangle$, where Q is a Sylow 2-subgroup of $G_{\zeta_{2'}} \simeq {}^3D_4(p^{3t/|\zeta_{2'}|})$.

Specifically, Carter subgroups of A are conjugate, that is, if $A_2 \in \mathcal{A}$ and $|F^*(A_2)| = \mathbf{Cmin}$, then A_2 does not satisfy the conditions of the theorem, whence $F^*(A_2) \not\simeq {}^3D_4(q^3)$.

Proof. Assume that the theorem is untrue and A is a counterexample such that $|O^{p'}(G)|$ is minimal. In view of [36, Thm. 1.2(vi)], every element of G is conjugate to its inverse. By [23; 35, Thms. 1.2 and 1.4], for every element $t \in G$ of odd prime order, all non-Abelian composition factors of $C_G(t)$ are simple groups of Lie type of order less than $\mathbf{Cmin}$. Thus $C_A(t)$ satisfies **(C)**, and by Lemma 1.5, $K \cap G$ is a 2-group. Now Lemma 3.4 implies that all cyclic groups generated by field automorphisms of same odd order in G are conjugate w.r.t. G. Since centralizer of every field automorphism in G is a group of Lie type of order less than $\mathbf{Cmin}$, again we use Lemma 1.5 and obtain the statement of the theorem by induction. Lemma 3.4 also implies that if $O^{p'}(G) \simeq D_4(q)$, then all cyclic groups generated by graph-field automorphisms are conjugate. Since centralizer of each graph-field automorphism in G is a group of Lie type of order less than $\mathbf{Cmin}$, again we appeal to Lemma 1.5 and arrive at (d) by induction. Thus we may assume that A does not contain a field or graph-field automorphism of odd order. Therefore either $G \simeq {}^3D_4(q^3)$ and A/G is a 2-group, or K contains an element s of order 3 such that $\langle s \rangle \cap A_1 = \{e\}$ (for groups ${}^3D_4(q^3)$, $\langle s \rangle \cap G = \{e\}$), $G \leftthreetimes \langle s \rangle = G \leftthreetimes \langle \tau \rangle$, and $K \cap G$ is a 2-group.

In the first case we obtain item (a) of the theorem provided that $(|A:G|,3)=1$. In the second case there exist two non-conjugate cyclic subgroups $\langle \tau \rangle$ and $\langle x \rangle$ of order 3 in A such that $\langle \tau \rangle \cap A_1 = \langle x \rangle \cap A_1 = \{e\}$ and $G \leftthreetimes \langle x \rangle = G \leftthreetimes \langle \tau \rangle$ (see [29, (9-1)]). Hence either $s = \tau \in K$, or $s = x \in K$. Assume that $q \neq 3^t$. In the former case items (b) and (c) follow from the known structure of Carter subgroups in a group of the set $\Gamma G_2(q)$, described in [10]; in the latter case we have $K \leq C_A(x)$. By [29, (9-1)], $C_G(x) \simeq \mathbf{P}GL_3^{\varepsilon}(q)$, where $q \equiv \varepsilon 1 \,(\mathrm{mod}\, 3)$, $\varepsilon = \pm$ and $\mathbf{P}GL_3^{+}(q) = \mathbf{P}GL_3(q)$, $\mathbf{P}GL_3^{-}(q) = \mathbf{P}GU_3(q)$. Consequently $K = (K \cap G) \times \langle y, \varphi \rangle$, where φ is a field automorphism of $O^{p'}(G)$ of order equal to a power of 2, y is an automorphism of order equal to a power of 3, and $x \in \langle y \rangle$. By the nilpotency of K, we obtain $y\varphi = \varphi y$, whence $C_{C_G(\varphi)}(x) = C_{C_G(x)}(\varphi)$. Now we have

$$O^{p'}(C_G(\varphi)) = \begin{cases} D_4(q^{1/|\varphi|}) & \text{if } O^{p'}(G) \simeq D_4(q), \\ {}^3D_4(q^{3/|\varphi|}) & \text{if } G \simeq {}^3D_4(q^3). \end{cases}$$

Hence $C_{C_G(x)}(\varphi) = C_{C_G(\varphi)}(x) \simeq \mathbf{P}GL_3^{\mu}(q^{1/|\varphi|})$, with $q^{1/|\varphi|} \equiv \mu 1 \,(\mathrm{mod}\, 3)$, where $\mu = \pm$ (ε and μ can be different). As indicated above, $K \cap G$ is a 2-group. On the other hand, by [26, Thm. 4], there exists an element z of order 3 centralizing a Sylow 2-subgroup of $C_G(x) = \mathbf{P}GL_3^{\varepsilon}(q)$ and belonging to $C_{C_G(x)}(\varphi) \simeq \mathbf{P}GL_3^{\mu}(q^{1/|\varphi|})$. Thus z centralizes K, that is, it lies in K. Since $K \cap G$ does not contain elements of odd order, this case is impossible.

Assume now that $q = 3^t$. Then $C_G(\tau) \simeq G_2(q)$ and we are done. In the second case $C_G(x) \simeq SL_2(q) \leftthreetimes U$, where U is a 3-group and $Z(C_G(x)) \cap U \neq \{e\}$, which clashes with Lemma 1.5. □

6. CARTER SUBGROUPS IN SEMILINEAR GROUPS OF LIE TYPE

THEOREM 6.1. Let G be a finite adjoint (not necessarily simple) group of Lie type over a field of characteristic p and $\overline{G}$ and σ be chosen so that $O^{p'}(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma$. Assume also that $G \not\simeq {}^3D_4(q^3)$. Choose a subgroup A in $\mathrm{Aut}(O^{p'}(\overline{G}_\sigma))$ so that $A \cap \overline{G}_\sigma = G$ and suppose that A is contained in the subgroup A_1 defined in Theorem 5.1, if $O^{p'}(G) = D_4(q)$. Let K be a Carter subgroup of A and $A = KG$. Then exactly one of the following statements holds:

(a) G is defined over a field of characteristic 2, $A = \langle G, \zeta g, t\rangle$, where t is a 2-element, K is contained in the normalizer of a t-stable Borel subgroup of G, and $K \cap \langle G, \zeta g\rangle$ satisfies one of the statements (a)-(f) of Lemma 3.8;

(b) $G \simeq \mathbf{P}SL_2(3^t)$, a field automorphism ζ is in A, $|\zeta| = t$ is odd, and, up to conjugation in G, $K = Q \rtimes \langle \zeta\rangle$, where Q is a Sylow 3-subgroup of $G_{\zeta_{3'}}$;

(c) $A = {}^2G_2(3^{2n+1}) \rtimes \langle\zeta\rangle$, $|\zeta| = 2n+1$, up to conjugation in G, $K = (K \cap G) \rtimes \langle \zeta\rangle$, $K \cap {}^2G_2(3^{2n+1}) = Q \times P$, where Q is of order 2, and $|P| = 3^{|\zeta|_3}$.

