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A GENERALIZATION OF FIBONACCI GROUPS

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We study the class of cyclically presented groups which contain Fibonacci groups and Sieradski groups. Conditions are specified for these groups to be finite, pairwise isomorphic, or aspherical. As a partial answer to the question of Cavicchioli, Hegenbarth, and Repovš, it is stated that there exists a wide subclass of groups with an odd number of generators which cannot appear as fundamental groups of hyperbolic three-dimensional manifolds of finite volume.

INTRODUCTION

In the present article, we deal with the class of cyclically presented groups introduced by Cavicchioli, Hegenbarth, and Repovš in [1]. Our interest in this class is motivated, on the one hand, by the fact that it contains well-known and extensively studied groups such as Fibonacci groups and Sieradski groups, and on the other hand, by the question posed in [1] asking whether groups in the class in question are fundamental groups of three-dimensional manifolds.

Fundamental groups of compact 2-dimensional manifolds are well known and quite thoroughly studied (cf. [2]). At the same time, we know from [3, Sec. 5.1] that, for $n \geqslant 4$, every finitely presented group can be realized as a fundamental group of some closed orientable n-dimensional manifold. The case of 3-dimensional manifolds is most complicated. It was shown in [4] that there exists no algorithm which given a finite presentation of a group determines whether that group is fundamental in a 3-dimensional manifold.

The recognition problem for groups of 3-manifolds is of interest both in topology (a fundamental group is a most valuable invariant of a manifold) and in group theory (a knowledge that a group is fundamental in a 3-manifold can give us more information about the structure of the group). In particular, if G is a fundamental group of a 3-manifold of constant negative curvature, then it is hyperbolic in the sense of Gromov, and therefore the equality and the conjugacy problems are decidable in G.

A class of groups for which the recognition problem for 3-manifolds seems most acute is the class of cyclically presented groups. We say that a group G is *cyclically presented* if it has the following presentation:

$$G_n(w) = \langle x_1, x_2, \ldots, x_n \mid w = 1,\ \eta(w) = 1,\ \ldots,\ \eta^{n-1}(w) = 1\rangle,$$

where w is a word in the alphabet $X = \{x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}\}$ and η is an automorphism of a free group $F_n = F_n(x_1, \ldots, x_n)$, whose action on generators is given by the following rule: $\eta(x_i) = x_{i+1}$, $i = 1, \ldots, n$, where all indices are taken modulo n.

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Algebraic properties of cyclically presented groups, for instance, finiteness conditions and algebraic invariants, were studied in [5-9]. Three-dimensional manifolds whose fundamental groups are cyclically presented were taken up in [10-17]. In particular, special attention was given to the case where an automorphism of $G_n(w)$, induced by the above automorphism η of F_n, corresponds to a cyclic branched covering of a 3-dimensional sphere (cf. [18-23]).

In the present article, we deal with cyclically presented groups corresponding to a defining word $w = x_1 x_{1+m} x_{1+k}^{-1}$, for some integers m and k, that is, with groups such as

$$G_n(m,k) = G_n(x_1 x_{1+m} x_{1+k}^{-1}) = \langle x_1, x_2, \ldots, x_n \mid x_i x_{i+m} = x_{i+k},\ i = 1, \ldots, n \rangle,$$

where all indices are taken modulo n and take up their values from the set $\{1, 2, \ldots, n\}$.

We recall that the class $G_n(m,k)$ contains many of the known and thoroughly studied groups. For $m = 1$ and $k = 2$, we have $G_n(1,2) \cong F(2,n)$, where $F(2,n)$ are the Fibonacci groups brought in sight by Conway in [5]. In [21], it was shown that if $n \geqslant 4$ is even then $F(2,n)$ are fundamental groups of 3-manifolds. Such a manifold can be presented as an $n/2$-fold cyclic branched covering of the 3-sphere, branched over the figure-eight knot. Moreover, for $n \geqslant 8$, the manifolds in question are hyperbolic. At the same time, it was shown in [24] that for n odd, $F(2,n)$ cannot be fundamental groups of hyperbolic 3-orbifolds (in particular, 3-manifolds) of finite volume. The various generalizations of Fibonacci groups were introduced and examined in [7, 9, 15, 16]. Asphericity and atoricity of the wide class of generalized Fibonacci groups were the subject matter of [25]. For $m = 2$ and $k = 1$, we have $G_n(2,1) \cong S(n)$, where $S(n)$ are the Sieradski groups treated in [26], in which it was shown that each $S(n)$ is a fundamental group of a 3-manifold. Moreover, it was proved in [27] that such a manifold can be obtained as an n-fold cyclic branched covering of the 3-sphere, branched over the trefoil knot (see also [12, 28]). For $k = 1$, the family of groups $G_n(m,1)$ was studied in [29], together with their HNN-extensions, finiteness, and asphericity. In [30], it was shown that an Abelianizer of $G_n(m,1)$ is infinite if and only if n is divisible by 6, and $m \equiv 2 \pmod 6$.

