THE CLASSIFICATION OF FINITE SIMPLE GROUPS AND ITS APPLICATIONS IN GROUP THEORY

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1 Preliminaries

1.1 Notations

The term "group" always means a "finite group".

$A \leqslant G$ and $A \trianglelefteqslant G$ means that A is a subgroup (respectively normal subgroup) of G.

If A is a group and $B \leqslant \mathrm{Sym}_n$, then $A \wr B$ is the permutation wreath product, i.e., $A \wr B = (A_1 \times \ldots \times A_n) : B$, where B permutes the A_i.

Let A, B be subgroups of G such that $B \trianglelefteqslant A$. Then $N_G(A/B) = N_G(A) \cap N_G(B)$ is the *normalizer* of A/B in G. If $x \in N_G(A/B)$, then x induces an automorphism of A/B by $Ba \mapsto Bx^{-1}ax$. Thus there exists a homomorphism $N_G(A/B) \to \mathrm{Aut}(A/B)$. The image of $N_G(A/B)$ under this homomorphism is denoted by $\mathrm{Aut}_G(A/B)$ and is called a *group of induced automorphisms* of A/B. If $B = 1$, then we write $\mathrm{Aut}_G(A)$.

A normal series

$$G = G_0 \trianglerighteqslant G_1 \trianglerighteqslant \ldots \trianglerighteqslant G_k = 1$$

is called a *chief series*, if, for each $i = 1, \ldots, n$, G_{i-1}/G_i is a minimal normal subgroup of G/G_i. A composition series

$$G = G_0 \trianglerighteqslant G_1 \trianglerighteqslant \ldots \trianglerighteqslant G_n = 1$$

is called an (rc)-series, if it is a refinement of a chief series of G.

1.2 Groups of induced automorphisms

Example 1.
Groups $\mathbb{Z}_6$ and Sym_3 has composition series with the same sections: $\mathrm{Sym}_3 \geqslant \mathbb{Z}_3 \geqslant 1$ and $\mathbb{Z}_6 \geqslant \mathbb{Z}_3 \geqslant 1$. However, $\mathrm{Aut}_{\mathrm{Sym}_3}(\mathbb{Z}_3) \simeq \mathbb{Z}_2$, while $\mathrm{Aut}_{\mathbb{Z}_6}(\mathbb{Z}_3) = 1$, so we can distinguish these groups, if we know composition series and groups of induced automorphisms.

Example 2 (Groups of induced automorphisms do depend on a composition series)
Let G be a finite simple group possessing an outer automorphism τ of order 2 (for example, $G \simeq \mathrm{Alt}_5$ and $\langle G, \tau\rangle = \mathrm{Sym}_5$). Consider an elementary abelian group $\langle x, y\rangle$ of order 4, then $\langle G, \tau\rangle$ acts on $\langle x, y\rangle$ by $x^\tau \mapsto x$, $y^\tau \mapsto x \cdot y$, and G is in the kernel of the action. Consider $L = \langle x, y\rangle : \langle G, \tau\rangle$. Then L possesses two composition series:

$$1 \leqslant \langle x\rangle \leqslant \langle x, y\rangle \leqslant \langle x, y\rangle \times G \leqslant L$$

$$1 \leqslant \langle y\rangle \leqslant \langle y\rangle \times G \leqslant \langle x, y\rangle \times G \leqslant L.$$

Clearly, L has the unique nonabelian composition factor G, but in the first series we have that $\mathrm{Aut}_L(G) = \mathrm{Aut}_L((\langle x, y\rangle \times G)/\langle x, y\rangle) = G : \langle\tau\rangle$, while in the second series we have $\mathrm{Aut}_L(G) = \mathrm{Aut}_L((\langle y\rangle \times G)/\langle y\rangle) = G$.