(d) p does not divide $|K \cap G|$, K contains a Sylow 2-subgroup of A, and by Lemma 3.5, A satisfies **(ESyl2)** if and only if G satisfies **(ESyl2)**.

Specifically, Carter subgroups of A are conjugate.

Remark. There exists a dichotomy for Carter subgroups in automorphism groups of finite groups of Lie type not containing a graph or graph-field automorphism of order 3. These are contained in the normalizer of a Borel subgroup, or else the characteristic is odd and a Carter subgroup contains a Sylow 2-subgroup of the whole group.

Suppose that the theorem is untrue and A is a counterexample with $|F^*(A)|$ minimal. Among such counterexamples, we take one for which $|A|$ is minimal. In this case for every almost simple group A_1 such that $|F^*(A_1)| < |F^*(A)|$, $F^*(A_1)$ is a finite simple group of Lie type, and A_1 satisfies the conditions of Theorem 6.1, Carter subgroups are conjugate. In fact, note that not more than one statement of the theorem can be fulfilled, since if either one of (b), (c) holds, then the condition $N_A(Q) = QC_A(Q)$ for a Sylow 2-subgroup Q of A is invalid, that is, (d) fails. (That no other two statements can hold simultaneously is evident.) Thus Carter subgroups of A_1 are conjugate. Note also that this fact immediately implies $|F^*(A)| \leqslant \mathbf{Cmin}$. Indeed, if $A_2 \in \mathcal{A}$ and $F^*(A_2) = \mathbf{Cmin}$, then either A_2 satisfies the conditions of Theorem 5.1, or A_2 meets the conditions of Theorem 6.1. As indicated in Theorem 5.1, the former case is impossible. The latter case, as we have just observed, is possible only if $|F^*(A)| \leqslant |F^*(A_2)| = \mathbf{Cmin}$ (since A is a counterexample to the theorem for which $|F^*(A)|$ is minimal).

We prove the theorem in the following way. If $F^*(A) \simeq \mathbf{P}Sp_{2n}(q)$, then the statement follows from Theorem 4.1. If $A = G$, then the statement follows from [3, 5, 6, 10]. We may therefore assume that $A/(A \cap G)$ is non-trivial. Let K be a Carter subgroup of A. First we show that if p divides $|K \cap G|$, then one of the items (a)-(c) specified in the theorem holds true. Next we prove that if p does not divide $|K \cap G|$, then K contains a Sylow 2-subgroup of A. Since both of the steps are quite cumbersome, we divide them between two sections. In view of [23], for every semisimple element $t \in G$, all non-Abelian composition factors of $C_G(t)$, and so also of $C_A(t)$, are simple groups of Lie type of order less than $|F^*(A)|$, and hence less than $\mathbf{Cmin}$. Therefore $C_A(t)$ satisfies **(C)**. In applying Lemmas 1.4 and 1.5, this fact will be used without further comment.

7. CARTER SUBGROUPS OF ORDER DIVISIBLE BY THE CHARACTERISTIC

Denote $K \cap G$ by K_G. For every group A satisfying the conditions of Theorem 6.1, the factor group A/G is Abelian, and is isomorphic to a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_t$ for some natural t, where $\mathbb{Z}_t$ denotes a cyclic group of order t. If the factor group A/G is not cyclic, then the group $O^{p'}(G)$ is split and A contains an element τa, where τ is a graph automorphism of $O^{p'}(G)$ and $a \in \overline{G}_\sigma$. It follows that every semisimple element of odd order is conjugate to its inverse in A (cf. Lemma 3.6). By Lemma 1.5, $|K_G|$ is divisible only by 2 and p. If $p = 2$, then K_G is a 2-group; it is contained in a proper K-stable parabolic subgroup P of G, and by Lemma 1.4, $KO_2(P)/O_2(P)$ is a Carter subgroup of $KP/O_2(P)$. Since $K_G \leq O_2(P)$, we have $(KO_2(P)/O_2(P)) \cap (P/O_2(P)) = \{e\}$. Hence P is a Borel subgroup of G, for otherwise $C_{P/O_2(P)}(KO_2(P)/O_2(P)) \neq \{e\}$, which clashes with the fact that $KO_2(P)/O_2(P)$ is a Carter subgroup of $KP/O_2(P)$. Thus P is a Borel subgroup and the theorem now follows from Lemma 3.8.

Now if $p \neq 2$, then again K_G is contained in a proper parabolic subgroup P of G such that $O_p(K_G) \leq O_p(P)$ and $O_2(K_G) \leq L$. By Lemmas 3.3 and 3.4, $H_2 \leq O_2(K \cap G) \leq K$. Lemma 2.3 implies that $O_p(K_G) \leq C_U(H_2) = \{e\}$. Therefore $K \cap G$ is a 2-group. By Lemmas 3.3 and 3.4, every element $x \in A \setminus G$ of odd order such that $\langle x \rangle \cap G = \{e\}$ centralizes some Sylow 2-subgroup of G. Hence K contains a Sylow 2-subgroup of A, that is, K satisfies (d). Therefore A/G is cyclic, and we may assume that $A = \langle G, \zeta g \rangle \in \Gamma G$.

Recall that we are in the conditions of Theorem 6.1, $A = \langle G, \zeta g \rangle$ is a counterexample to the theorem for which $|O^{p'}(G)|$ and $|A|$ are minimal, and K is a Carter subgroup of $\langle G, \zeta g \rangle$ such that p divides $|K_G|$. We have $K = \langle \zeta^k g, K_G \rangle$. Since $|O^{p'}(G)| \leqslant$ **Cmin**, Lemma 1.4 implies that KG/G is a Carter subgroup of A/G. Therefore $|\zeta^k| = |\zeta|$, and we may assume that $k = 1$ and $K = \langle K_G, \zeta g \rangle$.

