# Contents



## Lecture 1

Why finite groups are better (somehow) than others?

I Finite means that we can "count" elements of a group.

II Finite groups have "architecture", in particular, every finite group has a "fundament".

III Some subgroups are always in the "bottom" of a containing group. This means that if $X$ is already in the "bottom" of some subgroups, than it should be in the "bottom" of all other subgroups.

In our course we will try to understand all these advantages.

In this section we assume that $\mathbf{F}$ is a field of characteristic $p \geqslant 0$, $G = \mathrm{SL}_2(\mathbf{F})$ is a group of $2 \times 2$ matrices of determinant 1 over $\mathbf{F}$,

$$Z = Z(G) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \text{ where } \alpha = \pm 1 \right\},$$

$\overline{G} = G/Z$.

First we describe Abelian subgroups of $G$. Up to conjugation, there exist three type of maximal Abelian subgroups.

1. $U \times Z$, where $U$ is a maximal unipotent subgroup of $G$, i. e. $U$ is a group of upper triangular matrices with units on the diagonal.

$$U = \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \text{ where } \alpha \in \mathbf{F} \right\}.$$

Direct calculations shows that

$$N_G(U \times Z) = \left\{ \begin{pmatrix} \beta & \alpha \\ 0 & \beta^{-1} \end{pmatrix}, \text{ where } \alpha \in \mathbf{F}, \beta \in \mathbf{F}^\sharp \right\}$$

is a subgroup of upper triangular matrices. It is also called a Borel subgroup.

2. $H$, group of diagonal matrices, $H \simeq \mathbf{F}^\sharp$, where $F^\sharp$ is a multiplicative group of $\mathbf{F}$.

$$H = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \text{ where } \alpha \in \mathbf{F}^\sharp \right\}.$$

Sometimes $H$ is also called a maximal split torus or a Cartan subgroup.

$$N_G(H) = \left\langle H, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$$

is called a monomial subgroup.

3. $S$, maximal non-split torus. It appears from the quadratic extension of $\mathbf{F}$. Let $t \in G$ be chosen such that $t$ has an irreducible polynomial $\chi(x) \in \mathbf{F}[x]$. Then

$$\text{End}_{\,\text{Mat}_2(\mathbf{F})}(t) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = M, Mt = tM, \text{ where } a, b, c, d \in \mathbf{F} \right\}$$

is a commutative algebra isomorphic to $\mathbf{F}[x]/\langle \chi(x) \rangle = E$, where $E$ is the quadratic extension of $\mathbf{F}$. Then $S = E^\sharp \cap \text{SL}_2(\mathbf{F})$. $N_G(S) = \langle S, \sigma$, where $\sigma \in \text{Gal }(E/F)$ and $\sigma^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Easy calculations shows that if $A$ and $B$ are maximal Abelian subgroups of $G$, then either $A \cap B = Z$ or $A = B$. More over it is clear that every element of $G$ is contained in one of the subgroups mentioned above and subgroups of the same type are conjugate in $G$.

Incide $\text{SL}_2(\mathbf{F})$ there exists also an absolutely irreducible quaternion subgroup $Q$ such that $|Q| = 8$ and $N_G(Q) \simeq \text{SL}_2(3)$ or $N_G(Q) \simeq \text{GL}_2(3)$. Note that there are two classes of such subgroups in $\text{SL}_2(\mathbf{F})$ and just one class in $\text{GL}_2(\mathbf{F})$. A subgroup $Q$ exists if and only if $-1 \in \mathbf{F}^2 + \mathbf{F}^2$ and $p \neq 2$.

**Exercise 1**. Describe all solvable subgroups in $\text{SL}_2(\mathbf{F})$.

## Lecture 2

**Lemma 2.1.** Thompson transfer lemma. *Let $G$ be a finite group and $S$ be a Sylow 2-subgroup of $G$. Assume that $U$ is a maximal subgroup of $S$ and $t$ is an involution with $t^G \cap U = \emptyset$. Then $O^2(G) \neq 1$.*

PROOF. By the conditions of the lemma $t$ acts without fixed points on the cosets of $U$ in $G$, so there are $2n$ cosets, where $n = |G : S|$. Consider the permutation representation $\varphi : G \to \mathrm{Sym}(G : S)$. Then this representation contains an odd permutation $t^\varphi$, hence, $G \cap \mathrm{Alt}(G : S)$ is a nontrivial normal subgroup of $G$ of index 2. ⊠

Fix a finite subgroup $K$ of $G$ with $Z \leq K$ and $K/Z = \overline{K}$ is a non-Abelian finite simple group. Note the following evident properties of $K$.

1. If $r \neq 2, p$, then a Sylow $r$-subgroup of $K$ is cyclic.
2. If $2 \neq p$, then a Sylow 2-subgroup of $K$ is quaternion or generalized quaternion, i. e.
$$Q_n = \left\langle x, y | x^{2^n} = y^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, yxy^{-1} = x^{-1} \right\rangle.$$
Further all elements of order 4 are conjugate in $K$.
3. A Sylow $p$-subgroup $U_0$ of $K$ is elementary Abelian and, if $U_0 \neq 1$, then $U_0 \neq N_K(U_0) = U_0H_0$, where $H_0 = N_G(U_0) \cap K$ is a maximal Abelian subgroup of $K$.
4. If $A$ is a maximal Abelian subgroup of $K$ and $A$ is not conjugate with $U_0 \times Z$, then $|N_K(A) : A| = 2$.

Let $U_0, A_1, \ldots, A_r$ be all non-conjugate maximal Abelian subgroups of $K$ and $A_1 = H_0$. Then $g.c.d.(|\overline{A}_i|, |\overline{A}_j|) = 1$ and $g.c.d.(|\overline{U}_0|, |\overline{A}_i|) = 1$ for all $1 \leqslant i \neq j \leqslant r$. Denote $k = |\overline{K}|$, $|\overline{A}_i| = a_i$, $|\overline{U}_0| = u$. Clearly every element $\overline{k} \in \overline{K}$ is contained in some maximal Abelian subgroup, hence, up to conjugation, is contained in one of $\overline{U}_0, \overline{A}_1, \ldots, \overline{A}_r$. If $\overline{x} \neq 1$, then this subgroup is unique. The number of conjugated with $\overline{A}_i$ subgroups in $\overline{K}$ is equal to $|\overline{K} : N_{\overline{K}}(\overline{A}_i)| = \frac{k}{2a_i}$. The number of subgroups conjugated with $\overline{U}_0$ of $\overline{K}$ is equal to $|\overline{K} : N_{\overline{K}}(\overline{U}_0)| = \frac{k}{ua_1}$. Thus, the number of nonidentity elements of $\overline{K}$ is equal to

$$|\overline{K}| - 1 = k - 1 = \left( \sum_{i=1}^{r} \frac{k}{2a_i}(a_i - 1) \right) + \frac{k}{ua_1}(u - 1). \tag{1}$$

Dividing both part of the identity (1) by $k$ we obtain the following

$$1 - \frac{1}{k} = \left( \sum_{i=1}^{r} \frac{a_i - 1}{2a_i} \right) + \frac{u - 1}{a_1 u}. \tag{2}$$

**Lemma 2.2.** $r \leqslant 3$.

PROOF. Since $a_i \geqslant 2$ we obtain that $\frac{a_i - 1}{2a_i} \geqslant \frac{1}{4}$ and the lemma follows. ⊠

**Lemma 2.3.** $r \neq 1$.

PROOF. Assume that $r = 1$. Then we obtain that $k = \frac{2a_1u}{a_1u - u + 2}$. Since $a_1$ and $u$ divide $k$ and since $g.c.d.(a_1, u) = 1$ we obtain that either $k = 2a_1u$ or $k = a_1u$. In the second case we immediately obtain that $\overline{K} = \overline{A}_1 \ltimes \overline{U}_0$, a contradiction with the fact that $\overline{K}$ is simple. In the second case $\overline{K}$ contains a subgroup $\overline{A}_1 \ltimes \overline{U}_0$ of index 2. So this subgroup is normal, $\Rrightarrow\Lleftarrow$. ⊠

**Lemma 2.4.** *If $r = 3$, then $u = 1$ and $K \simeq \mathrm{SL}_2(5)$.*

PROOF. Write (2) in our case. We have

$$1 - \frac{1}{k} = \frac{1}{2}\left(1 - \frac{1}{a_1}\right) + \frac{1}{2}\left(1 - \frac{1}{a_2}\right) + \frac{1}{2}\left(1 - \frac{1}{a_3}\right) + \frac{u-1}{a_1 u}. \tag{3}$$

Making necessary computation we may rewrite (3) in the following form

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} = 1 + \frac{2}{k} + 2\frac{u-1}{a_1 u}. \tag{4}$$

The left part can be greater, than 1 only if $a_1, a_2, a_3 = \{2, 3, 5\}$. Thus (4) has the unique solution $u = 1$, $k = 60$. Since there exists a unique simple group of order 60, the lemma follows. ⊠

**Lemma 2.5.** *If $r = 2$, then $u \neq 1$ and either $K = SL_2(p^n)$ or $p = 3$, $K = SL_2(5)$. The second case appears from the following consideration. $PSL_2(5) \simeq \mathrm{Alt}_5$, $PSL_2(9) \simeq \mathrm{Alt}_6$ and $\overline{K}$ has two non-conjugate embedding into $\overline{G}$, as a stabilizer of a point and as a transitive subgroup acting on the set of lines.*

PROOF. Again rewrite (2) in our case.

$$1 - \frac{1}{k} = 1 - \frac{1}{2}\left(\frac{1}{a_1} + \frac{1}{a_2}\right) + \frac{u-1}{a_1 u}. \tag{5}$$

Since $|N_{\overline{K}}(\overline{A}_1)| = 2a_1 < k$ it follows that in the identity

$$\frac{1}{2a_1} + \frac{1}{2a_2} = \frac{1}{k} + \frac{u-1}{a_1 u} \tag{6}$$

the member $\frac{u-1}{a_1 u}$ is greater, than 0, hence, $u \neq 0$. Now

$$\frac{1}{2a_2} = \frac{1}{k} + \frac{1}{2a_1} - \frac{1}{a_1 u}. \tag{7}$$

Since $|N_{\overline{K}}(\overline{U}_0)| = a_1 u \leqslant k$, (7) implies that $\frac{1}{a_2} < \frac{1}{a_1}$, hence $a_2 > a_1$. Multiplying both parts of (6) by $2a_1a_2u$ we obtain

$$a_1 u = a + (u-2)a_2, \tag{8}$$

where $a = \frac{2a_1a_2u}{k}$. Thus $a = \frac{2a_1a_2u}{k}$ is integer. From the other hand $a_1, a_2, u$ divide $k$ and they are coprime, hence $a_1a_2k$ divides $k$. Thus either $a = 1$ or $a = 2$.