Theorem 1. (EV, 2007)
Let

$$G = G_0 \trianglerighteqslant G_1 \trianglerighteqslant \ldots \trianglerighteqslant G_n = 1$$

be an (rc)-series of G, denote the composition factor G_{i-1}/G_i by S_i. Assume that

$$G = H_0 \trianglerighteqslant H_1 \trianglerighteqslant \ldots \trianglerighteqslant H_n = 1$$

is a composition series of G and $T_i = H_{i-1}/H_i$. Then for every T_i there exists S_j such that $\mathrm{Aut}_G(T_i)$ is isomorphic to a subgroup of $\mathrm{Aut}_G(S_j)$ and $T_i \simeq S_j$.

1.3 Special embedding

Let $H_1, \ldots, H_n$ be groups. Recall that $H_1 \times \ldots \times H_n$ can be defined as a set of maps

$$\{f : \{1, \ldots, n\} \to \cup_{i=1}^n H_i \mid f(i) \in H_i\},$$

with multiplication $(f \cdot g)(i) = f(i) \cdot g(i)$.

Now given group H and a subgroup M of Sym_n we have $H \wr M = \{(f, m) \mid f \in H_1 \times \ldots \times H_n; m \in M\}$, where $(f_1, m_1) \cdot (f_2, m_2) = (f_1 \cdot f_2^{m_1^{-1}}, m_1 \cdot m_2)$ and $f^{m^{-1}}(i) = f(i^m)$.

Proposition 1.(Huppert, Endliche gruppen, Hauptsatz 1.4, p 413.)
Let H be a subgroup of G of index n, consider $G = \cup_{i=1}^n Hr_i$, the union of right cosets with representatives r_is. For every $g \in G$ define the permutation $\mathbf{P}(g)$ of $\{1, \ldots, n\}$ and $h_i(g) \in H$ $(i = 1, \ldots, n)$ by

$$r_i g = h_i(g) r_{i\mathbf{P}(g)}.$$

Thus $\mathbf{P}(G)$ is a subgroup of Sym_n.

Then the map $\mathbf{M}$ given by $g \mapsto \mathbf{M}(g) = (f, \mathbf{P}(g))$ with $f(i) = h_i(g)$ is a monomorphism from G into $H \wr \mathbf{P}(G)$. The map $\mathbf{M}$ is a monomial representation of G over H.

Proof.

By

$$h_i(gg')r_{i^{\mathbf{P}(gg')}} = r_i(gg') = (r_i g)g' = h_i(g)r_{i^{\mathbf{P}(g)}}g' = h_i(g)h_{i^{\mathbf{P}(g)}}(g')r_{i^{\mathbf{P}(g)\mathbf{P}(g')}}$$

it follows that $\mathbf{P}(gg') = \mathbf{P}(g)\mathbf{P}(g')$ and $h_i(gg') = h_i(g)h_{i^{\mathbf{P}(g)}}(g')$. Thus, $\mathbf{M}(gg') = (f, \mathbf{P}(g)\mathbf{P}(g'))$, where $f(i) = h_i(gg') = h_i(g)h_{i^{\mathbf{P}(g)}}(g')$. In view of identities $\mathbf{M}(gg') = (f, \mathbf{P}(gg'))$, where $f(i) = h_i(gg')$, and

$$\mathbf{M}(g)\mathbf{M}(g') = (f_1, \mathbf{P}(g)) \cdot (f_2, \mathbf{P}(g')) = (f_1 \cdot f_2^{\mathbf{P}(g')^{-1}}, \mathbf{P}(g)\mathbf{P}(g')),$$

where $f_1 \cdot f_2^{\mathbf{P}(g')^{-1}}(i) = h_i(g)h_{i^{\mathbf{P}(g)}}(g')$

we obtain $\mathbf{M}(gg') = \mathbf{M}(g)\mathbf{M}(g')$.

From $\mathbf{M}(g) = 1$ it follows that $r_i g = r_i$ for every i, hence, $g = 1$. $\square$

Induction often reduces general situation to the following case:

G possesses a unique minimal normal subgroup $T = L_1 \times \ldots \times L_k$, where $L_1, \ldots, L_k$ are nonabelian simple groups.