In view of Lemma 3.2, there exists a proper σ- and $\bar{\zeta}g$-stable parabolic subgroup $\overline{P}$ of $\overline{G}$ such that $O_p(K_G) \leq R_u(\overline{P})$ and $K_G \leq \overline{P}$. In particular, $\overline{P}$ and $\overline{P}^{\bar{\zeta}}$ are conjugate in $\overline{G}$. Let Φ be the root system of $\overline{G}$ and Π be the set of fundamental roots in Φ. In view of [12, Prop. 8.3.1], $\overline{P}$ is conjugate to some $\overline{P}_J = \overline{B} \cdot \overline{N}_J \cdot \overline{B}$, where J is a subset of Π and $\overline{N}_J$ is a complete preimage of W_J in $\overline{N}$ under the natural homomorphism $\overline{N}/\overline{T} \to W$. Now $\overline{P}_J$ is $\bar{\varphi}$-stable, whence $\overline{P}_J^{\bar{\zeta}} = \overline{P}_J^{\bar{\gamma}^\varepsilon}$ (recall that $\bar{\zeta} = \bar{\gamma}^\varepsilon \bar{\varphi}^k$ by definition). Consider a symmetry ρ of the Dynkin diagram of Φ corresponding to $\bar{\gamma}$. Let $\overline{J}$ be the image of J under ρ. Clearly, $\overline{P}_J^{\bar{\gamma}} = \overline{P}_{\overline{J}}$. Since $\overline{P}$ and $\overline{P}^{\bar{\zeta}}$ are conjugate in $\overline{G}$, $\overline{P}_J$ and $\overline{P}_J^{\bar{\zeta}}$ are conjugate in $\overline{G}$. In view of [12, Thm. 8.3.3], either $\varepsilon = 0$, or $J = \overline{J}$; that is, $\overline{P}_J$ is $\bar{\zeta}$-stable.

Now we have $\overline{P}^{\bar{y}} = \overline{P}_J$, for some $\bar{y} \in \overline{G}$. Hence $\langle \bar{\zeta}g, \overline{P} \rangle^{\bar{y}} = \langle (\bar{\zeta}g)^{\bar{y}}, \overline{P}_J \rangle$ and $\overline{P}_J^{(\bar{\zeta}g)^{\bar{y}}} = \overline{P}_J$. It follows that

$$(\bar{\zeta}g)^{\bar{y}} = \bar{y}^{-1}\bar{\zeta}g\bar{y} = \bar{\zeta}\left(\bar{\zeta}^{-1}\bar{y}^{-1}\bar{\zeta}g\bar{y}\right) = \bar{\zeta} \cdot h,$$

where $h = \left(\bar{\zeta}^{-1}\bar{y}^{-1}\bar{\zeta}g\bar{y}\right) \in \overline{G}$. Since $\overline{P}_J^{\bar{\zeta}} = \overline{P}_J = \overline{P}_J^{h^{-1}}$, we obtain $h \in N_{\overline{G}}(\overline{P}_J)$. By [12, Thm. 8.3.3], $N_{\overline{G}}(\overline{P}_J) = \overline{P}_J$, and so therefore $\langle \bar{\zeta}g, \overline{P} \rangle^y = \langle \bar{\zeta}, \overline{P}_J \rangle$. Both $\overline{P}$ and $\overline{P}_J$ are σ-stable. Hence $\bar{y}\sigma(\bar{y}^{-1}) \in N_{\overline{G}}(\overline{P}) = \overline{P}$. On the Lang–Steinberg theorem [19, Thm. 10.1], we may put $\bar{y} = \sigma(\bar{y})$, that is, $\bar{y} \in \overline{G}_\sigma$. Since $\overline{G}_\sigma = \overline{T}_\sigma \cdot O^{p'}(\overline{G}_\sigma)$ and $\overline{T} \leq \overline{P}_J$, we can suppose that $\bar{y} \in O^{p'}(\overline{G}_\sigma)$. Thus, up to conjugation in G, we may assume that $\overline{K} \leq \langle \bar{\zeta}, \overline{P}_J \rangle = \overline{P}_J \leftthreetimes \langle \bar{\zeta} \rangle$, and

$$K \leq \langle (\overline{P}_J \cap G), \zeta g \rangle = \langle P_J, \zeta g \rangle;$$

in particular, $g \in (\overline{P}_J)_\sigma$. Moreover, if $\overline{L}_J = \langle \overline{T}, \overline{X}_r \mid r \in J \cup -J \rangle$, then $\overline{L}_J$ is a σ- and $\bar{\zeta}$-stable Levi factor of $\overline{P}_J$ and $L_J = \overline{L}_J \cap G$ is a ζ-stable Levi factor of P_J. It follows that L_J^g is a ζg-stable Levi factor in P_J.

Since all Levi factors are conjugate w.r.t. $O_p(P_J)$, we may assume that L_J is a ζg-stable Levi factor. By Lemma 1.4,

$$KO_p(P_J)/O_p(P_J) = X$$

is a Carter subgroup of $\langle P_J, \zeta g\rangle/O_p(P_J)$, and

$$KZ(L_J)O_p(P_J)/Z(L_J)O_p(P_J) = \widetilde{X}$$

is a Carter subgroup of $\langle P_J, \zeta g\rangle/Z(L_J)O_p(P_J)$. Recall that $K = \langle \zeta g, K_G\rangle$; hence if v and $\tilde{v}$ are images of g under the natural homomorphisms

$$\omega : \langle P_J, \zeta g\rangle \to \langle L_J, \zeta v\rangle \simeq \langle P_J, \zeta g\rangle/O_p(P_J),$$

$$\tilde{\omega} : \langle P_J, \zeta g\rangle \to \langle P_J, \zeta g\rangle/Z(L_J)O_p(P_J) \simeq \langle L_J, \zeta v\rangle/Z(L_J),$$

then $X = \langle \zeta v, K_G^{\omega}\rangle$ and $\widetilde{X} = \langle \zeta\tilde{v}, K_G^{\tilde{\omega}}\rangle$. Note that $O_p(P)$ and $Z(L_J)$ are characteristic subgroups of P and L_J, respectively, and so we may treat ζ as an automorphism of $L_J \simeq P/O_p(P)$ and $\widetilde{L} = L_J/Z(L_J)$. We also observe that all non-Abelian composition factors of P are simple groups of Lie type of order less than **Cmin**, and hence $\langle P, \zeta g\rangle$ satisfies **(C)**. We can therefore apply Lemma 1.4 to $\langle \widetilde{L}, \zeta\tilde{v}\rangle$, $\langle L_J, \zeta v\rangle$, and $\langle P, \zeta g\rangle$.

