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ON THE CONJUGACY PROBLEM FOR CARTER SUBGROUPS

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Abstract: We prove that the Carter subgroups of a finite group are conjugate if so are the Carter subgroups of the group of induced automorphisms for every nonabelian composition factor.

Keywords: Carter subgroup, almost simple group, group of induced automorphisms

1. Introduction. Recall that a nilpotent self-normalizing subgroup is called a *Carter subgroup*. In the paper we consider the following

Problem. *Are any two Carter subgroups of a finite group conjugate?*

In [1] it is proven that a minimal counterexample to this problem should be almost simple. We intend to improve the results of [1] (see the theorem below). Actually, we will use the ideas of [1] in order to prove a stronger theorem.

Our notations are standard. Given a finite group G, we denote by $\mathrm{Aut}(G)$ the automorphism group of G. If $Z(G)$ is trivial then G is isomorphic to the group of its inner automorphisms and we may suppose that $G \leq \mathrm{Aut}(G)$. A finite group G is said to be *almost simple* if there is a simple group S with $S \leq G \leq \mathrm{Aut}(S)$, i.e., $F^*(G)$ is a simple group. We denote by $F(G)$ the Fitting subgroup of G and by $F^*(G)$, the generalized Fitting subgroup of G.

If G is a group while A, B, and H are subgroups of G, and B is normal in A ($B \trianglelefteq A$), then $N_H(A/B) = N_H(A) \cap N_H(B)$. If $x \in N_H(A/B)$ then x induces the automorphism $Ba \mapsto Bx^{-1}ax$ of A/B. Thus, there is a homomorphism of $N_H(A/B)$ into $\mathrm{Aut}(A/B)$. The image of this homomorphism is denoted by $\mathrm{Aut}_H(A/B)$ while its kernel is denoted by $C_H(A/B)$. In particular, if S is a composition factor of G then the group $\mathrm{Aut}_H(S)$ is defined for every $H \leq G$.

Definition. A finite group G is said to satisfy $(*)$ if, for its every nonabelian composition factor S and for its every nilpotent subgroup N, the Carter subgroups of $\langle \mathrm{Aut}_N(S), S\rangle$ are conjugate.

Clearly, if a finite group G satisfies $(*)$ then for every normal subgroup H and every soluble subgroup N of G the groups G/H and NH satisfy $(*)$. Our goal here is to prove the following

Theorem. *If a finite group G satisfies $(*)$ then the Carter subgroups of G are conjugate.*

Note that a finite group may fail to include Carter subgroups. In this case we also say that its Carter subgroups are conjugate. In Sections 2 and 3 we assume that X is a counterexample to the theorem of minimal order, i.e., that X is a finite group satisfying $(*)$, X contains nonconjugate Carter subgroups, but the Carter subgroups of every group M of order less than $|X|$, which satisfy $(*)$, are conjugate.

2. Preliminary results. Recall that X is a counterexample of minimal order to the theorem.

Lemma 1. *Let G be a finite group satisfying $(*)$ and $|G| \leqslant |X|$. Let H be a Carter subgroup of G. If N is a normal subgroup of G then HN/N is a Carter subgroup of G/N.*

Proof. Since HN/N is nilpotent, we have just to prove that it is self-normalizing in G/N. Clearly, this is true if $G = HN$. So, assume $M = HN < G$. By the minimality of X, $M^x = M$, $x \in G$, implies $H^x = H^m$ for some $m \in M$. It follows that $xm^{-1} \in N_G(H) = H$ and $x \in M$. This proves that HN/N is nilpotent and self-normalizing in G/N. □

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Lemma 2. *Let B be a minimal normal subgroup of X and let H and K be nonconjugate Carter subgroups of X. Then*

(i) *B is nonsoluble;*

(ii) *$X = BH = BK$;*

(iii) *B is the unique minimal normal subgroup of X.*

Proof. (i) We give a proof by contradiction. Assume that B is soluble and let $\pi : X \to X/B$ be the canonical homomorphism. Then H^π and K^π are Carter subgroups of X/B by Lemma 1. By the minimality of X, there exists $\bar{x} = Bx$ such that $(K^\pi)^{\bar{x}} = H^\pi$. It follows that $K^x \leq BH$. Since BH is soluble, K^x is conjugate to H in BH. Hence, K is conjugate to H in X; a contradiction.

(ii) Assume that $BH < X$. By Lemma 1 and the minimality of X, BH/B and BK/B are conjugate in X/B. So, there exists $x \in X$ such that $K^x \leq BH$. It follows that K^x is conjugate to H in BH. Hence, K is conjugate to H in X; a contradiction.