Since L_i-s are nonabelian and T is a minimal normal subgroup, we obtain that $C_G(T) = 1$ and G acts transitively by conjugation on $\{L_1, \ldots, L_k\}$. Now action of G by conjugation on $\{L_1, \ldots, L_k\}$ is permutationally equivalent to the action of G on right cosets by $N_G(L_1)$ by right multiplication. Denote the image of G in Sym_k by P. Then Proposition 1 implies that there exists the monomorphism $\mathbf{M} : G \to N_G(L_1) \wr P$. Clearly all normalizers $N_G(L_i)$ are conjugate in G, therefore we have

$$N_G(L_1) \wr P \simeq (N_G(L_1) \times \ldots \times N_G(L_k)) : P$$

and we assume that $\mathbf{M}$ maps G into $(N_G(L_1) \times \ldots \times N_G(L_k)) : P$. Now $C_G(L_1) \times \ldots \times C_G(L_k)$ is a normal subgroup of $N_G(L_1) \wr P$ and

$$((N_G(L_1) \times \ldots \times N_G(L_k)) : P) / (C_G(L_1) \times \ldots \times C_G(L_k)) \simeq \mathrm{Aut}_G(L_1) \wr P,$$

therefore there exist a homomorphism $\varphi : G \to \mathrm{Aut}_G(L_1) \wr P$. Moreover, since $\mathrm{Ker}(\varphi) \leqslant C_G(T) = 1$, it follows that φ is a monomorphism.

Thus we obtain the following

Theorem 2. (EV, 2008)
Assume that G possesses a unique minimal normal subgroup $T = L_1 \times \ldots \times L_k$, where $L_1, \ldots, L_k$ are nonabelian simple groups. In particular, G acts on $\{L_1, \ldots, L_k\}$ by conjugation. Denote the image of G in Sym_k by P and $N_G(L_1)/C_G(L_1)$ by $\mathrm{Aut}_G(L_1)$. Then there exists a monomorphism $\varphi : G \to \mathrm{Aut}_G(L_1) \wr P$.

Notice that the embedding $G \to \mathrm{Aut}(L_1) \wr \mathrm{Sym}_k$ is known and can be found in almost every textbook. However the embedding given by Theorem 2 is more accurate, it is close to isomorphic embedding.

2 Hall subgroups

2.1 Preliminaries

Symbol p always denotes a prime (or the set $\{p\}$), while π is a set of primes. By p' we denote the set of primes not equal to p, and by π' we denote the set of all primes not in π.

$\pi(n)$ is the set of all prime divisors of a positive integer n, for a group G we denote $\pi(|G|)$ by $\pi(G)$. A positive integer n with $\pi(n) \subseteq \pi$ is called a π-number, a group G with $\pi(G) \subseteq \pi$ is called a π-group. Given positive integer n denote by n_π the maximal divisor t of n with $\pi(t) \subseteq \pi$.

H is a Sylow p-subgroup of G if $\pi(H) \subseteq p$ and $\pi(\lvert G : H\rvert) \cap p = \varnothing$	H is a π-Hall subgroup of G if $\pi(H) \subseteq \pi$ and $\pi(\lvert G : H\rvert) \cap \pi = \varnothing$
$\mathrm{Syl}_p(G)$ is the set of Sylow p-subgroups of G	$\mathrm{Hall}_\pi(G)$ is the set of π-Hall subgroups of G
Sylow theorems	Hall properties
E-theorem: $\mathrm{Syl}_p(G) \neq \varnothing$	E_π-property: $\mathrm{Hall}_\pi(G) \neq \varnothing$
C-theorem: members of $\mathrm{Syl}_p(G)$ are conjugate	C_π-property: E_π+members of $\mathrm{Hall}_\pi(G)$ are conjugate
D-theorem: each p-subgroup of G is included in a Sylow p-subgroup	D_π-property: C_π+each π-subgroup of G is included in a π-Hall subgroup

	E_π	C_π	D_π
Normal subgroups	Yes	No	Yes (mod CFSG)
Factor groups	Yes	Yes (mod CFSG)	Yes
Extensions	No	Yes	Yes (mod CFSG)

2.2 E-property

Extension Lemma. (D. Revin, EV, 2010)
If $A \trianglelefteq G$, $\pi(G/A) \subseteq \pi$, and $M \in \mathrm{Hall}_\pi(A)$, then there exists $H \in \mathrm{Hall}_\pi(G)$ with $H \cap A = M$ if and only if G, acting by conjugation, leaves invariant the set $\{M^a \mid a \in A\}$.