Our plan is as follows. In Section 1, some results are presented concerning the structure and pairwise isomorphisms of groups $G_n(m,k)$; we specify conditions for these groups to be cyclic, factorable into free products, and pairwise isomorphic. In Section 2, we generalize the approach from [29], and using certain of the results from [31], give a condition for these groups to be aspherical. In Section 3, the question posed in [1] is answered in part; namely, we demonstrate that a wide subclass of groups $G_n(m,k)$ with an odd number n of generators cannot appear as fundamental groups of hyperbolic 3-orbifolds (in particular, 3-manifolds) of finite volume. In Section 4, we bring out certain of the results of computations obtained by using the computer program *GAP* from [32]: orders of groups $G_n(m,k)$ and their Abelianizers for small values of parameters. Finally, in Section 5, some open questions concerning properties of $G_n(m,k)$ are formulated.

1. ELEMENTARY PROPERTIES OF GROUPS $G_n(m,k)$

From the presentation of $G_n(m,k)$ we immediately obtain

LEMMA 1.1. Groups $G_n(m,k)$ have the following properties:
(1) if n and k are coprime then $G_n(0,k) \cong \mathbf{Z}_{2^n-1}$;
(2) if $k = 0$ or $m = k$ then $G_n(m,k)$ is trivial;
(3) $G_n(m,k) \cong G_n(n-m, n-m+k)$;

(4) $G_{2k}(k-1,k) \cong G_{2k}(k+1,1) \cong \mathbf{Z}_{2^k+1}$.

Proof. (1) The group $G_n(0,k)$ has the following presentation:

$$G_n(0,k) = \langle x_1, \dots, x_n \mid x_i^2 = x_{i+k}, \ \ i = 1, 2, \dots, n\rangle.$$

In particular, $x_{n-k}^2 = x_n$. Substituting this expression in $x_n^2 = x_k$ yields

$$G_n(0,k) = \langle x_1, \dots, x_{n-1} \mid x_{n-k}^4 = x_k, \, x_i^2 = x_{i+k}, \ i \neq n-k, \, i = 1, \dots, n-1\rangle.$$

Analogously, we can eliminate the generator x_{n-1}, and then x_{n-2}, etc. Ultimately,

$$G_n(0,k) = \langle x_1 \mid x_1^{2^n} = x_1\rangle \cong \mathbf{Z}_{2^n-1}.$$

(2) Follows immediately from the definition of $G_n(m,k)$.

(3) Put $y_i = x_i^{-1}$ for $i = 1, 2, \dots, n$. Notice that $G_n(m,k)$ can be given by the following presentation:

$$G_n(m,k) = \langle y_1, \dots, y_n \mid y_{i+m} y_i = y_{i+k}, \ \ i = 1, \dots, n\rangle.$$

Setting $j = i+m$, we rewrite this system of defining relations thus: $y_j y_{j-m} = y_{j-m+k}$, where $j = 1, \dots, n$. Since all indices are taken modulo n, we see that

$$G_n(m,k) = \langle y_1, \dots, y_n \mid y_j y_{j+(n-m)} = y_{j+(n-m+k)}, \ i = 1, \dots, n\rangle = G_n(n-m, n-m+k).$$

(4) The first isomorphism follows from item (3) and the second is established in [29, Prop. 2.2]. The lemma is proved.

In some cases $G_n(m,k)$ factors into a free product. Hereinafter, $(a_1, a_2, \dots, a_n)$ is the greatest common divisor of the integers $a_1, a_2, \dots, a_n$.

LEMMA 1.2. For a group $G_n(m,k)$, put $u = (n,k)$ and $r = (n, k-m)$. Then:

(1) For any positive integer ℓ, the group $G_{\ell n}(\ell m, \ell k)$ is isomorphic to a free product of ℓ copies of $G_n(m,k)$.

(2) If $(u,r) > 1$ then $(n,m,k) > 1$ and $G_n(m,k)$ factors into a non-trivial free product.