(iii) Suppose that M is a minimal normal subgroup of X different from B. By (i), M is nonsoluble. On the other hand, $MB/B \simeq M$ is a subgroup of the nilpotent group $X/B \simeq H/H \cap B$; a contradiction.

The following lemma is useful in many applications, and so we prove it here although we need only a part of the proof in our later arguments.

Lemma 3. *Let G be a finite group. Let H be a Carter subgroup of G. Assume that there exists a normal subgroup $B = T_1 \times \dots \times T_k$ of G such that $T_1 \simeq \dots \simeq T_k \simeq T$, $Z(T_i) = \{1\}$ for all i, and $G = H(T_1 \times \dots \times T_k)$. Then $\mathrm{Aut}_H(T_i)$ is a Carter subgroup of $\langle \mathrm{Aut}_H(T_i), T_i\rangle$.*

Proof. Assume that our statement is false and G is a counterexample with k minimal; then $k > 1$. Clearly, G acts transitively by conjugation on the set $\Omega := \{T_1, \dots, T_k\}$. We may assume that these T_j's are indexed so that G acts primitively on $\{\Delta_1, \dots, \Delta_p\}$, $p > 1$, where

$$\Delta_i := \{T_{1+(i-1)l}, \dots, T_{il}\}, \quad k = pl,$$

for each i. Denote by $\varphi : G \to \mathrm{Sym}_p$ the induced permutation representation. Clearly, $B \leq \ker\varphi$ so that $G^\varphi = (BH)^\varphi = H^\varphi$ is a primitive nilpotent subgroup of Sym_p. Hence p is prime and G^φ is a cyclic group of order p. In particular, $Y := \ker\varphi$ coincides with the stabilizer of any Δ_i, so that φ is permutationally equivalent to the representation of G on the right cosets of Y. Given $i = 1, \dots, p$, put $S_i = T_{1+(i-1)l} \times \dots \times T_{il}$. Then $Y = N_G(S_i)$ and $B = S_1 \times \dots \times S_p$. Consider $\xi : Y \to \mathrm{Aut}_Y(S_1)$ and put $A = Y^\xi$, $S = S_1^\xi$. Clearly, S is a normal subgroup of A. Moreover, S is isomorphic to S_1, since S_1 has trivial center. On the other hand, for each $i \neq 1$, $S_i \leq \ker\xi$, since S_i centralizes S_1.

Denote by $A \wr C_p$ the wreath product of A and a cyclic group C_p and let $\{x_1 = e, \dots, x_p\}$ be a right transversal of Y. Then the map $\eta : G \to A \wr C_p$ such that, for $x \in G$:

$$x \mapsto \big((x_1 x x_{1^{x^\varphi}}^{-1})^\xi, \dots, (x_p x x_{p^{x^\varphi}}^{-1})^\xi\big) x^\varphi$$

is a homomorphism. Clearly, Y^η is a subdirect product of the base subgroup A^p and

$$S_1^\eta = \{(s, 1, \dots, 1) \mid s \in S\}, \quad B^\eta = \{(s_1, \dots, s_p) \mid s_i \in S\} \leq Y^\eta.$$

Moreover, $\ker\eta = C_G(B) = \{e\}$, and so we may identify G with G^η. We choose $h \in H \setminus Y$. Then

$$G = \langle Y, h\rangle,\ h^p \in Y, \quad H = (Y \cap H)\langle h\rangle$$

and we may assume

$$h = (a_1, a_2, \dots, a_p)\pi,\ a_i \in A, \quad \pi = (1, 2, \dots, p) \in C_p.$$

For each i, $1 \leqslant i \leqslant p$, let $\psi_i : A^p \to A$ be the canonical projection and let $H_i := (H \cap Y)^{\psi_i}$. Clearly, $Y^{\psi_i} = A$. Moreover, for each $i \geqslant 2$, $H_i = H_1^{h^{i-1}} = H_1^{a_1 \dots a_{i-1}}$ since h normalizes $Y \cap H$. Put

$$N := (H_1 \times \dots \times H_p) \cap Y.$$

N is normalized by H, since $H = (N \cap H)\langle h \rangle$ and $H_i^h = H_{i+1 \pmod p}$. We claim that H_1 is a Carter subgroup of A. Assume that $n_1 \in N_A(H_1) \setminus H_1$. From $Y = (Y \cap H)B$ it follows that $n_1 = h_1 s$, $h_1 \in H_1$, $s \in N_S(H_1) \setminus H_1$. Let

$$b := (s, s^{a_1}, \ldots, s^{a_1 \ldots a_{p-1}}) \in B.$$

Then b normalizes N, for

$$H_i^b = H_i^{s^{a_1 \ldots a_{i-1}}} = H_1^{a_1 \ldots a_{i-1} s^{a_1 \ldots a_{i-1}}} = H_1^{s a_1 \ldots a_{i-1}} = H_1^{a_1 \ldots a_{i-1}} = H_i.$$