Assume that $a = 2$, i. e. $k = a_1a_2u$. If $a_1$ is even than $\overline{A}_1$ contains a Sylow 2-subgroup of $\overline{K}$. But $|N_{\overline{K}}(\overline{A}_1) : \overline{A}_1| = 2$. $\Rrightarrow\!\Lleftarrow$So either $a = 1$ or $u = 2^\alpha$. Clearly $\overline{A}_1\overline{U}_0$ is a Frobenius group, hence $u \equiv 1(\mod a_1)$, so, from (8), it follows that $a_2 \equiv a(\mod a_1)$. Now $a_1 = \frac{a}{u} + \frac{u-2}{u}a_2$, hence $a_2 = a_1 + a$. If $u \geqslant 4$, then (8) implies $2(a_1 + a) = a(u+1)$. Thus, if $a = 2$, then $a_1 = u - 1$, $a_2 = u + 1$. If $a = 1$, then $a_1 = \frac{u-1}{2}$, $a_2 = \frac{u+1}{2}$. So $K = \mathrm{SL}_2(p^\alpha)$, $p^\alpha = u$.

If $u = 3$ we obtain exceptional case. Case $u = 2$, clearly, impossible. ⊠

Now we prove that all elements of order 4 are conjugate in $K$.

**Proposition 2.6.** *Let $K$ be a finite group with quaternion Sylow 2-subgroup $Q$ and $K = [K, K]$, $Z(K) \simeq Z(Q) \simeq \mathbb{Z}_2$. Then all elements of order 4 are conjugate.*

PROOF. Consider $\overline{K} = K/Z(K)$. Then $\overline{Q}$ is dihedral. Since $[\overline{K}, \overline{K}] = \overline{K}$, we have that $O^2(\overline{K})$ is trivial. Let $\overline{M}$ be a maximal cyclic subgroup of $\overline{Q}$ and $\bar{t} \in \overline{Q} - \overline{M}$ is an involution. In view of Thompson transfer lemma $\bar{t}^{\overline{K}} \cap \overline{M} \neq \emptyset$, hence $\bar{t}$ is conjugate to a unique involution in $\overline{M}$. Hence, all involutions in $\overline{K}$ are conjugate, hence, all elements of order 4 in $K$ are conjugate as well. ⊠

**Exercise 2**. Find all finite subgroups of $\mathrm{SL}_2(\mathbf{F})$.

## Lecture 3

**Lemma 3.1.** *Let $t$ and $s$ be non-conjugate involutions in a finite group $G$. Then $\langle s, t\rangle$ is a dihedral group with maximal cyclic subgroup $\langle st\rangle$.*

PROOF. Since $G$ is finite, element $st$ has a finite order. If $|st|$ has an odd divider, than $s$ and $t$ are conjugate in $\langle s, t\rangle$. $\Rightarrow\Leftarrow$Hence $|st|$ is a power of 2 and the lemma follows. ⊠

**Theorem 3.2.** *Let $G$ be a finite group with more than one conjugacy class of involutions. Let $c = \max\{|C_G(s)|,$ where $t$ runs over the set of involutions in $G\}$. Then $|G| < c^3$.*

PROOF. Let $\Gamma$ be a graph of involutions in $G$, 2 vertices are connected if the involutions, corresponding to these vertices commute. Choose $t \in V(\Gamma)$ with $c = C_G(t)$. Let $s \in V(\Gamma)$ is chosen such that $s$ and $t$ are not conjugate. Then $|s^G| = \frac{|G|}{|C_G(s)|} \geqslant \frac{|G|}{c}$. Assume that $s_1 \in s^G$. By Lemma **3.1**, $\langle t, s_1\rangle$ is dihedral. Hence there exists $V(\Gamma) \ni z_1 \in Z(\langle t, s_1\rangle)$. Thus the distance beside $t$ and $s_1$ in $\Gamma$ is at most 2. There are at most $c-2$ involutions in the circle of radius 1, hence there are at most $(c-2)^2$ involutions in the circle of radius 2. Thus $(c-2)^2 \geqslant |s^G| \geqslant \frac{|G|}{c}$ and $|G| < c^3$. ⊠

**Theorem 3.3.** *Let $G$ be a finite simple group with one class of involutions. Assume that $c = |C_G(s)|$. Then $|G| \leqslant (c^2)!$.*

PROOF. Let $r$ be the number of conjugacy classes of $G$. Then

$$r = \frac{1}{|G|}\sum_{g\in G}|C_G(g)|. \tag{9}$$

Let $x \in G - \{1\}$ is such that $C_G(x)$ is maximal. Then (9) implies that

$$r \leqslant 1 + \frac{|G|-1}{|G|}|C_G(x)|. \tag{10}$$

Thus $r - 1 \leqslant |C_G(x)|$. Consider $S = \{(x, y) \in t^G \times t^G\}$, evidently $|S| = |t^G|^2 = \frac{|G|^2}{c^2}$. From the other hand

$$S = \{(x, y) \in t^G \times t^G | xy = yx\} \cup \{(x, y) \in t^G \times t^G | xy \neq yx\}. \tag{11}$$

We have

$$|\{(x, y) \in t^G \times t^G | xy = yx\}| \leqslant \frac{|G|}{c}(c-1)$$

and

$$\{(x,y)\in t^G\times t^G | xy\neq yx\} = \{(xy,y) | (x,y)\in t^G\times t^G, xy\neq yx\}.$$

Now $C_G(xy) \trianglelefteq C_G^{\pm}(xy) \trianglelefteq N_G(xy)$. Here $C_G^{\pm}(xy) = \{g\in G | (xy)^g = xy \text{ or } (xy)^g = (xy)^{-1}\}$. Therefore $|\{g\in G | g(xy)g^{-1} = (xy)^{-1}\}| \leqslant |C_G(xy)|$. Let $t_1,\ldots,t_s$ be representatives of conjugacy classes of $G$ such that they are conjugate to their inverses. Clearly $s\leqslant r-2$. Since $|C_G(xy)| < |G|$, (11) implies

$$\frac{|G|^2}{c^2} \leqslant |S| \leqslant |G|\left(\frac{c-1}{c}+s\right). \tag{12}$$

Hence, $\frac{|G|}{c^2} \leqslant 1+s \leqslant r-1 \leqslant |C_G(x)|$ and $|G:C_G(x)|\leqslant c^2$ for some element $x\in G$. Since $G$ is simple, permutation representation on cosets of $C_G(x)$ in $G$ is faithful. Thus, $G$ is isomorphic to a subgroup of $\mathrm{Sym}(|G:C_G(x)|)\leqslant \mathrm{Sym}(c^2)$ and the theorem follows. ⊠

**Corollary 3.4.** Brauer-Fauler theorem *There exist only finitely many finite simple groups with given centralizer of an involution.*

## Lecture 4

**Lemma 4.1.** *If $A$ is a subnormal subgroup of $G$ and $N$ is a minimal normal subgroup of $G$, then $N\leq N_G(A)$.*

Proof. Since $A$ is subnormal there exists a chain $A\trianglelefteq G_1\trianglelefteq G_2\trianglelefteq\ldots\trianglelefteq G_n = G$. We have that $N\leq G_{n-1}$. So $N = N_1\times\ldots\times N_k$, where $N_i\simeq N_j$ and $N_i\trianglelefteq G_{n-1}$ is a minimal normal subgroup of $G_{n-1}$. By induction all $N_i$ normalize $A$, hence $N$ normalizes $A$. ⊠

A finite group $L$ is called *quasisimple* if $[L,L]=L$ and $L/Z(L)$ is a finite simple group. A subnormal quasisimple subgroup of a group $G$ is called a *component* of $G$. *A layer* of $G$ ($E(G)$) is a subgroup generated by all components of $G$. $F(G)$ denotes a Fitting subgroup of $G$.

**Lemma 4.2.** Wieland Lemma. *If $A\lhd\lhd G$, $B\lhd\lhd G$, $[A,A]=A$ and $A$ has a unique maximal normal subgroup, then either $A\leq B$ or $B\leq N_G(A)$.*

Proof. Let $B^*$ be a maximal normal subgroup of $G$ containing $B$. By induction we may assume that $G=\langle A,B\rangle$ and $A\neq G\neq B$. Hence $G=B^*A$. By assumption $A\cap B^*$ is a unique maximal normal subgroup of $A$. Suppose, that there exists $b\in B$ with $A^b\neq A$. Consider $\overline{G}=G/B^*$. Then $\overline{A}=\overline{A}^b$. Note that $[A,A^b]$ is a proper normal subgroup of $A$, hence $[A,A^b]\leq A\cap B^*$, i. e. $[\overline{A},\overline{A}^b]=1$. But $1\neq\overline{G}=\overline{A}=[\overline{A},\overline{A}]=[\overline{A},\overline{A}^b]$, $\Rightarrow\Leftarrow$. ⊠

In view of Wieland Lemma any two components of a group commute. Thus $E(G)$ is a commutative product of components of $G$ and $[E(G),F(G)]=1$. Clearly $E(G)$ is a characteristic subgroup of $G$. Generalized Fitting subgroup is denoted by $F^*(G)$ and defined by $F^*(G)=E(G)F(G)$. Note for later use very simple fact that if $H\trianglelefteq G$, then $F^*(H)\leq F^*(G)$.

**Theorem 4.3.** $C_G(F^*(G))=Z(F^*(G))$.

Proof. Let $C = C_G(F^*(G))$. Clearly $F^*(C) \leq F^*(G)$. So $F^*(C) \leq Z(C)$. By definition, for any group $G$, $F^*(G/Z(G)) = F^*(G)/Z(G)$. That means that $F^*(C/F^*(C))$ is trivial, hence $C/F^*(C)$ is trivial. $\boxtimes$

**Corollary 4.4.** $|G| \leqslant |F^*(G)|!$.

Proof. Since $C_G(F^*(G)) = Z(F(G))$ we obtain that $G$ is embedded into

$$\text{Aut }(E(G)/Z(E(G))) \times \text{Aut }(F(G))$$

with kernel $Z(F(G))$. Clearly

$$|Aut(E(G)/Z(E(G)))| \leqslant |E(G)/Z(E(G))|!.$$

Now $F(G)/Frat(F(G)) \simeq p_1^{k_1} \times \ldots \times p_m^{k_m}$ is the direct product of elementary Abelian groups. So $G/C_G(F(G))F(G)$ is embedded into

$$GL_{k_1}(p_1) \times \ldots \times GL_{k_m}(p_m),$$

i.e.

$$|G/C_G(F(G))F(G)| \leqslant p_1^{k_1^2} \cdot \ldots \cdot p_m^{k_m^2}$$

and the corollary follows. $\boxtimes$

**Exercise 3**. Let $N$ be a finite nilpotent $\pi$-group, where $\pi$ is a set of primes. Then $C_{\text{Aut }(N)}(N/Frat(N))$ is a nilpotent $\pi$-group.