If P_J is a Borel subgroup of G, then the statement of the theorem follows from Lemma 3.8. Therefore we may suppose that $L_J \neq Z(L_J)$, that is, P_J is not a Borel subgroup of G. Consequently $L_J = H(G_1 * \ldots * G_k)$, where G_i are subsystem subgroups of G, $k \geqslant 1$, and H is a Cartan subgroup of G. Let $\zeta g = (\zeta_2 g_2)\cdot(\zeta_{2'} g_{2'})$ be the product of 2- and $2'$-parts of ζg (with $g_2, g_{2'} \in (\overline{P}_J)_\sigma$). Now $\zeta_{2'} = \varphi^k$, for some k, is a field automorphism (recall that the triality automorphism is not considered), and $\zeta_{2'}$ normalizes each G_i since φ normalizes each G_i. Moreover, in view of Lemma 3.3, $\zeta_{2'}$ centralizes a Sylow 2-subgroup of H. In particular, $\zeta_{2'}$ centralizes a Sylow 2-subgroup of $Z(L_J) \leq H$. Therefore every element of odd order in $\langle L_J, \zeta_{2'} v_{2'}\rangle$ centralizes a Sylow 2-subgroup of $Z(L_J)$ (here, $v_{2'}$ is an image of $g_{2'}$ under ω).

Now $\widetilde{L} = (\mathbf{P}G_1 \times \ldots \times \mathbf{P}G_k)\widetilde{H}$, where $\widetilde{H} = H^{\tilde{\omega}}$ and $\mathbf{P}G_1, \ldots, \mathbf{P}G_k$ are canonical finite groups of Lie type with trivial center. Put $M_i = C_{\widetilde{L}}(\mathbf{P}G_i)$; clearly, $M_i = (\mathbf{P}G_1 \times \ldots \times \mathbf{P}G_{i-1} \times \mathbf{P}G_{i+1} \times \ldots \times \mathbf{P}G_k)C_{\widetilde{H}}(\mathbf{P}G_i)$. Denote by L_i a factor group $\widetilde{L}/M_i$, and by π_i the corresponding natural homomorphism. Then L_i is a finite group of Lie type and $\mathbf{P}G_i \leq L_i \leq \widehat{\mathbf{P}G}_i$.

Put $M_{i,j} = C_{\widetilde{L}}(\mathbf{P}G_i \times \mathbf{P}G_j)$; then

$$M_{i,j} = (\mathbf{P}G_1 \times \ldots \times \mathbf{P}G_{i-1} \times \mathbf{P}G_{i+1} \times \ldots \times \mathbf{P}G_{j-1} \times \mathbf{P}G_{j+1} \times \ldots \times \mathbf{P}G_k)C_{\widetilde{H}}(\mathbf{P}G_i \times \mathbf{P}G_j).$$

Denote by $\pi_{i,j}$ the corresponding natural homomorphism $\widetilde{L} \to \widetilde{L}/M_{i,j}$. If M_i (resp., $M_{i,j}$) is ζ-stable, then M_i (resp., $M_{i,j}$) is normal in $\langle \widetilde{L}, \zeta\tilde{v}\rangle$, and we write π_i (resp., $\pi_{i,j}$) for the natural homomorphism $\pi_i : \langle \widetilde{L}, \zeta\tilde{v}\rangle \to \langle \widetilde{L}, \zeta\tilde{v}\rangle/M_i$ ($\pi_{i,j} : \langle \widetilde{L}, \zeta\tilde{v}\rangle \to \langle \widetilde{L}, \zeta\tilde{v}\rangle/M_{i,j}$).

Now we handle ζ. Since ζ^2 is a field automorphism, there are two cases to consider: either ζ normalizes $\mathbf{P}G_i$, or ζ^2 normalizes $\mathbf{P}G_i$ and $\mathbf{P}G_i^{\zeta} = \mathbf{P}G_j$ for some $j \neq i$. We treat these two cases separately.

Let ζ normalize $\mathbf{P}G_i$. Then ζ normalizes M_i, and Lemma 1.4 implies that $\widetilde{X}^{\pi_i} = K_i$ is a Carter subgroup of $\langle L_i, (\zeta\tilde{v})^{\pi_i}\rangle$. Since $\langle L_i, (\zeta\tilde{v})^{\pi_i}\rangle$ is a semilinear group of Lie type satisfying the conditions of Theorem 6.1 (by construction, ζ^2 is a field automorphism, and so we are not in the conditions of Theorem 6.1), $|L_i| < |G|$, and p does not divide $|K_i|$, we see that K_i contains a Sylow 2-subgroup Q_i of $\langle L_i, (\zeta\tilde{v})^{\pi_i}\rangle$ (in particular, $p \neq 2$), and by Lemma 1.6, the group $\langle L_i, (\zeta\tilde{v})^{\pi_i}\rangle$ satisfies **(ESyl2)**.

Let ζ^2 normalize $\mathbf{P}G_i$ and $\mathbf{P}G_i^\zeta = \mathbf{P}G_j$. Then $M_{i,j}$ is normal in $\langle \widetilde{L}, \zeta\tilde{v}\rangle$. We want to show that $\langle \widetilde{L}, \zeta\tilde{v}\rangle^{\pi_{i,j}}$ satisfies **(ESyl2)**. Since $M_{i,j}$ is a normal subgroup of $\langle \widetilde{L}, \zeta\tilde{v}\rangle$, $(\widetilde{X})^{\pi_{i,j}}$ is a Carter subgroup of $\langle \widetilde{L}, \zeta\tilde{v}\rangle^{\pi_{i,j}}$ by Lemma 1.4. Consider the subgroup

$$\langle (\mathbf{P}G_i)^{\pi_{i,j}} \times (\mathbf{P}G_j)^{\pi_{i,j}}, \widetilde{X}^{\pi_{i,j}}\rangle$$

of $\langle \widetilde{L}, \zeta\tilde{v}\rangle^{\pi_{i,j}}$ (note that $(\mathbf{P}G_i)^{\pi_{i,j}} \simeq \mathbf{P}G_i$ and $(\mathbf{P}G_j)^{\pi_{i,j}} \simeq \mathbf{P}G_j$, and till the end of this paragraph, these groups will be identified for brevity). Now we are in the conditions of Lemma 1.8: namely, we have a finite group $\widetilde{G} = (\widetilde{X})^{\pi_{i,j}}(\mathbf{P}G_i \times \mathbf{P}G_j)$, where $\mathbf{P}G_i \simeq \mathbf{P}G_j$ has trivial center. Consequently $\mathrm{Aut}_{(\widetilde{X})^{\pi_{i,j}}}(\mathbf{P}G_i) \simeq \mathrm{Aut}_{\widetilde{X}}(\mathbf{P}G_i)$ is a Carter subgroup of $\mathrm{Aut}_{\widetilde{G}}(\mathbf{P}G_i)$. Furthermore, $\mathbf{P}G_i$ is a canonical finite group of Lie type, and