Now $[b, h^{-1}] := b^{-1} h b h^{-1} \in Y$ is such that:

$$[b, h^{-1}]^{\psi_i} = 1, \text{ if } i \neq p, \quad [b, h^{-1}]^{\psi_p} = [s, (a_1 \cdot \ldots \cdot a_p)^{-1}]^{a_1 \cdot \ldots \cdot a_{p-1}},$$

where $a_1 \cdot \ldots \cdot a_p = (h^p)^{\psi_1} \in H_1$. Since $s \in N_S(H_1)$, it follows that

$$[s, (a_1 \cdot \ldots \cdot a_p)^{-1}] \in H_1, \quad [s, (a_1 \cdot \ldots \cdot a_p)^{-1}]^{a_1 \cdot \ldots \cdot a_{p-1}} \in H_p.$$

So $[b, h^{-1}] \in N$ and $b \in N_G(N\langle h \rangle)$. But $H \leq N\langle h \rangle$ implies $N_G(N\langle h \rangle) = N\langle h \rangle$. Indeed, if $g \in N_G(N\langle h \rangle)$ then H^g is a Carter subgroup of $N\langle h \rangle$. However, $N\langle h \rangle$ is soluble. Hence, there exists $y \in N\langle h \rangle$ with $H^g = H^y$. Now, H is a Carter subgroup of G, thus $gy^{-1} \in H$ and $g \in N\langle h \rangle$. Therefore, $b \in N, s \in H_1$; i.e., $n_1 \in H_1$; a contradiction.

Now $A = H_1(T_1 \times \cdots \times T_l)$ and $l < k$. By induction we see that $\mathrm{Aut}_{H_1}(T_1)$ is a Carter subgroup of $\langle \mathrm{Aut}_{H_1}(T_1), T_1 \rangle$. By construction, $\mathrm{Aut}_H(T_1) = \mathrm{Aut}_{H_1}(T_1)$ and the lemma follows. $\square$

3. Proof of the theorem. Write $B = T_1 \times \cdots \times T_k$, $T_i \simeq T$, a nonabelian simple group. What remains to prove is $k = 1$. In the notations of the proof of Lemma 3, we have shown that H_1 is a Carter subgroup of A. Clearly, since each H_i is conjugate to H_1 in A, $N_A(H_i) = H_i$, $i = 1, \ldots, p$. It follows easily that N is a Carter subgroup of Y. Let $y := (y_1, \ldots, y_p) \in N_Y(N)$. From $N^{\psi_i} = H_i$ we derive $y_i \in N_A(H_i) = H_i$ for each i. Hence, $y \in N$.

We have seen that, to each Carter subgroup H of X, we can assign a Carter subgroup $N = N_H$ of Y such that H normalizes N_H. Clearly, $N_H \neq \{e\}$, otherwise X would have order p. So let K be a Carter subgroup of X not conjugate to H, and let N_K be the Carter subgroup of Y corresponding to K. By the minimality of X, for each $x \in X$ we have $\langle H^x, K \rangle = X$. On the other hand, by the inductive hypothesis, there exists $x \in Y$ such that $N_K = (N_H)^x$. Hence, N_K is normal in $\langle K, H^x \rangle = X$; a contradiction, since $\{e\} \neq N_K$ is nilpotent.

4. Some properties of Carter subgroups. Here we prove some lemmas of use in studying Carter subgroups in finite groups, in particular, in almost simple groups.

Lemma 4. *Let G be a finite group satisfying $(*)$, let H be a normal subgroup of G, and let K be a Carter subgroup of G. Then KH/H is a Carter subgroup of G/H.*

Proof. This fact is proven in Lemma 1 under the additional assumption $|G| \leqslant |X|$. By the theorem we have that $|X| = \infty$, and so the lemma is true for any finite group G. $\square$

Lemma 5. *Assume that G is a finite group. Let K be a Carter subgroup of G with center $Z(K)$. Assume also that $e \neq z \in Z(K)$ and $C_G(z)$ satisfies $(*)$. Then*

(1) *Every subgroup Y containing K and satisfying $(*)$ is self-normalizing in G.*

(2) *No conjugate of z in G, but z, lies in $Z(G)$.*

(3) *If H is a Carter subgroup of G, nonconjugate to K, then z is not conjugate to any element in the center of H.*

In particular the centralizer $C_G(z)$ is self-normalizing in G, and z is not conjugate to any power $z^k \neq z$.

Proof. This lemma is proven in [2, Lemma 3.1] for a minimal counterexample to the problem and therefore its use for finding Carter subgroups depends heavily on the classification of finite simple groups. We state here a stronger version of the lemma in order to avoid such dependence.