## Lecture 5

Later we consider important case $F^*(H) = O_p(H)$.

**Theorem 5.1.** *Let $G$ be a finite simple group of Lie type in characteristic $p$, $H$ is a parabolic subgroup of $G$ (i.e. $H \geq N_G(U)$, where $U \in Syl_p(G)$ is a maximal unipotent subgroup of $G$). Then $F^*(H) = O_p(H)$.*

In classical groups of Lie type maximal parabolic subgroups are stabilizers of some totally isotropic subspace. A subgroup $H$ of a finite group $G$ is called *$p$-local* if $H$ is the normalizer of a nontrivial $p$-subgroup of $G$.

**Theorem 5.2.** Borel-Tits Theorem. *Let $G$ be a finite simple group of Lie type in characteristic $p$. Let $H$ be a $p$-local subgroup of $G$. Then $F^*(H) = O_p(H)$.*

We divide the proof of Borel-Tits theorem into three steps.

1. If $N$ is a maximal parabolic subgroup of $G$ then $F^*(N) = O_p(N)$.

2. If $H$ is a $p$-local subgroup of $G$ then $H \leq N$ for some maximal parabolic subgroup of $G$.

3. Thompson $A \times B$-lemma.

The first step is Theorem **5.1** above.

Proof. (of step 2) (We prove this theorem only for classical groups in odd characteristic. But it remains true for any group of Lie type.) Recall that in classical groups $M$ is a maximal parabolic subgroup if and only if $M$ is a stabilizer of some isotropic subspace $U$ of $V$, i. e. $M = St_G(U)$.

Since $H$ is $p$-local there is a $p$-subgroup $R$ of $G$ with $H = N_G(R)$. Let $W$ be the set of $R$-stable vectors. Assume that $Rad(W) = W \cap W^{\perp} = \{0\}$. Then $V = W \oplus W^{\perp}$ and $W, W^{\perp}$ are $R$-invariant. Since $R$ is a $p$-group, there exists an $R$-stable vector of $W^{\perp}$, $\Rrightarrow\Lleftarrow$.

Thus $Rad(W)$ is nontrivial, hence $H \leq St_G(Rad(W))$. But $Rad(W)$ is a totaly isotropic subspace of $V$, hence $St_G(Rad(W))$ is a maximal parabolic subgroup of $G$. ⊠

**Lemma 5.3.** *Let $B$ be a $p'$-group acting on a $p$-group $R$. Assume that $B$ centralizes $U \leq R$ with $C_R(U) \leq U$. Then $B$ centralizes $R$.*

Proof. Induction on $|R : U|$. Consider $N = N_R(U) \leq N_{RB}(U)$. By conditions of the lemma, $B \leq C_{RB}(U) \trianglelefteq N_{RB}(U)$. Now $[N, B] \leq C_{RB}(U) \cap R = C_R(U) \leq U$. By conditions of the lemma $[U, B] = 1$. Consider $\overline{N} = N/Frat(N)$, so $B$ has an embedding $\varphi : B \to GL(\overline{N})$. We obtain that $[\overline{N}, B] \leq \overline{U}$ and $[\overline{U}, B] = 0$, hence $B^{\varphi}$ is a unipotent subgroup of $GL(\overline{N})$. But $|B|$ is coprime to $p$, hence $B^{\varphi}$ is trivial. Thus $B$ centralizes $\overline{N}$, hence, $N$. Since $C_R(N) \leq C_R(U) \leq U$ and since $N \gneqq U$ we are done by induction. ⊠

Proof. The second proof. Clearly $[B, U, R] = [U, R, B] = 1$. Hence $[R, B, U] = 1$. This means that $[R, B] \leq C_R(U) \leq U$. So $[B, R, R] = 1$. Since $g.c.d.(|B|, |R|) = 1$, we obtain that $[R, B] = 1$. ⊠

**Lemma 5.4.** Thompson $A \times B$-lemma. *Let $A$ be a $p$-group and $B$ be a $p'$-group. Assume that $A \times B$ acts on a $p$-group $P$ and $C_P(A) \leq C_P(B)$. Then $B$ centralizes $P$.*

Proof. Consider $R = PA$. Let $U = C_P(A)A$. We have that $C_R(U) \leq C_R(A) \leq U$. By Lemma **5.3** we obtain that $B$ centralizes $R$, hence $P$. ⊠

**Corollary 5.5.** *Let $H$ be a finite group with $F^*(H) = O_p(H)$. Assume that $A$ is a $p$-subgroup of $H$. Then $F^*(C_H(A)) = O_p(C_H(A))$.*

Proof. Let $B$ be a $p'$-subgroup of $F^*(C_H(A))$. Clearly $A \times B$ acts on $P = F^*(H)$. Now $C_P(A) = C_H(A) \cap P \trianglelefteq C_H(A)$. Therefore $C_P(A) \leq O_P(C_H(A))$. Since every $p'$-element of $F^*(C_H(A))$ centralizes $O_p(C_H(A))$, we obtain that $[O_p(C_H(A)), B] = 1$, so $[C_P(A), B] = 1$. By Thompson $A \times B$-lemma, $[P, B] = 1$, thus, by Fitting Lemma $B \leq F^*(H) = P$, i. e. $B$ is trivial. ⊠

**Corollary 5.6.** *Let $G$ be a finite group of Lie type in characteristic $p$. Let $A$ be a non-trivial $p$-subgroup of $G$. Then $F^*(N_G(A)) = O_p(N_G(A))$.*

Proof. Let $B$ be a $p'$-subgroup of $F^*(N_G(A))$. Clearly $A \leq O_p(N_G(A)) \leq F^*(N_G(A))$, hence $B \leq C_G(A)$. Since $C_G(A) \trianglelefteq N_G(A)$, we obtain that $F^*(C_G(A)) = C_G(A) \cap F^*(N_G(A))$, thus $B \leq F^*(C_G(A))$. But $C_G(A)$ is contained in some maximal parabolic subgroup of $G$, hence, by the corollary above, $B$ is trivial. ⊠

**Corollary 5.7.** Thompson $O$-balance theorem. *Let $G$ be a finite group, $p$ a prime. Assume that every $p$-local subgroup of $G$ is solvable. Let $A, A_1$ be two commuting $p$-subgroups of $G$. Then $O_{p'}(C_G(A)) \cap C_G(A_1) = C_G(A) \cap O_{p'}(C_G(A_1))$.*

PROOF. Consider $\overline{C}_1 = C_G(A_1)/O_{p'}(C_G(A_1))$. It is solvable (by conditions) and $O_{p'}(\overline{C}_1)$ is trivial. Thus $F^*(\overline{C}_1) = F(\overline{C}_1) = O_p(\overline{C}_1) = \overline{P}$. By Fitting Lemma $C_{\overline{C}_1}(\overline{P}) = Z(\overline{P})$. Assume that $B \in O_{p'}(C_G(A)) \cap C_G(A_1)$. Since $A, A_1$ commute, then $A \le C_G(A_1)$. It is also clear that $C_{\overline{P}}(\overline{A}) = \overline{C_P(A)}$. Hence, there exists an $A$-invariant Sylow $p$-subgroup $P$ of $O_{p',p}(C_G(A_1))$ that is mapped onto $\overline{P}$. Then $[B, C_P(A)] \le O_{p'}(C_G(A)) \cap O_{p',p}(C_G(A_1)) \le O_{p'}(C_G(A_1))$. Thus $[C_{\overline{P}}(\overline{A}), \overline{B}] = 1$, i. e. $C_{\overline{P}}(\overline{B}) \ge C_{\overline{P}}(\overline{A})$. By Thompson $A \times B$-lemma, $[\overline{B}, \overline{P}] = 1$. ⊠

**Corollary 5.8.** Thompson Critical subgroup Lemma. *Let $B$ be a $p'$-group of automorphisms of a $p$-group $P$. Then there exists a characteristic subgroup $C \le P$ such that $B$ acts faithfully on $C$ and nilpotency class of $C$ is at most 2. Moreover, if $p$ is odd than* $\exp(C) = p$.

PROOF. Choose a minimal characteristic subgroup $C$ of $P$ such that $B$ acts faithfully on $C$. We want to prove that $C$ satisfies all other condition of the corollary. Let $D = Frat(C)C_C(Frat(C))$. Then $C_C(D) \le D$. By $A \times B$-lemma $B$ acts faithfully on $D$, hence $C = D$, so $C = C_C(Frat(C))$, i. e. $Frat(C) \le Z(C)$. Hence nilpotency class of $C$ is at most 2.

Now $x^p \in Z(C)$ for all $x \in C$, hence $[x, y]^p = [x^p, y] = 1$ for all $x, y \in C$. This implies that $[C, C]$ is elementary Abelian. Assume that $\Omega_1(C) \lneqq C$. In view of minimality of $C$, $B$ can not act faithfully on $\Omega_1(C)$, hence $C_B(\Omega_1(C)) \ne 1$ and there exists $1 \ne \alpha \in C_B(\Omega_1(C))$. From the other hand $1 \ne [C, \alpha] = \langle x^{-1}x^\alpha | x \in C\rangle$. Now

$$(x^{-1}x^\alpha)^p = (x^{-1})^p(x^\alpha)^p[x^{-1}, x^\alpha]^{\binom{p}{2}} = (x^{-1})^p(x^\alpha)^p = (x^p)^{-1}(x^p)^\alpha = 1.$$

(Here we use fact that $p$ is odd in order to conclude that $p$ divides $\binom{p}{2}$.) Thus $[C, \alpha, \alpha] = 1$. Since $g.c.d.(|C|, |\alpha|) = 1$, it follows that $[C, \alpha] = 1$. $\Rrightarrow\Lleftarrow$. ⊠

## Lecture 6

In this lecture we shell try to understand the difference between sporadic groups and other simple groups.

Assume that $G$ is a finite group of Lie type in characteristic $p$ over a field $GF(p^n)$. Then $G$ contains a Borel subgroup $B = H \ltimes U$. If $p^n > 2$, there exists a prime $r$ dividing $p^n - 1$ and an elementary Abelian $r$-subgroup $D \le H$. If we take $A \le D$ of corank at least 2, then $C_G(A)$ has a component $K$ such that $K$ is a group of Lie type in characteristic $p$.

If $G = \mathrm{Alt}_n$, let $r = 2$ and take an Abelian subgroup $D = \langle (1,2)(3,4), (1,3)(2,4), \ldots \rangle$, $A \le D$ of corank at least 2. Then $C_G(A)$ contains a component isomorphic to $\mathrm{Alt}_m$.