$$\mathbf{P}G_i \leq \mathrm{Aut}_{\widetilde{G}}(\mathbf{P}G_i) \leq \mathrm{Aut}(\mathbf{P}G_i);$$

that is, $\mathrm{Aut}_{\widetilde{G}}(\mathbf{P}G_i)$ satisfies the conditions of Theorem 6.1 (by construction, ζ^2 is a field automorphism, and so we are not in the conditions of Theorem 5.1) and $(\widetilde{X})^{\pi_{i,j}} \cap (\mathbf{P}G_i \times \mathbf{P}G_j)$ is not divisible by the characteristic. By induction, $\mathrm{Aut}_{(\widetilde{X})^{\pi_{i,j}}}(\mathbf{P}G_i)$ contains a Sylow 2-subgroup of $\mathrm{Aut}_{\widetilde{G}}(\mathbf{P}G_i)$ (in particular, $p \neq 2$). A similar argument shows that $\mathrm{Aut}_{\widetilde{X}}(\mathbf{P}G_j)$ contains a Sylow 2-subgroup of $\mathrm{Aut}_{\widetilde{G}}(\mathbf{P}G_j)$. Therefore $\mathrm{Aut}_{\widetilde{G}}(\mathbf{P}G_i)$ and $\mathrm{Aut}_{\widetilde{G}}(\mathbf{P}G_j)$ satisfy **(ESyl2)**. Since $\mathrm{Aut}_{\widetilde{G}}(\mathbf{P}G_i) \leq \mathrm{Aut}_{\langle \widetilde{L}, \zeta\tilde{v}\rangle^{\pi_{i,j}}}(\mathbf{P}G_i)$ and $\mathrm{Aut}_{\widetilde{G}}(\mathbf{P}G_j) \leq \mathrm{Aut}_{\langle \widetilde{L}, \zeta\tilde{v}\rangle^{\pi_{i,j}}}(\mathbf{P}G_j)$, Lemmas 2.6 and 3.5 imply that induced automorphisms groups $\mathrm{Aut}_{\langle \widetilde{L}, \zeta\tilde{v}\rangle^{\pi_{i,j}}}(\mathbf{P}G_i)$ and $\mathrm{Aut}_{\langle \widetilde{L}, \zeta\tilde{v}\rangle^{\pi_{i,j}}}(\mathbf{P}G_j)$ satisfy **(ESyl2)**. Consider $N_{\langle \widetilde{L}, \zeta\tilde{v}\rangle^{\pi_{i,j}}}(\mathbf{P}G_i)$ and $N_{\langle \widetilde{L}, \zeta\tilde{v}\rangle^{\pi_{i,j}}}(\mathbf{P}G_j)$. Since

$$|\langle \widetilde{L}, \zeta\tilde{v}\rangle^{\pi_{i,j}} : N_{\langle \widetilde{L}, \zeta\tilde{v}\rangle^{\pi_{i,j}}}(\mathbf{P}G_i)| = |\langle \widetilde{L}, \zeta\tilde{v}\rangle^{\pi_{i,j}} : N_{\langle \widetilde{L}, \zeta\tilde{v}\rangle^{\pi_{i,j}}}(\mathbf{P}G_j)| = 2,$$

it is easy to see that for every element h of $\langle \widetilde{L}, \zeta\tilde{v}\rangle^{\pi_{i,j}}$, the coset equality $hN_{\langle \widetilde{L}, \zeta\tilde{v}\rangle^{\pi_{i,j}}}(\mathbf{P}G_i) = hN_{\langle \widetilde{L}, \zeta\tilde{v}\rangle^{\pi_{i,j}}}(\mathbf{P}G_j)$ holds, whence $N_{\langle \widetilde{L}, \zeta\tilde{v}\rangle^{\pi_{i,j}}}(\mathbf{P}G_i) = N_{\langle \widetilde{L}, \zeta\tilde{v}\rangle^{\pi_{i,j}}}(\mathbf{P}G_j)$. By construction, $C_{\langle \widetilde{L}, \zeta\tilde{v}\rangle^{\pi_{i,j}}}(\mathbf{P}G_i) \cap C_{\langle \widetilde{L}, \zeta\tilde{v}\rangle^{\pi_{i,j}}}(\mathbf{P}G_j) = \{e\}$, and by Lemma 1.7 (with $C_{\langle \widetilde{L}, \zeta\tilde{v}\rangle^{\pi_{i,j}}}(\mathbf{P}G_i)$ and $C_{\langle \widetilde{L}, \zeta\tilde{v}\rangle^{\pi_{i,j}}}(\mathbf{P}G_j)$ treated as normal subgroups), the normalizer $N_{\langle \widetilde{L}, \zeta\tilde{v}\rangle^{\pi_{i,j}}}(\mathbf{P}G_i)$ satisfies **(ESyl2)**. Now $|\langle \widetilde{L}, \zeta\tilde{v}\rangle^{\pi_{i,j}} : N_{\langle \widetilde{L}, \zeta\tilde{v}\rangle^{\pi_{i,j}}}(\mathbf{P}G_1)| = 2$, and so $\langle \widetilde{L}, \zeta\tilde{v}\rangle^{\pi_{i,j}}$ satisfies **(ESyl2)** by Lemma 1.9.

We claim that $\langle L_J, \zeta v\rangle$ satisfies **(ESyl2)**. Since $\widetilde{L} \neq \{e\}$, $p \neq 2$ as noted above. Let Q be a Sylow 2-subgroup of $\langle L_J, \zeta v\rangle$. Consider an element $x \in N_{\langle L_J, \zeta v\rangle}(Q)$ of odd order. We need to prove that x centralizes Q. Elsewhere above we observed that every element of odd order in $\langle L_J, \zeta v\rangle$ centralizes $Q \cap Z(L_J)$; hence if $\tilde{x} = x^{\tilde{\omega}}$ centralizes $\widetilde{Q} = Q^{\tilde{\omega}} \simeq Q/(Q \cap Z(L_J))$, then x centralizes Q. Now either M_i is normal in $\langle \widetilde{L}, \zeta\tilde{v}\rangle$, or $M_{i,j}$ is normal in $\langle \widetilde{L}, \zeta\tilde{v}\rangle$ and $\left(\bigcap_i M_i\right) \bigcap \left(\bigcap_{i,j} M_{i,j}\right) = \{e\}$. Moreover, by the above, $\tilde{x}^{\pi_i}$ centralizes $\widetilde{Q}M_i/M_i$ and $\tilde{x}^{\pi_{i,j}}$ centralizes $\widetilde{Q}M_{i,j}/M_{i,j}$. By Lemma 1.7 (with normal subgroups M_i and $M_{i,j}$), $\tilde{x}$ centralizes $\widetilde{Q}$.