Let $\pi = \{2, 3\}$, $G = \mathrm{GL}_3(2) = \mathrm{SL}_3(2)$ be a group of order $168 = 2^3 \cdot 3 \cdot 7$. Then G has exactly two classes of π-Hall subgroups with representatives

$$\begin{pmatrix} \boxed{\mathrm{GL}_2(2)} & * \\ 0 & \boxed{1} \end{pmatrix} \text{ and } \begin{pmatrix} \boxed{1} & * \\ 0 & \boxed{\mathrm{GL}_2(2)} \end{pmatrix}.$$

The first one consists of line stabilizers in the natural representation of G, and the second one consists of plain stabilizers. The map $\iota : x \in G \mapsto (x^t)^{-1}$ is an automorphism of order 2 of G. It interchanges classes of π-Hall subgroups, hence the group $\widehat{G} = G \leftthreetimes \langle \iota \rangle$ does not possess a π-Hall subgroup.

Proposition 2. (V.Zenkov, EV, 2009)
Let G be a finite group satisfying conditions of Theorem 2 and $\varphi : G \to \mathrm{Aut}_G(L_1) \wr P$ be the monomorphism constructed in the proof of Theorem 2. Then for every π-Hall subgroup H of G there exists a π-Hall subgroup M of $\mathrm{Aut}_G(L_1) \wr P$ such that $H^\varphi = M \cap G^\varphi$. Moreover, if $\mathrm{Aut}_G(L_1) \wr P \in E_\pi$, then there exists a π-Hall subgroup M of $\mathrm{Aut}_G(L_1) \wr P$ such that $M \cap G^\varphi$ is a π-Hall subgroup of G^φ. In particular, $G \in E_\pi$ if and only if $\mathrm{Aut}_G(L_1) \wr P \in E_\pi$.

It is a standard fact that $\mathrm{Aut}_G(L_1) \wr P \in E_\pi$ if and only if $\mathrm{Aut}_G(L_1) \in E_\pi$. Thus Proposition 2 allows us to obtain a criterion of E_π-property in finite groups.

Theorem 3 (F.Gross; D.Revin, EV)
Let

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \ldots \trianglerighteq G_n = 1$$

be an (rc)-series of a finite group G. Then $G \in E_\pi$ if and only if $\mathrm{Aut}_G(G_{i-1}/G_i) \in E_\pi$ for every $i = 1, \ldots, n$.

2.3 C-property

Let $\pi = \{2, 3\}$, $G = \mathrm{GL}_5(2)$, $\iota : x \in G \mapsto (x^t)^{-1}$ and $\widehat{G} = G \leftthreetimes \langle \iota \rangle$. Then $G \in E_\pi$ and if $H \in \mathrm{Hall}_\pi(G)$, then H is a stabilizer of a series of subspaces $V = V_0 < V_1 < V_2 < V_3 = V$, where V is a natural G-module and $\dim V_k/V_{k-1} \in \{1, 2\}$ for every $k = 1, 2, 3$. Therefore G has three classes of conjugate π-Hall subgroups

$$H_1 = \begin{pmatrix} \boxed{\mathrm{GL}_2(2)} & & * \\ & \boxed{1} & \\ 0 & & \boxed{\mathrm{GL}_2(2)} \end{pmatrix}, H_2 = \begin{pmatrix} \boxed{1} & & * \\ & \boxed{\mathrm{GL}_2(2)} & \\ 0 & & \boxed{\mathrm{GL}_2(2)} \end{pmatrix}, H_3 = \begin{pmatrix} \boxed{\mathrm{GL}_2(2)} & & * \\ & \boxed{\mathrm{GL}_2(2)} & \\ 0 & & \boxed{1} \end{pmatrix}.$$

The class containing H_1 is ι-invariant, so Extension Lemma implies that there exists $H \in \mathrm{Hall}_\pi(\widehat{G})$ with $H \cap \widehat{G} = H_1$. Classes containing H_2 or H_3 are interchanged by ι. So Extension Lemma implies that H_2, H_3 are not contained in π-Hall subgroups of $\widehat{G}$.