Let $m_p(H)$ be the maximum of ranks of Abelian $p$-subgroups of $H$,

$$m_{2,p}(G) = \max_{H \text{ is a 2-local subgroup of } G} \{m_p(H)\},$$

$$e(G) = \max_{p \text{ odd}} \{m_{2,p}(G)\}.$$

If $m_2, p(G) = e(G)$ let

$\mathcal{B}_p(G) = \{B | B \text{ elementary Abelian}, m_p(B) = e(G),\ B \text{ is in some 2-local subgroup of } G\}$.

A simple group $G$ is called *large* sporadic group if

(a) $m_2(G) \geqslant 3$ and for any elementary Abelian 2-subgroup $E \leq G$ of rank at least $m_2(G) - 2$, we have $F^*(C_G(E)) = O_2(C_G(E))$.

(b) $e(G) \geqslant 3$ and for any $B \in \mathcal{B}_p(G)$ and any $A \leq B$ of corank 2, we have $F^*(C_G(A)) = O_p(C_G(A))$.

These two conditions exclude the possibility for $G$ to be an alternating group or a group of Lie type.

**Theorem 6.1.** *There exist only finitely many large sporadic groups.*

**Definition 6.2.** A finite group $G$ is called *of characteristic $p$-type* if for any $p$-local subgroup $H \leq G$ we have $F^*(H) = O_p(H)$.

**Theorem 6.3.** Klingei-Mason Theorem. *Let $G$ be a finite group of characteristic 2-type.*

1. *Assume that there exists $p$ such that $G$ is of characteristic $p$-type.*
2. *Assume that there exists a 2-local subgroup $H \leq G$ such that $m_p(H) \geqslant 2$ (i. e. $m_{2,p}(G) \geqslant 2$).*

*Then*

(1) $p \in \{3, 5\}$.

(2) *$H$ can be chosen with $F^*(H)$ is a 2-group of symplectic type (i. e. it does not contain non-cyclic characteristic subgroups).*

(3) $|G| \leqslant (((2^{62})!)^2)!$.

**Remark**. Since $H$ acts faithfully on $F^*(H) = O_2(H)$, we obtain that $m_2(H) \geqslant 3$, hence, $m_2(G) \geqslant 3$.

PROOF. (of remark) Consider $\overline{P} = O_2(H)/Z(H)$. Since $O_2(H)$ is of symplectic type, $\overline{P}$ is elementary Abelian and $m_2(\overline{P}) \leqslant m_2(O_2(H))$. $H$ acts faithfully on $\overline{P}$, hence $H/O_2(H)$ is isomorphic to a subgroup of $GL(\overline{P})$. In view of our conditions, $H/O_2(H)$ contains an Abelian $p$-subgroup of rank 2. But in $\mathrm{GL}_1(2)$ and $\mathrm{GL}_2(2)$ all Abelian subgroups of odd order are cyclic. So, $m_2(O_2(H)) \geqslant \dim(\overline{P}) = m_2(\overline{P}) \geqslant 3$. ⊠

**Exercise 4**. Under conditions of the theorem $F^*(G)$ is a finite non Abelian simple group.

**Lemma 6.4.** Thompson Dihedral Lemma. *Let $q$ and $r$ be distinct primes and $Q$ is a $q$-group, $E$ is an elementary Abelian $r$ group acting on $Q$ such that $C_E(Q) = 1$ and $m_r(E) = n$. Assume further that $r | q - 1$.*

*Then $QE \geq Q_1E_1 \times \ldots \times Q_nE_n$, where $Q_iE_i$ is a non Abelian group of order $qr$ (i. e. it is a Frobenius group with kernel $Q_i$).*

PROOF. Let $Q_0$ be a maximal $E$-invariant subgroup of $Q$ containing $Frat(Q)$. Consider $\overline{Q} = Q/Frat(Q)$. Clearly $E$ has an embedding into $\mathrm{GL}(\overline{Q})$ and, moreover, $E$ is conjugate to a subgroup of diagonal matrices of $\mathrm{GL}(\overline{Q})$. Choose $\overline{Q}_0$ to be of corank 1 in $\overline{Q}$ and let $Q_0$ be its complete preimage in $Q$. Thus $|Q : Q_0| = q$.

If $C_E(Q_0) = 1$ we are done by induction. Therefore assume that $E_1 = C_E(Q_0) \neq 1$. We have that $E_1$ acts faithfully on $Q/Q_0$, hence $|E_1| = r$. Consider $Q_1 = [Q, E_1]$.

Note that $|Q_1| = q$. Indeed, $Q_1$ is $QE$-invariant and $Q_1 = [Q_1, E_1]$. Hence $Q_1 \trianglelefteq Q$. Recall that $Q_0 \trianglelefteq Q$ and $|Q : Q_0| = q$. Now $[Q_1, Q_0, E_1] = 1$, $[Q_0, E_1, Q_1] = 1$, hence $[E_1, Q_1, Q_0] = [Q_1, Q_0] = 1$. Thus $Q_1 \cap Q_0 \leq Z(Q)$. Since $|Q_1 : (Q_1 \cap Q_0)| = |Q : Q_0| = p$ we have that $Q_1$ is Abelian. Therefore $Q_1 = [Q_1, E_1] \oplus C_{Q_1}(E_1)$, so $C_{Q_1}(E_1) = Q_0 \cap Q_1 = 1$, hence $|Q_1| = q$. Moreover $Q = Q_1 \times Q_0$, $E = E_1 \times E_0$ ($E_0 = C_E(Q_1)$) and we may apply induction. ⊠

**Remark**. Analogous statement holds if we do not assume that $r | q - 1$. In this case $QE \geq Q_1E_1 * \ldots * Q_nE_n$, where $Q_i$ is a special group, $|E_i| = r$ and $Q_i = [Q_i, E_i]$. Moreover, if $C_Q(E) = 1$, then $QE \geq Q_1E_1 \times \ldots \times Q_nE_n$.

Turn to our theorem.

Choose a 2-local subgroup $H \leq G$ with

(1) $m_p(H) = m_{2,p}(G)$.

(2) Fix $E \in \mathcal{B}_p(G)$ with $E \leq H$.

(3) If possible, $Z(H) = 1$.

(4) Fix $Q = F^*(H) = O_2(H)$.

**Lemma 6.5.** *$Q$ is a group of symplectic type. (Then $Z(Q)$ is cyclic and $Z(Q) \leq C_Q(E)$.)*

Proof. Assume by contradiction that $Q$ contains a characteristic elementary Abelian subgroup of rank at least 2. So $A \trianglelefteq H$ and $A$ is an $E$-module. Let $A_0 = C_A(E) \leq C_G(E)$, denote $m_2(A_0) = m$. Since $G$ is of characteristic $p$-type, then $F^*(C_G(E)) = P_E = O_p(F^*(C_G(E)))$ is a $p$-group. By Thompson Dihedral Lemma $P_EA_0$ contains a subgroup $P_1A_1 \times \ldots \times P_mA_m$, where $P_iA_i$ is a dihedral group. Clearly $P_E \geq E$. Thus $E \cap (P_1A_1 \times \ldots \times P_mA_m) = 1$. Now $P_EA_0 \geq E \times (P_1A_1 \times \ldots \times P_mA_m)$. Thus $m_p(C_G(A_0)) \geqslant m_p(E) + m - 1$. From the other hand we choose $E \in \mathcal{B}_p(G)$, hence $m_p(E) \geqslant m_p(C_G(A_0))$, so $m \leqslant 1$.

Let $E_1$ be a subgroup of $E$ of corank 1. Let $B = C_A(E_1) \leq C_G(E_1)$. Like above

$$O_p(C_G(E_1))B \geq E_1 \times \underbrace{D_{2p} \times \ldots \times D_{2p}}_{l \text{ times}},$$

where $l = m_2(B)$. Let $t$ be an involution in one of $D_{2p}$. Then $m_p(E) \geqslant m_p(C_G(t)) \geqslant m_p(E_1) + l - 1$, hence $m_2(C_A(E_1)) \leqslant 2$.

Now by Shur's Lemma $A = \langle C_A(E_1)$, where $E_1$ runs over hyperplanes of $E\rangle$. If $E$ centralizes all of $C_A(E_1)$, then $E$ centralizes $A$, hence $m_2(C_A(E)) = m_2(A) \geqslant 2$, $\Rrightarrow\Lleftarrow$. Hence there exists a hyperplane $E_1$ of $E$ and such that $E$ does not centralizes $C_A(E_1)$. Since $E_1 \trianglelefteq E$, we have that $C_A(E_1)$ is $E$-invariant. If $m_2(C_A(E_1)) = 1$, we take $C_G(C_A(E_1))$ instead of $H$ and proceed by induction. So we may assume that $m_2(C_A(E_1)) \geqslant 2$. Let $V = C_A(E_1)$. As we noted before, $m_2(V) \leqslant 2$, hence $m_2(V) = 2$, i. e. $V = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle t_1, t_2 \rangle$, where $t_1, t_2$ are involutions. Moreover, if we choose $H = N_G(V)$, then $C_{O_2(H)}(E) = 1$. By Thompson Dihedral Lemma $C_G(E_1)V \geq E_1 \times D_{2p} \times D_{2p}$. Thus $m_p(C_G(t_1)) = m_p(E)$. Hence we can choose $H = C_G(t_1)$. But then $Z(H) \geqslant \langle t_1 \rangle$, and we already noted that $H$ can be chosen such that $Z(H) = 1$. A contradiction with (3). ⊠

**Remark**. Let $B \in \mathcal{B}_p(G)$, $H$ a 2-local subgroup with $B \leq H$ and $O_2(H) = Q$. From the above proof it follows that if $A \leq B$ has corank $r \geqslant 1$, then $m_2(C_Q(A)) \leqslant r + 1$.

# Lecture 7

Continue the proof of Theorem **6.3**. As we prove in the previous lecture, we can choose $H$ so that $Q = F^*(H) = O_2(H)$ is a 2-group of symplectic type. P.Hall gave the classification of such groups. He proved that $Q = Q_0 * Z$, where $Z$ is cyclic or of maximal nilpotency class and $Q_0$ is extraspecial (i. e. $Q_0$ is a central product of dihedral groups and quaternion groups).

**Lemma 7.1.** *$p \in \{3,5\}$ and, if $p = 5$, then $|Z(Q)| = 2$.*

PROOF. Let $\overline{Q} = Q/Z$. Then $\overline{Q}$ is a vector space. Clearly $\overline{B}$ acts faithfully of $\overline{Q}$. Otherwise we can substitute $Q$ by a proper $\overline{B}$-invariant subgroup. If $|Z(Q)| = 2$, then we can define a quadratic form $F$ on $\overline{Q}$ by

$$F(\overline{x}) = \begin{cases} 1, & \text{if } x^2 \neq 1 \\ 0, & \text{otherwise.} \end{cases}$$

If $|Z(Q)| > 2$ we can define an antisymmetric form $(,)$ on $\overline{Q}$ by $(\overline{x}, \overline{y}) = [x, y]$ (clearly, any commutator is of order 2).