Thus $\langle L, \zeta v\rangle$ satisfies **(ESyl2)**, and by Lemma 1.6, there exists a Carter subgroup F of $\langle L, \zeta v\rangle$ containing Q. Since $\langle L, \zeta v\rangle$ satisfies **(C)**, Theorem 1.1 implies that $X = K^\omega$ and F are conjugate; that is, X contains a Sylow 2-subgroup of $\langle L, \zeta v\rangle$, and up to conjugation in $\langle P_J, \zeta g\rangle$, K contains a Sylow 2-subgroup of $\langle P_J, \zeta v\rangle$. In particular, the Sylow 2-subgroup Q_1 of a Cartan subgroup H is in K and Q_1 centralizes $K \cap O_p(P_J) \neq \{e\}$, which contradicts Lemma 2.3.

8. CARTER SUBGROUPS OF ORDER NOT DIVISIBLE BY THE CHARACTERISTIC

Again we are in the conditions of Theorem 6.1. As noted in the previous section, for every group A satisfying the conditions of Theorem 6.1, the factor group A/G is Abelian, and is isomorphic to a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_t$, for some natural t. If the factor group A/G is not cyclic, then $O^{p'}(G)$ is split and A contains an element τa, where τ is a graph automorphism of $O^{p'}(G)$ and $a \in \overline{G}_\sigma$. Thus if A/G is not cyclic, or $\Phi(\overline{G}) \neq A_n, D_{2n+1}, E_6$, then every semisimple element of G is conjugate to its inverse, as follows by Lemma 3.6 and [10, Lemma 2.2]. By Lemma 1.5, $K_G = K \cap G$ is a 2-group. Under the conditions of Theorem 6.1, the group A/G is Abelian, and if $\overline{A}_1$ is a Hall $2'$-subgroup of A/G, then $\overline{A}_1$ is cyclic. Let x be the preimage of a generating element of $\overline{A}_1$ taken in K. Then $\langle x\rangle \cap G \leq \langle x\rangle \cap \overline{G}_\sigma \leq K \cap \overline{G}_\sigma = K \cap (A \cap \overline{G}_\sigma) = K \cap G$. As indicated above, $K \cap G$ is a 2-group; hence $\langle x\rangle \cap \overline{G}_\sigma = \{e\}$. By Lemma 3.4, the element x w.r.t. $\overline{G}_\sigma$ is conjugate to a field automorphism of odd order, and by Lemma 3.3, x centralizes a Sylow 2-subgroup of G. Hence K_G is a Sylow 2-subgroup of G (in particular, $p \neq 2$), and since A/G is Abelian, Lemma 1.7 implies that K contains a Sylow 2-subgroup of A. Thus Theorem 6.1 is true in this instance. Therefore we may assume that $A = \langle \zeta g, G\rangle$ is a semilinear group of Lie type, $K = \langle \zeta^k g, K_G\rangle$ is a Carter subgroup of A, and $\Phi(\overline{G}) \in \{A_n, D_{2n+1}, E_6\}$.

As in the previous section, we can put $k = 1$. Since G_ζ is non-trivial, the centralizer $C_G(\zeta g)$ is as well; hence the subgroup K_G, too, is non-trivial. Therefore $Z(K) \cap K_G \neq \{e\}$. Consider an element $x \in Z(K) \cap K_G$ of prime order. Then $K \in C_A(x) = \langle \zeta g, C_G(x)\rangle$. Now $C_{\overline{G}}(x)^0 = \overline{C}$ is a connected σ-stable reductive subgroup of maximal rank in $\overline{G}$. Moreover, $\overline{C}$ is a characteristic subgroup of $C_{\overline{G}}(x)$ and $C_{\overline{G}}(x)/\overline{C}$ is isomorphic to a subgroup of Δ (see [22, Prop. 2.10]). Thus K is contained in $\langle K, C\rangle$, where $C = \overline{C} \cap G$. Moreover, by Lemma 3.1, the subgroup $C = \overline{C} \cap G = T(G_1 * \ldots * G_m)$ is normal in $C_A(x)$ and $K_G C/C$ is isomorphic to a subgroup of Δ. Assume that $|K_G|$ is not divisible by 2.

If $m = 0$, then $C = T = Z(C)$ is a maximal torus. Consequently $\overline{T}$ is $\bar{\zeta} g$-stable. In view of Lemma 3.7, $N_A(C_A(x)) \neq C_A(x)$. Since $C_A(x)$ is solvable in this instance, we arrive at a contradiction with Lemma 1.5.

If $m \geqslant 1$, then $Z(C)$ and $G_1 * \ldots * G_m$ are normal subgroups of $\langle K, C\rangle$. Hence $\widetilde{G} = \langle K, G_1 * \ldots * G_m * Z(C)\rangle/Z(C) \leq \langle K, C\rangle/Z(C)$. It follows that $\widetilde{G} = \widetilde{K}(\mathbf{P}G_1 \times \ldots \times \mathbf{P}G_m)$, where $\widetilde{K} = KZ(C)/Z(C)$ is a Carter subgroup of $\widetilde{G}$ (cf. Lemma 1.4) and $Z(\mathbf{P}G_i)$ is trivial. Now $\widetilde{K}$ acts by conjugation on $\{\mathbf{P}G_1, \ldots, \mathbf{P}G_m\}$, and without lost of generality, we may assume that $\{\mathbf{P}G_1, \ldots, \mathbf{P}G_m\}$ is a $\widetilde{K}$-orbit. Thus we are in the conditions of Lemma 1.8 and $\mathrm{Aut}_{\widetilde{K}}(\mathbf{P}G_1)$ is a Carter subgroup of $\mathrm{Aut}_{\widetilde{G}}(\mathbf{P}G_1)$. Moreover, $|\widetilde{K} \cap \mathbf{P}G_1 \times \ldots \times \mathbf{P}G_m|$ is not divisible by the characteristic. By induction, either $\mathrm{Aut}_{\widetilde{K}}(\mathbf{P}G_1)$ contains a Sylow 2-subgroup of $\mathrm{Aut}_{\widetilde{G}}(\mathbf{P}G_1)$, or $\mathrm{Aut}_{\widetilde{G}}(\mathbf{P}G_1)$ satisfies the conditions of Theorem 5.1 and $\mathrm{Aut}_{\widetilde{K}}(\mathbf{P}G_1) \cap \mathbf{P}G_1$ is a non-trivial 2-group; in particular, p is odd. In any case $|K \cap G|$ is divisible by 2, which contradicts our assumption. Therefore the order $|K_G|$ is even, and we may conceive of $x \in Z(K) \cap K_G$ as an involution.