Proposition 3. (D.Revin, EV, 2010)
Let G be a finite group satisfying conditions of Theorem 2 and $\varphi : G \to \mathrm{Aut}_G(L_1) \wr P$ be the monomorphism constructed in the proof of Theorem 2. Then $G \in C_\pi$ if and only if $\mathrm{Aut}_G(L_1) \wr P \in C_\pi$.

Again, it is easy to see that $\mathrm{Aut}_G(L_1) \wr P \in C_\pi$ if and only if $\mathrm{Aut}_G(L_1) \in C_\pi$ and $P \in C_\pi$. Thus we obtain an algorithm to check, whether $G \in C_\pi$.

Let $\{e\} = G_0 < G_1 < \ldots < G_n = G$ be a composition series of a finite group G which is a refinement of a chief series $\{e\} = G_0 < G_{k_1} < \ldots < G_{k_m} = G$. If $G_{k_j} = G_{k_{j-1}+s_j}$, then $G_{k_j}/G_{k_{j-1}} = L_{j,1} \times \ldots \times L_{j,s_j}$, where $L_{j,1} \simeq \ldots \simeq G_{i_j+1}/G_{i_j} \simeq L_j$ is simple.

Set $\overline{H}_m = \{e\}$ and $H_m = G_n = G_{k_m} = G$. Assume that we have obtained $\overline{H}_j \in \mathrm{Hall}_\pi(G/G_{k_j})$ and H_j is its complete preimage in G. We check, if $\mathrm{Aut}_{H_j}(G_{k_{j-1}+r}/G_{k_{j-1}+r-1}) \in C_\pi$ for all $r = 1, \ldots, s_j$.

If not, then $G \notin C_\pi$.

If yes, then $G/G_{k_{j-1}} \in C_\pi$ and we take $\overline{H}_{j-1}$ to be any π-Hall subgroup of $G/G_{k_{j-1}}$.

3 Carter subgroups

3.1 Notation

Recall that a nilpotent self-normalizing subgroup K of G is called a Carter subgroup.

Theorem 4. (R.Carter, 1962)
Let G be a finite solvable group. Then G possesses a Carter subgroup and all Carter subgroups of G are conjugate.

Problem 1.
Are Carter subgroups of a finite group conjugate?

Problem 2.
When given finite group possesses a Carter subgroup?

3.2 Conjugacy

Lemma 1. (R.Carter, 1962)
Assume that Carter subgroups of G exist and conjugate. Then, for every normal subgroup N of G, we have that Carter subgroups of G/N exist and conjugate.

Example
Consider $G = \mathrm{Sym}_3$ and its normal subgroup Alt_3. Then $K = \langle (1,2) \rangle$ is a Carter subgroup of G, while Alt_3 is a Carter subgroup of Alt_3. So the intersection of a Carter subgroup with normal subgroup is not necessary a Carter subgroup of the normal subgroup.

Example
Consider $G = \mathrm{Alt}_5$. Then $M_1 = \langle (1,2)(3,4), (1,3)(2,4) \rangle$; $M_2 = \langle (1,2,3) \rangle$; $M_3 = \langle (1,2,3,4,5) \rangle$ are all (up to conjugation) nilpotent subgroup of G. Direct calculations show that $N_G(M_i) \neq M_i$ for $i = 1,2,3$, thus G does not possess a Carter subgroup.