By Machke's theorem $B$ acts completely reducible on $\overline{Q}$ and, since $Z(Q)$ is $B$-invariant, preserves antisymmetric or quadratic form defined above. Hence $\overline{Q} = \overline{Q}_1 \oplus^{\perp} \ldots \oplus^{\perp} \overline{Q}_r$ is a direct sum of orthogonal irreducible $B$-submodules. Since $B$ is Abelian, $C_B(\overline{Q}_i) = B_i$ has corank 1. Like in the proof of Lemma **6.5** we obtain that $\dim(\overline{Q}_i) \leqslant 4$. Indeed, we have that $Q_i$ has an Abelian subgroup of rank $\dim(\overline{Q}_i)/2$. Thus for $Q_i$ we have the following possibilities: $Q_8$, $Q_8 * D_8$, or $Q_8 * \mathbb{Z}_{2^n}$. Since $B/B_i$ has an embedding into Aut $(\overline{Q}_i)$ (that is orthogonal or symplectic in dimension $\leqslant 4$), and the only odd primes dividing $|\text{Aut } (\overline{Q}_i)|$ are $\{3,5\}$, we obtain that $p \in \{3,5\}$. Moreover, if $p = 5$, then the only possibility is $Q_i = Q_8 * D_8$, hence, $|Z(Q)| = 2$. ⊠

**Lemma 7.2.** $|Q/Z(Q)| \leqslant 2^{|B|-1}$.

PROOF. Immediate from the previous lemma. If $p = 3$, then $\overline{Q}_i \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$; and if $p = 5$, then $\overline{Q}_i \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. In any case $\dim(\overline{Q}_i) \leqslant |B| - 1$, hence $|Q/Z(Q)| \leqslant 2^{|B|-1}$. ⊠

**Lemma 7.3.** *If $p = 5$, then $B \simeq \mathbb{Z}_5 \times \mathbb{Z}_5$.*

PROOF. If not, choose $B_2 = C_B(\overline{Q}_1 \oplus \overline{Q}_2)$. Clearly $B_2$ has corank 2 in $B$. By Thompson dihedral lemma, $m_2(C_Q(B_2)) \leqslant 3$. From the other hand, $C_G(B_2) \geq Q_8 * D_8 * Q_8 * D_8 \simeq D_8 * D_8 * D_8 * D_8$ contains a subgroup of rank 5, $\Rightarrow\Leftarrow$. ⊠

As a corollary we immediate obtain that $|Q| \leqslant 2^{25}$.

**Lemma 7.4.** *Let $R$ be a $p$-group of $\Phi$-class $\leqslant 2$ (i. e. $Frat(R) \leq Z(R)$ is elementary Abelian) and* $\exp(R) = p$. *If $m_p(R) \leqslant m$ then $|R| \leqslant p^{m^2}$.*

PROOF. If $m = 1$ the statement is trivial. Assume that $A \leqslant R$, $A$ is Abelian and rank $(A) = m$. Moreover, under our conditions $A \geq Frat(R)$, hence $A$ is normal in $R$. Then $A = C_R(A)$ and $R/A$ is isomorphic to a subgroup of $GL_m(A)$. Thus $|R/A| \leqslant p^{\frac{n^2}{2}}$ and the lemma follows. ⊠

**Lemma 7.5.** *If $p = 3$ then $m_3(B) \leqslant 8$.*

Proof. Let $x \in B^\sharp = B - \{1\}$ is chosen such that $C_Q(x)$ is maximal. Let $C_Q(x) = Q_1 * \ldots * Q_l * Z$, where $Q_i$ is either quaternion or dihedral. Let $Q_0 = Q_1 * \ldots * Q_\ell$. Note that $C_B(Q_0) = \langle x \rangle$. If not, let $\langle x, y \rangle \in C_B(Q_0)$, hence $Q_0 = C_Q(x')$ for any $x \in \langle x, y \rangle - \{1\}$ (we have equality in view of maximality of $C_Q(x)$). But $Q = \langle C_Q(x') : x' \in \langle x, y \rangle - \{1\} = Q_0$ (Shur's lemma), $\Rrightarrow\Lleftarrow$. Thus $B/\langle x \rangle$ acts faithfully on $Q_0/Z$. Note that $m_3(B/\langle x \rangle) = \ell$, hence $m_3(B) = \ell + 1$.

Consider $C_G(x)$. Since $G$ of characteristic $p$-type, $F^*(C_G(x)) = O_p(C_G(x)) = P$ is a 3-group. Also $Q_0 \in C_G(x)$ acts faithfully on $P$. In view of Critical subgroups Lemma there exists a subgroup $C$ of $P$ of $\Phi$-class at most 2 such that $Q_0$ acts faithfully on this subgroup. Hence $Q_0$ acts faithfully on $C/Frat(C)$. We obtain an extraspecial 2-group acting on $\overline{C}$. Let $\overline{C}_1$ be a faithful $Q_0 = Q_1 * \ldots * Q_e ll$-submodule. This representation is a tensor product of $\ell$ 2-dimensional irreducible faithful representations for $Q_8$. Thus $\dim(\overline{C}_) = 2^\ell$. There exists an involution $t \in Q_1 * Q_2 - Z(Q_1)$. Hence $m_3(C_{\overline{C_1}}(t)) = \frac{1}{2}\dim(\overline{C}_1) = 2^{\ell-1}$. Hence $m_3(C_{\overline{C}_1}(t)) \geqslant 3^{2^{\ell-1}}$. By previous lemma $|C_{C_1}|(t)| \leqslant 3^{\ell^2}$. Thus $\ell \leqslant 7$. $\boxtimes$

Note that we actually bound also $|Z(Q)|$, since $B$ acts faithfully on $Z(Q)$. Now applying Brauer-Fauler theorem we complete the proof.

## Lecture 8

**Definition 8.1.** By $L_{p'}(H)$ we denote a subgroup of complete preimage in $H$ of $E(H/O_{p'}(H))$ generated by all $p$-elements. $L_{2'}(H)$ can be written in the form $E(H) * B(H)$.

The following theorem we state without proof.

**Theorem 8.2.** $B$-Theorem. *Let $G$ be a finite group, $H$ a 2-local subgroup of $G$. Then $B(H) \leq B(G)$. In particular, if $O_{2'}(G) = 1$, then $L_{2'}(H) = E(H)$ for any 2-local subgroup $H$.*

Let $\mathbb{I}_2(G)$ be the set of all involutions in $G$,

$$\mathcal{L}_2(G) = \{K | K \text{ is a component of } C_G(t) \text{ for some } t \in \mathbb{I}_2(G)\}.$$

On the set $\mathcal{L}_2(G)$ define a linear order by $K < L$ if and only if $|K| < |L|$. Let $\mathcal{L}_2^*(G)$ be the set of maximal elements of $\mathcal{L}_2(G)$ under given order. Assume that $G$ is a simple group and there exists $K \in \mathcal{L}_2^*(G)$ (i. e. $G$ is not of characteristic 2-type). Assume further that $K \in \mathcal{L}_2^*(G)$ satisfies to the following conditions:

1. $K$ is a group of Lie type in odd characteristic.
2. $K$ is simple.
3. $K \not\simeq L_2(5), L_2(7), L_2(9), G_2(3)', PSp_4(3)$.

For $H \leq G$ define $K(H)$ as a product o all components $K_i$ of $H$ such that $K_i \in \mathcal{L}_2^*(G)$ and $K_i \simeq K$ for all $i$.

**Theorem 8.3.** Component theorem (Pauell, Ashbacher, Gilman). *Under above conditions on $G$ and $K$, for any $H \leq G$ we have that $K(H)$ is a product of at most 2 components.*

Before proving this theorem we prove some additional results.

**Theorem 8.4.** $L$-Balance theorem (Gorenstein, Walter). *Let $C$ be a finite group with $O_{2'}(C) = 1$. Assume that $B$-theorem holds for $C$. Let $H$ be a 2-local subgroup of $C$. Then $E(H) \leq E(C)$.*

Proof. (Here we give an outline of the proof.) Assume that $H = C_C(t)$ and $t \in \mathbb{I}_2(C)$. Note (without proof) that if $K$ is a quasisimple group and $t \in \mathbb{I}_2(\mathrm{Aut}\ (K))$ then $C_K(t) \neq Z(K)$.

Clearly, $E(H) \cap E(G) \trianglelefteq H$. Let $E(H)/Z(E(H)) = \overline{E}_1 \times \overline{E}_2 \times \ldots \times \overline{E}_r$, where $\overline{E}_i$ is a non Abelian finite simple group for all $i$. Hence, every normal subgroup of $\overline{E(H)}$ is a direct product of some of $\overline{E}_i$. Denote $E^1 = E(H) \cap E(C)$, in view of above consideration there exists $E^2$ with $E^1 * E^2 = E$ and $E_1 \cap E_2 = Z(E_1) \cap Z(E_2)$.

We state that $E^2$ centralizes $F^*(C)$, hence is trivial. The proof of this fact we divide into several propositions.

**Proposition 8.5.** $[E^2, O_2(C)] = 1$.

Proof. Since $H = C_C(t)$, it follows that $H$ contains a subgroup $E^2 \times \langle t \rangle$. Let $X$ be a $2'$-subgroup in $E^2$ and consider $X \times \langle t \rangle$. Let $R = C_{O_2(C)}(t) \trianglelefteq H$, so $R \leq O_2(H)$. Since $X \leq F^*(H)$ we have that $[X, O_2(H)] = 1$, i. e. $[X, R] = 1$. Applying Thompson $A \times B$-lemma to $X \times \langle t \rangle$ acting on $O_2(C)$ we obtain that $X$ centralizes $O_2(C)$. But $E^2$ is semisimple, hence it is generated by its $2'$-elements (this follows from the fact that $[E^2, E^2] = E^2$). Thus $E^2$ acts trivially on $O_2(C)$. ⊠

Now $t$ interchanges some components in $E(C)$ and stabilizes others. Thus we have the following picture $L_1 * L_1^t$, $L_2 * L_2^t, \ldots, L_k * L_k^t$, $L_{2k+1}, \ldots, L_s$.