We write $\zeta g = \zeta_2 g_1 \cdot \zeta_{2'} g_2$, where $\zeta_2 g_1$ is a 2-part, and $\zeta_{2'} g_2$ is a $2'$-part, of ζg. By Lemma 3.3, the element $\zeta_{2'}$ centralizes a Sylow 2-subgroup Q_G of G, and so we may assume that the order of g_2 is odd. Up to conjugation in G, we can suppose that $\zeta_{2'}$ centralizes a Sylow 2-subgroup of K_G. In particular, $\zeta_{2'}$ centralizes x. Let Q be a Sylow 2-subgroup in $C_G(x)$. Then there exists $y \in G$ such that $Q^y \leq Q_G$. Replacing the subgroup K by its conjugate K^y, we may think of $\zeta_{2'}$ as centralizing a Sylow 2-subgroup of $C_G(x)$. Since $\zeta_{2'} g_2$ centralizes x, we obtain $g_2 \in C_{\overline{G}_\sigma}(x)$. Moreover, $g_2 \in C_{\overline{G}}(x)^0$ by Lemma 2.1. In particular, g_2 normalizes each G_i and centralizes $Z(C)$ and $Z(C_G(x))$.

Note that $\zeta_{2'}$ normalizes each G_i and centralizes a Sylow 2-subgroup of $Z(C_G(x))$. (Recall that $\zeta_{2'}$ centralizes a Sylow 2-subgroup of $C_G(x)$.) Indeed, $\zeta_{2'}$ normalizes C; hence it normalizes characteristic subgroups $O^{p'}(C) = G_1 * \ldots * G_m$ and $Z(C)$ of C. We may so consider an induced automorphism $\zeta_{2'}$ of

$$O^{p'}(C)/(Z(C) \cap O^{p'}(C) = \mathbf{P}G_1 \times \ldots \times \mathbf{P}G_m.$$

Since each group $\mathbf{P}G_i$ has trivial center and is not representable as a direct product of proper subgroups, the corollary to the Krull–Remak–Schmidt theorem [37, 3.3.10] implies that $\zeta_{2'}$ permutes distinct $\mathbf{P}G_i$.

Since $\zeta_{2'}$ centralizes a Sylow 2-subgroup of $C_G(x)$ and $C \trianglelefteq C_G(x)$, $\zeta_{2'}$ centralizes a Sylow 2-subgroup of C; hence it centralizes the Sylow 2-subgroup $Q_1 \times \ldots \times Q_m$ of $\mathbf{P}G_1 \times \ldots \times \mathbf{P}G_m$, where Q_i is a Sylow 2-subgroup in $\mathbf{P}G_i$. If $\zeta_{2'}$ induced a non-trivial permutation on the set $\{\mathbf{P}G_1, \ldots, \mathbf{P}G_m\}$, then it would induce one on $\{Q_1, \ldots, Q_m\}$. This is impossible since each Q_i is non-trivial. Thus every element of odd order in $\langle K, C\rangle$ centralizes a Sylow 2-subgroup of $Z(C)$ and normalizes each G_i.

If $\Phi(\overline{G}) = E_6$, then centralizer of every involution of G in $\overline{G}$ is connected by Lemma 2.1. In view of Lemma 2.8, every involution of G is contained in a maximal torus T such that $N(G,T)/T \simeq W$, where W is a Weyl group of $\overline{G}$. It is well known that $\overline{C}$ is generated by the torus $\overline{T}$ and by $\overline{T}$-root subgroups. We write $\overline{C} = \overline{T}(\overline{G}_1 * \ldots * \overline{G}_k)$. Since $\overline{T}_\sigma$ either is obtained from a maximal split torus $\overline{H}$ by twisting with an element w_0 of order 2 or coincides with $\overline{H}_\sigma$, and each field automorphism acts trivially on the factor group $N_{\overline{G}}(\overline{H})/\overline{H}$, we conclude that $\bar{\zeta}_{2'}$ normalizes every subgroup $\overline{G}_i$. Hence if $\Phi(\overline{G}_i) = D_4$, then $\bar{\zeta}_{2'}$ induces a field (but not graph or graph-field) automorphism of $\overline{G}_i$. Moreover, since σ acts trivially on the factor group $N_{\overline{G}}(\overline{T})/\overline{T}$ (see Lemma 3.3), [23, Prop. 6] implies that σ normalizes each $\overline{G}_i$. Therefore no one of G_i is isomorphic to ${}^3D_4(q^3)$. If $\Phi(\overline{G})$ coincides with A_n or D_n, then no one of G_i is isomorphic to ${}^3D_4(q^3)$ by [15, Props. 7, 8, 10]. Therefore none of the groups G_i are isomorphic to ${}^3D_4(q^3)$ in any case. Moreover, Lemma 2.1 implies that $|K_G : (K_G \cap C)|$ divides $|C_{\overline{G}}(x)/C_{\overline{G}}(x)^0|$ and $C_{\overline{G}}(x)/C_{\overline{G}}(x)^0$ is a 2-group. In [16] it was proved that if the root system Φ has type D_n and Ψ is its subsystem of type D_4, then no element of $N_{W(\Phi)}(W(\Psi))$ induces an order 3 symmetry of the Dynkin diagram of Ψ. Since ζ^2 is a field automorphism, the lack of order 3 symmetry, together with [23, Prop. 6], implies that $\zeta_{2'}$ is a field (but not graph or graph-field) automorphism, for any G_i. Therefore the induced automorphism group $\langle \mathrm{Aut}_{\widetilde{K}}(\mathbf{P}G_i), \mathbf{P}G_i\rangle$ satisfies the conditions of Theorem 6.1, for all i.