Proposition 4.
Let G be a finite group satisfying conditions of Theorem 2 and $\varphi : G \to \mathrm{Aut}_G(L_1) \wr P$ be the monomorphism constructed in the proof of Theorem 2. Assume also that G possesses a Carter subgroup K and $G = (L_1 \times \ldots \times L_k)K$. Then $\mathrm{Aut}_K(L_1)$ is a Carter subgroup of $\mathrm{Aut}_G(L_1)$.

Using this proposition it is easy to prove, the following

Theorem 5 (F.Dalla Volta, M.C.Tamburini, 1997; EV 2006)
Let

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \ldots \trianglerighteq G_n = 1$$

be an (rc)-series of a finite group G. Assume that Carter subgroups of $\mathrm{Aut}_G(G_{i-1}/G_i)$ are conjugate for each $i = 1, \ldots, n$ (we do not assume, that Carter subgroups exist). Then Carter subgroups of G are conjugate.

Theorem 6. (EV, 2007)
Carter subgroups of a finite group are conjugate.

3.3 Existence

Definition
Let $G = G_0 \geq G_1 \geq \ldots \geq G_n = \{e\}$ be a chief series of G. Then $G_i/G_{i+1} = L_{i,1} \times \ldots \times L_{i,k_i}$, where $L_{i,1} \simeq \ldots \simeq L_{i,k_i} \simeq L_i$ and L_i is a simple group. If $i \geqslant 1$, then denote by $\overline{K}_i$ a Carter subgroup of G/G_i (if it exists) and by K_i its complete preimage in G/G_{i+1}. If $i = 0$, then $\overline{K}_0 = \{e\}$ and $K_0 = G/G_1$ (note that $\overline{K}_0$ always exists). A finite group G is said to satisfy **(E)**, if for each i, j, either $\overline{K}_i$ does not exists, or $\mathrm{Aut}_{K_i}(L_{i,j})$ contains a Carter subgroup.

Now Lemma 2 allows us to get an existence criterion for Carter subgroups.

Theorem 7. (EV, 2008)
Let G be a finite group. Then G contains a Carter subgroup if and only if G satisfies **(E)**.

4 Base size

4.1 Group action

Assume that G acts on X. Elements of X are called points.

G_x is the stabilizer of the point x, x^G is the orbit of the point x.

If H is a subgroup of G, then we always assume that G acts on $G : H$ by right multiplication. Clearly, the action of G on x^G "is the same" as the action of G on $G : G_x$ by right multiplication $(G_x\tau)\sigma = G_x(\tau\sigma)$. Clearly, H_G is the kernel of the action of G on $G : H$.

If G acts on X, then the action of G on X^k is defined by $\sigma : (x_1, \ldots, x_k) \mapsto (x_1^\sigma, \ldots, x_k^\sigma)$.

4.2 Main problem

We consider the following problem

Main problem
Given G acting transitively on X find the minimal k such that there exist points, $x_1, \ldots, x_k \in X$, with $\forall i = 1, \ldots, k \ \ x_i\sigma = x_i \Rightarrow \forall x \in X, x\sigma = x$. This set of points $x_1, \ldots, x_k$ is called a base of G, while the number k is called the base size.

Using the fact that for every $x \in X$ and $y = x\sigma$ the identity $G_y = G_x^\sigma := \sigma^{-1}G_x\sigma$ holds, the main problem can be formulated in the following way:

Main problem
Given subgroup H of G find the minimal number k of conjugated with H subgroups such that their intersection equals H_G.

Using the action of G on X^k we may give another equivalent form of the main problem.

Main problem
Given subgroup $H \leqslant G$ consider the action of G on $X = G : H$ by right multiplication. Find the minimal k such that X^k possesses a regular G/H_G-orbit.