**Proposition 8.6.** *If $L_i \neq L_i$, then $[L_i, E^2] = 1$.*

Proof. $C_{L_i L_i^t}(t) = K_i$ is a component in $E(C)$, thus $K_i \leq E^1$ and $[K_i, E^2] = 1$. ⊠

Thus we are left to prove that $E^2$ acts trivially on $L = L_{2k+1} * \ldots * L_s$. As we note above, $C_{L_j}(t) \nleq Z(L_j)$. Now $[C_L(t), E^2, E^2] = 1$, since $[C_L(t), E^2] \leq E(C) \cap E^2 \leq Z(E^2)$. By the same reason, $[E^2, C_L(t), E^2] = 1$. By 3-subgroups Lemma, $[E^2, E^2, C_L(t)] = 1$. Since $E^2$ is semisimple, it follows that $[E^2, E^2] = E^2$, hence $[E^2, E^2, C_L(t)] = [E^2, C_L(t)] = 1$. Thus $E^2$ centralizes $C_{L_{2k+1}}(t) * \ldots * C_{L_s}(t) \nleq Z(L_{2k+1}) * \ldots * Z(L_s)$, hence $E^2$ normalizes each of $L_j$, $2k+1 \leqslant j \leqslant s$. Since for any simple group $K$, Out $(K)$ is solvable, we obtain that $E^2 \leq L$, $\Rrightarrow\Lleftarrow$. ⊠

**Remark**. In this proof we use the fact that Out $(K)$ is solvable for any finite simple group $K$. But this fact up to now has only one proof using the classification of finite simple groups. Actually, we can avoid using of this fact by more detail consideration. Actually, we have that $E^2$ centralizes a Sylow 2-subgroup in $L$, and Glauberman's theorem states that $E^2$ should be solvable.

**Corollary 8.7.** $K$-Balance Theorem. *Assume that $s, t \in \mathbb{I}_2(G)$ and $[s, t] = 1$. Let $L$ be a component in $K(K(C_G(t)) \cap C_G(s))$ (i. e. $L \simeq K$). Then $L \leq K(C_G(s))$.*

We use it without proof because for the proof we need more general statement of $L$-Balance theorem.

## Lecture 9

**Exercise 5**. Let $p, r$ be odd primes and $p < r$. Assume that $G$ is a finite group of characteristic $p$-type and of characteristic $r$-type. Assume that $m_{p,r} \geqslant 2$. Then $p = 2$ and $r \in \{3, 5\}$.

**Definition 9.1.** We say that a finite group $G$ is of *weak characteristic p-type* if $F^*(C_G(A))$ is a $p$-group for any $p$-subgroup $A$ of $p$-rank $\geqslant 2$ of $G$.

**Exercise 6**. Prove the following. Let $p$ and $r$ be odd primes and $p < r$. Assume that $G$ is a finite group of weak characteristic $p$-type and of characteristic $r$-type with $m_{p,r}(G) \geqslant 2$. Then $G \simeq Ly$ (Lyons simple sporadic group). Hint: one should use induction on the fact that every proper section is not a simple group.

Consider a group $E = K_1 \times \ldots \times K_r$, where $K_i \simeq K$ and $K$ is a finite simple group of Lie type in odd characteristic $p$ and $K \not\simeq L_2(5), L_2(7), L_2(9), G_2(3)', PSp_4(3)$. Consider $A = \mathbb{Z}_2 \times \mathbb{Z}_2$ acting on $E$. Then the following theorem holds.

**Theorem 9.2.** Generation Theorem. $E = \langle C_E(a) | a \in A - \{1\} \rangle$.

Note that this theorem is not true in general case. Assume that $A = K \simeq \mathrm{Alt}_5$ and let $A$ be a Klein 4-group. Then for any $a \in A - \{1\}$ we have $C_E(a) = A$, hence $\langle C_E(a) | a \in A - \{1\} \rangle = A$.

Let $\pi_i : E \to K_i$ be a projection. For $I \subseteq \{1, \ldots, r\}$ define $K_I = \prod_{i \in I} K_i$. By $\Delta$ we denote a diagonal of $E$, $\Delta_I$ denotes a diagonal of $K_I$. The set $\{1, \ldots, r\}$ is a union of $A$-orbits and we, clearly, may assume that it consists of just one orbit.

**Lemma 9.3.** *Let $D \leq E$ is such that $\pi_i|_D$ is a surjection for all $i$. Then there exists a partition $I_1 \cup \ldots \cup I_m = \{1, \ldots, r\}$ such that $D = \prod_{i=1}^m \Delta_{I_i}$.*

**Exercise 7**. Prove this lemma.

**Lemma 9.4.** Generation lemma. *Let $A$ be an elementary Abelian 4-group in* Aut $(E)$ *and assume that $A$ interchanges components of $E$ transitively (i. e. $r = 2$ or $r = 4$). Then $E = \langle C_G(a) | a \in A - \{1\} \rangle$.*

Proof. Consider first the case $r = 4$, so $E = K_1 \times K_2 \times K_3 \times K_4$ and $A$ is a Klein 4-group in $\mathrm{Sym}_4$. Since $A$ is unique up to conjugation in $\mathrm{Sym}_4$, we may consider elements $a_1, a_2 \in A$ with $C_E(a_1) = \Delta_{\{1,2\}} \times \Delta_{\{3,4\}}$ and $C_E(a_2) = \Delta_{\{1,3\}} \times \Delta_{\{2,4\}}$. By using above lemma we obtain that $\langle C_E(a_1), C_E(a_2) \rangle = E$.

Consider now the case $r = 2$. So $E = K_1 \times K_2$, and we have a homomorphism $\varphi : A \to \mathrm{Sym}_2$. Define $Ker(\varphi) = A_0$, then $A - A_0 = \{a, a'\}$. So $C_E(a) = \Delta$, $C_E(a') = \Delta'$, $A_0 - \{1\} = \{a_0\}$. Let $E_0 = \langle C_E(a), C_E(a') \rangle$. In view of above Lemma we have that either $E_0 = \Delta = \Delta'$ or $E_0 = E$. If we have the second possibility, we are done. So we may assume that $E_0 = \Delta = \Delta'$, i. e. $\forall k \in K_1, k^a = k^{a'}$ and $\forall k \in K_1, k^{aa'} = k$. But this implies that $C_E(a_0) \geq K_1$ and we obtain the statement. ⊠

In view of this lemma we may think that $E = K$ and $A \leq$ Aut $(K)$. The following two lemmas complete the proof of Generation theorem.

**Lemma 9.5.** Seitz Lemma. *Let $K$ be a finite simple group of Lie type in odd characteristic $p$, $A$ an elementary Abelian 2-subgroup of rank $\geqslant 2$ of* Aut $(K)$ *and assume that $A$ normalizes nontrivial $p$-subgroup of $K$. Then $K = \langle C_K(a) | a \in A - \{1\} \rangle$.*

PROOF. Here we give just a general idea. Let $R$ be a nontrivial $A$-invariant $p$-subgroup in $K$. Consider the chain $R_1 = O_p(N_K(R))$, $R_2 = O_p(N_K(R_1))$, .... Since $K$ is finite this sequence become stable after a finite number of steps. Thus we get $A$-invariant $p$-subgroup $U_J$ and its normalizer is $P_J = U_J \rtimes C_J$ an $A$-invariant parabolic subgroup of $K$. Here $C_J = L_J H \Phi_K \Gamma_J$, where $\Phi_K$ is a group of field automorphisms and $\Gamma_J$ is a group of graph automorphisms. Note that $C_J$ contains a Sylow 2-subgroup of $P_J$, hence, up to conjugation, we may assume that $A \leq C_J$.

There exists also "opposite" parabolic subgroup $P_{-J} = U_{-J}C_J$, so $A$ normalizes $U_J$ and $U_{-J}$. By Shur's Lemma we have that $U_J = \langle C_{U_J}(a) | a \in A - \{1\}\rangle$ and $U_{-J} = \langle C_{U_{-J}}(a) | a \in A - \{1\}\rangle$ (here we use that $A$ is not cyclic).

Consider $K_0 = \langle U_J, U_{-J}\rangle$. Clearly $N_K(K_0) \geq \langle P_J, P_{-J}\rangle \geq \langle B_J, B_{-J}\rangle$. But $\langle B_J, B_{-J}\rangle = K$, hence $N_K(K_0) = K$. Since $K$ is simple, we obtain that $K_0 = K$ and the lemma follows. $\boxtimes$

**Lemma 9.6.** Lyons-Seitz Theorem. *Let $K$ be a finite simple group of Lie type in odd characteristic $p$. Let $A$ be a Klein 4-group in* Aut $(K)$. *Then $K = \langle C_K(a) | a \in A - \{1\}\rangle$ if $K \not\simeq L_2(5), L_2(7), L_2(9), G_2(3)', PSp_4(3)$.*

PROOF. (Just comments of the proof.) Note first that if $K \not\simeq L_2(p^n)$ and $a$ is an involution in Aut $(K)$, then $C_K(a)$ contains a nontrivial component of Lie type in the same characteristic $p$. Since $A$ is Abelian, $A$ normalizes this centralizer. Consider another involution $a' \in A$ and the component, say $L_1$. Then either $L_1^{a'} = L_2$, so $C_{C_K(a)}(a') = C_K(A)$ contains an element of order $p$, thus we may apply Seitz Lemma. If $L_1^{a'} = L_1$, then, by $K$-Balance Theorem, we obtain that $C_K(a)$ and $C_K(a')$ contain common component of Lie type in characteristic $p$. The lemma follows. $\boxtimes$

## Lecture 10

The main goal of the lecture is to prove the following

**Theorem 10.1.** Component Theorem. *Let $G$ be a finite simple group, $H \leq G$. Then $K(H)$ is a product of at most 2 components.*

Recall the definitions. $\mathbb{I}_2(G)$ is the set of all involutions of $G$,

$$\mathcal{L}_2(G) = \{K | K \text{ is a component of } C_G(t) \text{ for some } t \in \mathbb{I}_2(G)\}.$$

If $K$ is a finite group of Lie type in odd characteristic $p$ and $K \not\simeq L_2(5), L_2(7), L_2(9), G_2(3)', PSp_4(3)$, then for $H \leq G$ define $K(H)$ by the set of all components $K_i$ of $H$ with $K_i \simeq K$. Note that $K(H) \trianglelefteq N_G(H)$, but $K(H)$ is not characteristic in general case.

Two lectures before we state the following theorem.

**Theorem 10.2.** $K$-Balance Theorem. *Assume that $s, t \in \mathbb{I}_2(G)$ and $[s,t] = 1$. Let $L$ be a component in $K(K(C_G(t)) \cap C_G(s))$ (i. e. $L \simeq K$). Then $L \leq K(C_G(s))$.*

We also need the following result that we give without proof.

**Theorem 10.3.** Ashbacher's ZD-Theorem. *Let $G$ be a finite group, $M \leq G$ and $z \in \mathbb{I}_2(G)$.Assume that following two conditions hold*

(a) $z^g \in M$ *implies* $g \in M$;

(b) *If* $u = zz_1 = z_1 z \in M$, $z_1 = z^g$, *and* $u^g \in M$, *then* $g \in M$.