Now consider $\widetilde{G} = \widetilde{K}(\mathbf{P}G_1 \times \ldots \times \mathbf{P}G_m) \leq \langle K, C\rangle/Z(C)$ (possibly with $m = 0$), where $\widetilde{K} = KZ(C)/Z(C)$ is a Carter subgroup of $\widetilde{G}$ (see Lemma 1.4), $\mathbf{P}G_i$ does not factor into a direct product of proper subgroups for all i, and $Z(\mathbf{P}G_i) = \{e\}$. By Lemma 1.8, $\mathrm{Aut}_{\widetilde{K}}(\mathbf{P}G_1)$ is a Carter subgroup of $\mathrm{Aut}_{\widetilde{G}}(\mathbf{P}G_1)$. Since $\mathbf{P}G_1$ is a finite group of Lie type and $\mathrm{Aut}_{\widetilde{G}}(\mathbf{P}G_1)$ satisfies the conditions of Theorem 6.1, by induction we derive that $\mathrm{Aut}_{\widetilde{G}}(\mathbf{P}G_1)$ satisfies **(ESyl2)**. Similarly, $\mathrm{Aut}_{\widetilde{G}}(\mathbf{P}G_i)$ satisfies **(ESyl2)** for all i. We have

$$\mathrm{Aut}_{\langle K,C\rangle/Z(C)}(\mathbf{P}G_i) \geq \mathrm{Aut}_{\widetilde{G}}(\mathbf{P}G_i);$$

so $\mathrm{Aut}_{\langle K,C\rangle/Z(C)}(\mathbf{P}G_i)$ satisfies **(ESyl2)** by Lemmas 2.6 and 3.5. Since $C_{\langle K,C\rangle/Z(C)}(\mathbf{P}G_1 \times \ldots \times \mathbf{P}G_m) = \{e\}$, Lemma 1.7 (with normal subgroups $C_{\langle K,C\rangle/Z(C)}(\mathbf{P}G_1) \cap N_{\langle K,C\rangle/Z(C)}(\mathbf{P}G_1), \ldots, C_{\langle K,C\rangle/Z(C)}(\mathbf{P}G_m) \cap N_{\langle K,C\rangle/Z(C)}(\mathbf{P}G_1))$ implies that $N_{\langle K,C\rangle/Z(C)}(\mathbf{P}G_1)$ satisfies **(ESyl2)**. Now

$$|\langle K, C\rangle/Z(C) : N_{\langle K,C\rangle/Z(C)}(\mathbf{P}G_1)| = 2^t,$$

and each element of odd order in $\langle K, C\rangle/Z(C)$ normalizes $\mathbf{P}G_1$; hence the factor group $\langle K, C\rangle/Z(C)$ satisfies **(ESyl2)** by Lemma 1.9, and so does $\langle K, C\rangle$ by Lemma 1.7. Since $|\mathbf{P}G_i| < \mathbf{Cmin}$ for all i, $\langle K, C\rangle$ satisfies **(C)**. By Lemma 1.6, there exists a Carter subgroup F of $\langle K, C\rangle$ containing a Sylow 2-subgroup of $\langle K, C\rangle$. By Theorem 1.1, the subgroups F and K are conjugate in $\langle K, C\rangle$, and so K contains a Sylow 2-subgroup Q of $\langle K, C\rangle$. Since $|C_G(x) : C|$ is a power of 2 and $\langle K, C\rangle$ normalizes $C_G(x)$, we conclude that $|\langle K, C_G(x)\rangle : \langle K, C\rangle|$ is a power of 2. Moreover, by construction, each element of odd order in $\langle K, C_G(x)\rangle$ is in $\langle K, C\rangle$. By Lemma 1.9, therefore, $\langle K, C_G(x)\rangle$ satisfies **(ESyl2)** and K contains a Sylow 2-subgroup Q of $\langle K, C_G(x)\rangle$.

Let ΓQ be a Sylow 2-subgroup of $\langle G, \zeta g\rangle$ containing Q and let $t \in Z(\Gamma Q) \cap G$. Then $t \in C_G(x)$, hence $t \in Z(Q)$, and so $t \in Z(K)$. Thus if we replace x by t in the above argument we obtain $Q = \Gamma Q$, that is, K contains a Sylow 2-subgroup of $\langle G, \zeta g\rangle$, which completes the proof of Theorem 6.1. □

9. PROOF OF THE MAIN THEOREM

Before we formulate the main theorem, we point out a consequence of Theorem 6.1.

COROLLARY 9.1. **Cmin** $= \infty$, that is, $\mathcal{A} = \varnothing$.

Proof. In fact, let $\mathcal{A} \neq \varnothing$ and $A \in \mathcal{A}$ be such that $|F^*(A)| =$ **Cmin**. Since $F^*(A) = O^{p'}(\overline{G}_\sigma)$ for some adjoint simple connected algebraic group $\overline{G}$ and some Frobenius map σ, we denote the intersection $A \cap \overline{G}_\sigma$ by G. As noted at the beginning of Sec. 4, we may assume that $A = KF^*(A) = KG$. Therefore the group A satisfies the conditions of Theorem 5.1, or of Theorem 6.1. In both of the cases, however, Carter subgroups of A have been proven to be conjugate, and we are therefore led to a contradiction with the choice of A. □

In order to state the main theorem not appealing to the classification of finite simple groups, we couch the following definition: A finite group is a *K-group* if all of its non-Abelian composition factors are known simple groups.

THEOREM 9.2 (Main Theorem). Let G be a finite K-group. Then Carter subgroups of G are conjugate.

Proof. By [6, Thm. 1.1; 9, Table; 10, Thms. 3.3-3.5] and Theorems 4.1, 5.1, and 6.1 above, for every known simple group S and every nilpotent subgroup N of its automorphism group, Carter subgroups of $\langle N, S\rangle$ are conjugate. Thus G satisfies **(C)**. Hence Carter subgroups of G are conjugate by Theorem 1.1. □

By Lemma 1.4 and Theorem 9.2, a homomorphic image of a Carter subgroup is a Carter subgroup.

THEOREM 9.3. Let G be a finite K-group, H be a Carter subgroup of G, and N be a normal subgroup of G. Then HN/N is a Carter subgroup of G/N.

Acknowledgement. I express my sincere gratitude to Prof. M. C. Tamburini for an extremely helpful discussion of the paper, which allowed me to substantially improve on its first version. Thanks also are due to A. V. Vasiliev, M. A. Grechkoseeva, A. V. Zavarnitsine, and V. D. Mazurov for pointing out some misprints and errors.

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