(1) Take $x \in N_G(Y)$. Then K^x is a Carter subgroup of Y. By the theorem, the Carter subgroups of Y are conjugate. Therefore, there exists $y \in Y$ with $K^x = K^y$. Hence, $xy^{-1} \in N_G(K) = K \leq Y$ and $x \in Y$.

(2) Assume $z^{x^{-1}} \in Z(K)$ for some $x \in G$. Then z belongs to the center of $\langle G, G^x \rangle \leq C_G(z)$. Since $C_G(z)$ satisfies $(*)$, there exists $y \in C_G(z)$ such that $K^x = K^y$. From $xy^{-1} \in C_G(z)$ we get $z^{xy^{-1}} = z$. Hence, $z^x = z^y = z$. We conclude that $z^{x^{-1}} = z$.

(3) If our claim were false, replacing H with some conjugate H^x (if need be), we may assume $z \in Z(K) \cap Z(H)$, i.e. $z \in Z(\langle K, H \rangle) \leq C_G(z)$. Again since $C_G(z)$ satisfies $(*)$, there would exist $y \in C_G(z)$ such that $H = K^y$. A contradiction. $\square$

Note that for every known finite simple group G (and hence almost simple, since the group of outer automorphisms is soluble) and for most elements $z \in G$ of prime order we see that the composition factors of $C_G(z)$ are among the known simple groups. Indeed, for sporadic groups this statement can be checked by using [3]. Composition factors of $C_{A_n}(z)$ are alternating groups. If G is a finite simple group of Lie type over a field of characteristic p and $(|z|, p) = 1$, then z is semisimple and composition factors of $C_G(z)$ are finite groups of Lie type. If $|z| = p$ and p is a good prime for G then [4, Theorems 1.2 and 1.4] implies that all composition factors of $C_G(z)$ are finite groups of Lie type. The only case in which the structure of centralizers of unipotent elements of order p is not completely known is as follows: p is a bad prime for G.

Therefore, if we classify the Carter subgroups of an almost simple finite group A by induction then we may assume that $C_A(z)$ satisfies $(*)$ for most elements of prime order $z \in A$. In particular, we can improve the table from [2] using the results of the present paper and [5]. In the table below A is an almost simple group with conjugate Carter subgroups.

$\mathrm{Soc}(A) = G$	Conditions for A
alternating, sporadic; $A_1(r^t),\ B_\ell(r^t),\ C_\ell(r^t),\ t$ even if $r = 3$; ${}^2B_2(2^{2n+1}),\ G_2(r^t),\ F_4(r^t),\ {}^2F_4(2^{2n+1})$; $E_7(r^t),\ r \neq 3;\ E_8(r^t),\ r \neq 3, 5$	none
$D_{2\ell}(r^t),\ {}^3D_4(r^{3t}),\ {}^2D_{2\ell}(r^{2t})$, where t even, if $r = 3$ and if $G = D_4(r^t)$ then $\lvert(\mathrm{Field}(G) \cap A) : (\widehat{G} \cap A)\rvert_{2'} > 1$	$A/(A \cap \widehat{G})$ is a 2-group or $\lvert\widehat{G} : (A \cap \widehat{G})\rvert \leq 2$
$B_\ell(3^t),\ C_\ell(3^t),\ D_{2\ell}(3^t),\ {}^3D_4(3^{3t}),\ {}^2D_{2\ell}(3^{2t})$, $D_{2\ell+1}(r^t),\ {}^2D_{2\ell+1}(r^{2t}),\ {}^2G_2(3^{2n+1})$, $E_6(r^t),\ {}^2E_6(r^{2t}),\ E_7(3^t),\ E_8(3^t),\ E_8(5^t)$	$A = G$
$A_\ell(r^t),\ {}^2A_\ell(r^{2t}),\ \ell > 1$	$G \leq A \leq \widehat{G}$

References

1. *Dalla Volta F., Lucchini A., and Tamburini M. C.*, "On the conjugacy problem for Carter subgroups," Comm. Algebra, **26**, No. 2, 395–401 (1998).
2. *Tamburini M. C. and Vdovin E. P.*, "Carter subgroups of finite groups," J. Algebra, **255**, No. 1, 148–163 (2002).
3. *Conway J. H., Curtis R. T., Norton S. P., Parker R. A., and Wilson R. A.*, Atlas of Finite Groups, Clarendon Press, Oxford (1985).
4. *Seitz G. M.*, "Unipotent elements, tilting modules, and saturation," Inv. Math., **141**, No. 3, 467–503 (2000).
5. *Previtali A., Tamburini M. C., and Vdovin E. P.*, "The Carter subgroups of some classical groups," Bull. London Math. Soc., **36**, No. 1, 145–155 (2004).

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