*Then either* $\langle z^G \rangle \leq M$ *or* $\langle z^G \rangle \cap M$ *is a strongly embedded subgroup of* $\langle z^G \rangle$.

**Definition 10.4.** A proper subgroup $M$ of $G$ is called *strongly embedded* in $G$, if $|M|$ is even but $|M \cap M^g|$ is odd for all $g \in G - M$. In particular, $M = N_G(M)$, $M \geq S \in Syl_2(G)$, $M \geq N_G(S)$, $M \geq C_G(x)$ for any $x \in \mathbb{I}_2(G)$.

The following theorem we again state without proof.

**Theorem 10.5.** Suzuki Theorem. *If $M$ is a strongly embedded subgroup of a finite simple group $G$, then $G \simeq L_2(2^n), Sz(2^{2n+1}), PSU-3(2^{2n})$.*

Now we are able to prove Component Theorem. Assume that Component Theorem is false and $H$ is chosen so that $K(H)$ is maximal. Since $H$ is a counterexample, we obtain that $K(H) = K_1 \times \ldots \times K_r, \geqslant 3$. Denote the set $I = \{1, \ldots, l\}$. As in previous lecture define for a subset $J \subseteq I$ define $K_J$ and $\Delta_J$. Clearly we may assume that $H = N_G(K(H))$.

**Lemma 10.6.** *Let $t \in \mathbb{I}_2(C_G(K_i))$. Then $K_i \leq K(C_G(T))$.*

Proof. Let $S \in Syl_2(C_G(K_i))$. Assume that $K_i \in \mathcal{L}_2^*(G)$. Hence there exists $s \in S$ such that $K_i$ is a component in $C_G(s)$. Thus $K_i \in K(C_G(s))$. We obtain a subgroup $S \times K_i$ and $s, t \in S$. Choose $z \in \mathbb{I}_2(Z(S))$. In view of $K$-Balance theorem we have that $K_i \leq K(C_G(z) \cap C_G(s)) \leq K(C_G(z))$. There are two possibilities: either $K_i \trianglelefteq K(C_G(z))$ or $K_i \leq K(C_G(z))$ and $K_i$ is a diagonal in $K_{i_1} \times K_{i_2} \trianglelefteq K(C_G(z))$. In any case $K$-Balance theorem implies $K_i \leq K(C_G(z) \cap C_G(t)) \leq K(C_G(t))$. ⊠

**Lemma 10.7.** (a) *If $\emptyset \neq J \subseteq I$, then $N_G(K_J) \leq H = N_G(K_I)$.*
(b) *If $t \in \mathbb{I}_2(K_i)$, then $K(C_G(t)) = K_{\{i'\}}$ and $C_G(t) \leq H$.*

Proof. Consider $C_J = K_J C_G(K_J) \trianglelefteq N_G(K_J)$. Clearly it is enough to prove that $K(C_J) = K_I$. Indeed, in this case $N_G(K_J) \leq N_G(C_J) \leq N_G(K(C_J)) \leq N_G(K_I) = H$. By definition $K_J \trianglelefteq C_J$. If $j \in J$, then $K_j \lhd \lhd C_J$, hence $K_i \trianglelefteq K(C_J)$. Now $K_{J'} \leq C_G(K_J) \leq C_J$.

Without lost of generality we may assume $1 \in J$. Consider $t \in \mathbb{I}_2(K)$. In view of above lemma $K_2 \times \ldots \times K_r \leq K(C_G(t))$.

Assume by contradiction that $K_2 \times \ldots \times K_r \lneqq K(C_G(t))$. In view of maximality of $r$ we have that either $K_2 \times \ldots \times K_r \times K_{r+1}$ or, up to renumbering of components, $K_r$ is a diagonal in $K_{r_1} \times K_{r_2}$, hence $K(C_G(t)) = K_2 \times \ldots \times K_{r-1} \times K_{r_1} \times K_{r_2}$. We have that $K_2 \times \ldots \times K_r \leq C_G(K_1) \leq C_G(t)$, hence if $K_2 \times \ldots \times K_r \lhd \lhd C_G(t)$, then $K_2 \times \ldots \times K_r \lhd \lhd C_J$ and we obtain point (a). So we may assume that we have the second case. But in the second case $K_2 \times \ldots \times K_{r-1} \times K_{r_1} \times K_{r_2} \lhd \lhd C_J$, hence $K_1 \times \ldots \times K_{r-1} \times K_{r_1} \times K_{r_2} \leq K(C_J)$, $\Rrightarrow\Lleftarrow$with the maximality of $r$. Thus the second case is impossible. Clearly we also prove point (b) of the lemma. ⊠

**Lemma 10.8.** *If $t \in \mathbb{I}_2(K_I)$ and $supp(t) = J \neq I$, then $K(C_G(t)) = K_{J'}$ and $C_G(t) \leq H$. Here $supp(t) = \{i | \pi_i(t) \neq 1\}$.*

Proof. Let $t \in \mathbb{I}_2(K_I)$. Assume that $1 \notin supp(t)$. Hence $[K_1, t] = 1$. Choose $U \leq K_1$ such that $U = \mathbb{Z}_2 \times \mathbb{Z}_2$ is a Klein 4-group (clearly such a subgroup always exists in groups of Lie in odd characteristic). In view of Lyons-Seitz Theorem $K(C_G(t)) = \langle C_{K(C_G(t))}(u) | u \in U - \{1\}\rangle$. By Lemma **10.7**(b) we have that $\forall u \in U^\sharp, C_G(u) \leq H$. Hence, $K(C_G(t)) \leq H$. Clearly $K_1 \leq K(C_G(t))$. There can be two possibilities. First $K_1 \trianglelefteq K(C_G(t))$, by Lemma **10.7**(a) we obtain that $(K(C_G(t)) \leq K(N_G(K_1)) = K_I$. If $K_1$ is not normal $K(C_G(t))$, then $K_1$ is a diagonal in $K_{1_1} \times K_{1_2}$.?????? From here it follows that $K(C_G(t)) \leq K(H)$, thus Lemma **10.7**(b) implies $K(C_G(t)) = K_{J'}$. It also follows that $\forall t \in \mathbb{I}_2(K(H)), C_G(t) \leq H$ ⊠

**Corollary 10.9.** *If* $t, t^g \in \mathbb{I}_2(K_I)$ *and* $supp(t) \neq I \neq supp(t^g)$*, then* $|supp(t)| = |supp(t^g)|$.

Proof. We have $C_G(t) \leq H$, $C_G(t^g) \leq H$. Further $K(C_G(t)) = K_{supp(t)'}$, $K(C_G(t^g)) = K(C_G(t))^g = K_{supp(t^g)'}$. Hence $|supp(t)| = |supp(t^g)|$. ⊠

**Lemma 10.10.** *Let* $t \in \mathbb{I}_2(K_I)$ *and* $supp(t) = J \neq I$*. If* $t^g \in M$*, then* $g \in M$.

## Lecture 11

**Exercise 8**. If $G$ has a strongly embedded subgroup, then $G$ has just one class of involutions.

Recall that $L_{2'}(H) = O^{2'}(E(H \mod O_{2'}(H)))$, $L_{2'}(H) = E(H) * B(H)$. In Lecture 8 we state the following

**Theorem 11.1.** $B$-Theorem. *Let $G$ be a finite group, $H$ a 2-local subgroup of $G$. Then $B(H) \leq B(G)$.*

**Corollary 11.2.** *If $O_{2'}(G) = 1$ and $H$ is a 2-local subgroup of $G$, then $L_{2'}(H) = E(H)$.*

Note that there are groups with the same 2-structure, but non isomorphic. Consider $G = PSL_2(43)$ and $\widetilde{G} = (\mathbb{Z}_{11} \times \mathbb{Z}_{11} \times \mathbb{Z}_1 1) \rtimes \mathrm{Alt}_5$. $\mathrm{Alt}_5$ acts on $F_{11}^3$ in view of the following embedding

$$\mathrm{Alt}_5 \simeq PSL_2(5) \leq PSL_2(11) \simeq O_3(11)'.$$

If $T \in Syl_2(G)$, then $T \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. If $\widetilde{T} \in Syl_2(\widetilde{G})$, then $\widetilde{T} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. Now $C_G(T) = T$ and $C_{\widetilde{G}}(\widetilde{T}) = \widetilde{T}$, $N_G(T) \simeq N_{\widetilde{G}}(\widetilde{T}) = \widetilde{T} \simeq \mathrm{Alt}_4$. There exists just one class of involutions and $C_G(t) \simeq C_{\widetilde{G}}(\widetilde{t}) \simeq D_{44} = \mathbb{Z}_{22} \rtimes \mathbb{Z}_2$. So we can not recognize these two groups just by their 2-structure. From the other hand there exists

**Theorem 11.3.** Brauer-Suzuki-Wolf Theorem. *If $G$ is a finite simple group such that $C_G(t)$ is a dihedral group, then $G \simeq PSL_2(q)$ with $q$ odd.*

**Definition 11.4.** (Gorenstein, Goldshmidt.) Let $A$ be an elementary Abelian $p$-subgroup of $G$. A function $\theta$ on $A^\sharp = A - \{1\}$ is called *an $A$-signalizing functor* if

(i) for any $a \in A^\sharp$, we have $\theta(a)$ is an $A$-invariant $p'$-subgroup in $C_G(a)$;

(ii) $\forall a, a' \in A^\sharp$ we have $\theta(a) \cap C_G(a') = C_G(a) \cap \theta(a')$.

Let $V$ be an $A$-invariant $p'$-subgroup of $G$. Define $\forall a \in A^\sharp, \theta_V(a) = C_V(a)$. Clearly, $\theta_V$ is an $A$-signalizing functor.

The following theorem we state without proof.

**Theorem 11.5.** Signalizing functor theorem. *Let $G$ be a finite group, $A$ be an elementary Abelian $p$-subgroup of rank $\geqslant 3$. Assume that $\theta$ is a $p$-signalizing functor. Then $\forall a \in A^\sharp, \theta(a) = C_V(a)$, where $V = \langle \theta(a) | a \in A^\sharp \rangle$.*

Note that rank $\geqslant 3$ is essential. Counterexample in lower rank. Consider $G \simeq PSL_2(11)$ for $p = 2$. $T = A = \mathbb{Z}_2 \times \mathbb{Z}_2$, $\theta(a) = O_{2'}(C_G(a))$. But $\langle \theta(a), a \in A^\sharp \rangle = G$.

By using Signalizing functor theorem one can prove $B$-Theorem. ?????

## Lecture 12

Here we give a brief idea about classification of finite simple non Abelian groups.

By Alt we denote the set of alternating groups of degree at least 5. Chev($p$) is the set of groups of Lie type in characteristic $p$. Spor is the set of 26 sporadic groups.