The number k from the main problem is called the *base size* of G with respect to H, and is denoted by $\mathrm{Base}_H(G)$. If $G \leqslant \mathrm{Sym}(X)$ is transitive, then put $\mathrm{Base}(G) = \mathrm{Base}_{G_x}(G)$, where $x \in X$. The number of regular G/H_G-orbits on X^k is denoted by $\mathrm{Reg}_H(G, k)$. Clearly, $\mathrm{Reg}_H(G, k) = 0$ if $k < \mathrm{Base}_H(G)$.
Let again $X = G : H$ and $k = \mathrm{Base}_H(G)$. If $(x_1, \ldots, x_k)$ is a G-regular point, then, clearly, $(x_1, \ldots, x_k)^G \leqslant |X| \cdot (|X| - 1) \cdot \ldots \cdot (|X| - k + 1) < |X|^k$. Since $|X| = |G : H|$, we obtain $|G/H_G| < |G : H|^k$ and, equivalently $|H/H_G| < |G : H|^{k-1}$.

Index bounding problem
Given subgroup H of G find the minimal k such that $|H/H_G| < |G : H|^{k-1}$.

4.3 Solvable subgroups

Theorem (L.Babai, A.J.Goodman, L.Pyber, 1997).
There exists a constant c such that every G possessing a solvable subgroup of index n possesses a normal solvable subgroup of index at most n^c.

Conjecture (L.Babai, A.J.Goodman, L.Pyber, 1997).
The constant c in the theorem is not greater than 7.

Kourovka notebook, 17.41
Let S be a maximal solvable subgroup of a finite group G.
(a) Is it true that $\mathrm{Base}_S(G) \leqslant 7$?
(b) Is it true that $\mathrm{Base}_S(G) \leqslant 5$?

Proposition 5. (EV, 2012)
Let G be a finite group satisfying conditions of Theorem 2 and $\varphi : G \to \mathrm{Aut}_G(L_1) \wr P$ be the monomorphism constructed in the proof of Theorem 2. Assume also that G possesses a solvable subgroup S such that $G = (L_1 \times \ldots \times L_k)S$. Then there exists a solvable subgroup $\overline{S}$ of $\mathrm{Aut}_G(L_1) \wr P$ such that $\mathrm{Aut}_G(L_1) \wr P = (L_1 \times \ldots \times L_k)\overline{S}$ and $\overline{S} \cap G^\varphi = S^\varphi$.

Theorem 8. (EV, 2012)
Let G be a group and let

$$\{e\} = G_0 < G_1 < G_2 < \ldots < G_n = G$$

be a composition series of G that is a refinement of a chief series. Assume that, for some k, the following condition holds: If G_i/G_{i-1} is nonabelian, then for every solvable subgroup T of $\mathrm{Aut}_G(G_i/G_{i-1})$ we have

$$\mathrm{Base}_T(\mathrm{Aut}_G(G_i/G_{i-1})) \leqslant k \text{ and } \mathrm{Reg}_T(\mathrm{Aut}_G(G_i/G_{i-1}), k) \geqslant 5.$$

Then, for every maximal solvable subgroup S of G, we have $\mathrm{Base}_S(G) \leqslant k$.

Example
Consider $G = \mathrm{Sym}_5 \wr \mathrm{Sym}_2$ and $S = \mathrm{Sym}_4 \wr \mathrm{Sym}_2$. It is evident that Alt_5 is the unique nonabelian composition factor of G (however there are two nonabelian composition factors isomorphic to Alt_5). It is also easy to see, that for every solvable subgroup T of $\mathrm{Sym}_5 = \mathrm{Aut}(\mathrm{Alt}_5)$ we have $\mathrm{Base}_T(\mathrm{Sym}_5) \leqslant 4$. However $\mathrm{Reg}_{\mathrm{Sym}_4}(\mathrm{Sym}_5, 4) = 1$ and $\mathrm{Base}_S(G) = 5$.

Example
Consider $G = \mathrm{Sym}_8$ and $S = \mathrm{Sym}_4 \wr \mathrm{Sym}_2$. Then $\mathrm{Base}_S(G) = 5$. Notice also that the inequality $\mathrm{Reg}_S(G, 5) \geqslant 12$ is proven.

Lemma 3. (EV, 2012)
Let G be a transitive permutation group acting on $X = \{1, \ldots, n\}$ and let the stabilizer S of 1 be solvable. Assume that $k = \max\{\mathrm{Base}(G), 6\}$. Then $\mathrm{Reg}(G, k) \geqslant 5$.