**Theorem 12.1.** (Classification theorem) *Let $G$ be a finite non Abelian simple group. Then $G \in \mathrm{Alt} \cup \mathrm{Chev}(p) \cup \mathrm{Spor}$.*

**Definition 12.2.** Let $G$ be a non Abelian finite simple group (here and below), a prime $r$ is called *semisimple prime* if there exists a (semisimple) element $x \in G$ such that $C_G(x)$ has a component $K$. Note that this definition is not equivalent to the definition of semisimple element in groups of Lie type, but close to it.

A prime $p$ is called *characteristic prime* for $G$ if $\forall x \in G$ of order $p, F^*(C_G(x)) = O_p(C_G(x))$.

Denote by $\pi(G)$ the set of prime divisors of the order of $G$. $\pi_{ss}(G)$ is the set of semisimple divisors, $\pi_{ch}(G)$ is the set of characteristic divisors of the order or $G$. Clearly $\pi_{ss}(G) \cap \pi_{ch}(G) = \emptyset$. In general, $\pi_{ss}(G) \cup \pi_{ch}(G) = \pi(G)$ is not true. For example, consider $G = PSL_2(11)$, let $t \in \mathbb{I}_2(G)$, then $C_G(t) = D_{12}$. Thus $2 \notin \pi_{ss}(G) \cup \pi_{ch}(G)$.

Define $\pi(G)^* = \{p \in \pi | m_p(G) \geqslant 3\}$, $\pi_{ss}(G)^* = \pi_{ss}(G) \cap \pi(G)^*$, $\pi_{ch}(G)^* = \pi_{ch}(G) \cap \pi(G)^*$. In this case the identity $\pi_{ss}(G)^* \cup \pi_{ch}(G)^* = \pi(G)^*$ is true. The main part of the proof of this identity is the $B_p$-Theorem.

**Theorem 12.3.** ($B_p$-Theorem) *For any $p$-local $H \leq G$ we have $B_p(H) \leq B_p(G)$.*

Here for a prime $p$ we define $L_{p'}(H) = O^{p'}(E(H \mod O_{p'}(H)))$ (see Lecture 8), then $L_{p'}(H) = O^{p'}(E(H)) * B_p(H)$. This defines subgroup $B_p(H)$.

**Corollary 12.4.** *If $O_{p'}(G) = 1$, then for any $p$ dividing $|H|$, $L_{p'}(H)$ is semisimple.*

**Corollary 12.5.** *If $p \notin \pi_{ss}(G)$, then for ny $x$ of order $p$, $F^*(C_G(x))/O_{p'}(C_G(x))$ is a $p$-group.*

**Definition 12.6.** A proper subgroup $M \leq G$ is called *strongly $p$-embedded* if $p$ divides $|M|$ but $\forall g \in G - M$, $p$ does not divides $|M \cap M^g|$.

**Theorem 12.7.** (Strongly $p$-embedded $p'$-local theorem) *Let $G$ be a finite group, $p \in \pi(G)^*$, and $M$ is a strongly $p$-embedded subgroup of $G$. Then $O_{p'}(M) = O_{p'}(G)$.*

There are four more or less different ways to prove the Classification theorem.

First proof (historical proof). Consists of the following steps.

I. The classification of simple groups with $m_2(G) \leqslant 2$.

II. Prove the fact that if $m_2(G) \geqslant 3$, then either $2 \in \pi_{ss}(G)^*$ or $2 \in \pi_{ch}(G)^*$.

III. The classification of finite simple groups with $2 \in \pi_{ss}(g)^*$.

IV. The classification of finite simple groups of characteristic 2-type with $e(G) \leqslant 2$ (so-called quasithin groups). Stive Smith and Michael Ashbacher.

V. The classification of finite simple groups of characteristic 2-type with $e(G) \geqslant 3$.

VI. The classification of finite simple groups of characteristic 2-type with $p \in \pi_{ss}(G)^*$.

VII. The classification of finite simple groups of characteristic 2-type and of characteristic $p$-type with $m_{2,p}(G) \geqslant 3$. (There are no such groups.)

Second proof (Gorenstein-Lyons-Solomon).

This proof is based on the following generalization of semisimple and characteristic primes.

**Definition 12.8.** A prime $r$ is called *strongly semisimple prime* for $G$ if there exists $x \in G$ of order $r$ and a component $K$ of $C_G(x)$ with $K \notin \mathrm{Chev}(p) \cup \mathrm{Spor}$.

A prime $p$ is called *weakly characteristic prime* for $G$ if $\forall x \in G$ of order $p$, $F^*(C_G(x)) = O_p(C_G(x))E(C_G(x))$. If $K$ is a component in $C_G(x)$, then $K/Z(K) \in \mathrm{Chev}(p) \cup \mathrm{Spor}$.

This way has the advantage, that we deal with not only simple, but also almost simple groups. Note that $\pi(G)^* = \pi_{sss}(G)^* \cup \pi_{wch}(G)^*$. In some small or sporadic groups there is a possibility that $\pi_{sss}(G)^* \neq \pi_{ss}(G)^*$ and $\pi_{wch}(G)^* \neq \pi_{ch}(G)^*$, but usually we have the identities.

Third proof (Stellmacher). We can union steps IV–VII of the first proof. The idea is: the proof in general case is not much more difficult than in case of quasithin groups.

Fourth proof. Union of steps III and IV of the third proof.

# Solution of exercises

Exercise **1**. Assume that $S$ is a solvable subgroup of $SL_2(\mathbf{F})$. Clearly we may consider algebraic closure $\mathbb{F}$ of $\mathbf{F}$ and assume that $S \leq SL_2(\mathbb{F})$. Consider $F(S)$. If $F(S)$ has a nontrivial unipotent element, say $x$, then $F(S)$ has a central unipotent element ($F(S)$ is nilpotent), hence $F(S)$ is contained in a centralizer of some unipotent element. As we already noted, $F(S) \leq Z(SL_2(\mathbf{F})) \times U$ and $S$ is a subgroup of Borel subgroup.

So assume that $F(S)$ consists of semisimple elements, i. e. it is completely reducible. If $F(S)$ is not irreducible, then it is Abelian and we obtain that $S$ is contained in the normalizer of some Abelian subgroup of $SL_2(\mathbf{F})$. So assume that $F(S)$ is irreducible. Therefore $Z(F(S)) = Z(SL_2(\mathbf{F}))$ and, hence $F(S)$ is a 2-group. Since $F(S)$ is non Abelian, it follows that $F(S)$ is extraspecial. Since the rank of Abelian subgroups of $SL_2(\mathbf{F})$ is not greater than 2, we have that $F(S)$ is quaternion. Thus either $F(S)$ contains a characteristic Abelian subgroup and so $S$ is in the normalizer of some Abelian subgroup, or $F(S) = Q_8$ and we obtain exceptional subgroup described in Lecture 1.

Exercise **2**. Let $S$ be a finite subgroup of $SL_2(\mathbf{F})$ and consider $F^*(S)$. In view of description Solvable subgroups and quasisimple subgroups we wee that either $F^*(S) = F(S)$ or $F^*(S) = E(S)$. In the first case $S$ is solvable, in the second case $S$ is the normalizer of a quasisimple subgroup.

Exercise **3**. Assume first that $N$ is a $p$-group. Assume that $x \in C_{\mathrm{Aut}\ (N)}(N/Frat(N))$ and $g.c.d.(|x|, p) = 1$. It is known that in this case $x$ centralizes $N$. Hence, $C_{\mathrm{Aut}\ (N)}(N/Frat(N))$ is a $p$-group in this case. Now, if $N = P_1 \times \ldots \times P_k$ is a direct product of its Sylow subgroups, then $N/Frat(N) = P_1/Frat(P_1) \times \ldots \times P_k/Frat(P_k)$ and $C_{\mathrm{Aut}\ (N)}(N/Frat(N)) = C_{\mathrm{Aut}\ (P_1)}(P_1/Frat(P_1)) \times \ldots \times C_{\mathrm{Aut}\ (P_k)}(P_k/Frat(P_k))$ and the statement follows.

Exercise **4**. First we need to prove the following lemma.

**Lemma 12.9.** *Let $P$ be a $p$-group and $A$ an elementary Abelian $r$-group of rank $\geqslant 2$ and $A$ acts on $P$. Then there exists an element $a \in A^\sharp$ such that $C_P(a) \neq 1$.*

Proof. If $A$ does not acts faithfully, then the statement is trivial, since $C_P(A) \neq 1$. Hence we may assume that $A$ acts faithfully. By Thompson Dihedral Lemma (generalized variant) we obtain a subgroup $A_1P_1 \times \ldots \times A_kP_k \leq A \ltimes P$, where $k$ is the rank of $A$, $A_i = \langle a_i \rangle$ is cyclic for all $i$. So $C_P(a_1) \geq P_2 \times \ldots \times P_k \neq 1$. ⊠

Now turn to our exercise. Assume first that $R = O_r(G) \neq 1$ for some $r \neq p$. Let $B \in \mathcal{B}_p(G)$. In view of Lemma **12.9**, there exists an element $b \in B$ such that $C = C_R(b) \neq 1$. Consider $N_G(\langle b \rangle)$. Since $G$ is of characteristic $p$-type, $F^*(N_G(\langle b \rangle)) = O_p(N_G(\langle b \rangle))$. But $F^*(N_G(\langle b \rangle)) \geq N_G(\langle b \rangle) \cap R \geq C \neq 1$, $\Rrightarrow\Lleftarrow$. Thus $O_r(G) = 1$ for all $r \neq p$.

Assume now that $P = O_p(G) \neq 1$. As we noted, $G$ contains an elementary Abelian 2-subgroup $A$ of rank at least 3. Like above we obtain a contradiction with the fact that $G$ is of characteristic 2-type.

So, $F^*(G) = E(G) = K_1 \times \ldots \times K_l$ is a direct product of simple groups. If $l \geqslant 2$, take an involution $t$ in $K_1$. Then $F^*(C_G(t))$ contains $K_2 \times \ldots \times K_l$. From the other hand, $G$ is of characteristic 2-type, hence $F^*(C_G(t)) = O_2(C_G(t))$, $\Rrightarrow\Lleftarrow$. Thus $l = 1$ and the statement follows.

Exercise **5**. In view of Klingei-Mason Theorem it is enough to prove that there does not exist a finite group $G$ such that $G$ is of characteristic $r$-type and of characteristic $p$-type for some odd primes $p < r$ and $m_{p,r}(G) \geqslant 2$. Assume by contradiction that such a group $G$ exists. Let $H$ be a $p$-local subgroup with $m_r(H) = m_{p,r}(G) = k \geqslant 2$ and let $B$ be an elementary Abelian $r$-subgroup of $H$ of rank $k$. Let $P = O_p(H)$.

Exercise **6**.

Exercise **7**.