Proof. (1) It is easy to note that for each $j = 1, \dots, \ell$, the subgroup $G_j = \langle x_j, x_{j+\ell}, \dots, x_{j+\ell(n-1)}\rangle$ of $G_{\ell n}(\ell m, \ell k)$ is isomorphic to $G_n(m,k)$. Obviously, if $j \neq j'$ then the above sets of generators for G_j and $G_{j'}$ are disjoint. From the presentation of $G_{\ell n}(\ell m, \ell k)$, we see that this group is isomorphic to a free product $G_1 * G_2 * \dots * G_\ell$.

(2) In this case n, k, and m have the common divisor $d = (u,r) > 1$, and the statement then follows from the previous item. The lemma is proved.

Groups $G_n(t,1)$ were investigated in [29]. The following statement shows that, in many cases, groups $G_n(m,k)$ reduce to $G_n(t,1)$, that is, are isomorphic to them.

LEMMA 1.3. If $(n,k) = 1$ or $(n, m-k) = 1$ then $G_n(m,k)$ is isomorphic to $G_n(t,1)$ for some positive integer t.

Proof. Let $(n,k) = 1$. We reorder the generators of $G_n(m,k)$ by setting

$$c_1 = x_1, \ c_2 = x_{1+k}, \ \dots, \ c_i = x_{1+(i-1)k}, \ \dots, \ c_n = x_{1+(n-1)k}.$$

Obviously, the set $\{c_1, \dots, c_n\}$ coincides with the set $\{x_1, \dots, x_n\}$. The first relation $x_1 x_{1+m} = x_{1+k}$ is rewritten into $c_1 c_{1+t} = c_2$, where $c_{1+t} = x_{1+m} = x_{1+tk}$ and t is determined from the condition $tk \equiv$

$m \pmod n$. The next relation $c_2 c_{2+t} = c_3$ complies with $x_{1+k} x_{1+k+m} = x_{1+2k}$ because $c_{2+t} = x_{1+(1+t)k} = x_{1+k+m}$. Analogously, $c_j c_{j+t} = c_{j+1}$ complies with $x_{1+(j-1)k}\ x_{1+(j+t-1)k} = x_{1+jk}$, which is equivalent to $x_{1+(j-1)k}\ x_{1+(j-1)k+m} = x_{1+(j-1)k+k}$.

If j runs through the set $\{1, \dots, n\}$ then $1 + (j-1)k$, taken modulo n, also runs through $\{1, \dots, n\}$. Therefore $G_n(m,k) \cong G_n(t,1)$.

For $(n, m-k) = 1$, by Lemma 1.1(3), the proof reduces to the case treated above.

In virtue of Lemmas 1.1 and 1.2, we may limit ourselves to groups $G_n(m,k)$ with parameters n, m, and k satisfying the following conditions:

$$0 < m < k < n, \ \ (n,m,k) = 1. \tag{1}$$

The group $G_n(m,k)$ with parameters as in (1) is referred to as *irreducible*, since otherwise it is either trivial, or cyclic, or factorable into a free product. Nevertheless, among irreducible groups we can find pairwise isomorphic ones. Namely, the following statement holds.

THEOREM 1.1. Let $G_n(m,k)$ and $G_n(m',k')$ be two irreducible groups. If k' is divided by $r = (n, k-m)$, and if there exist positive integers i and j such that

$$\begin{cases} i + j(k-m) \equiv 1 - m \pmod n, \\ m' + 1 \equiv i + jk' \pmod n, \\ 1 \leqslant i \leqslant r, \ 1 \leqslant j \leqslant n/r, \end{cases} \tag{2}$$

then $G_n(m,k)$ is isomorphic to $G_n(m',k')$.

Before embarking on the proof, we give two illustrative examples.

Example 1.1. Let $n = 4$, $m = m' = 1$, $k = 2$, and $k' = 3$. Then $r = 1$, and we can take $i = 1$ and $j = 3$ satisfying (2). Therefore, by Theorem 1.1, $G_4(1,2) \cong G_4(1,3)$. In fact, it is not hard to verify that $G_4(1,2) \cong G_4(1,3) \cong Z_5$.

Example 1.2. Let $n = 5$, $m = 1$, $k = 2$, $m' = 2$, and $k' = 3$. Then $r = 1$, and we can take $i = 1$ and $j = 4$ satisfying (2). Therefore, by Theorem 1.1, $G_5(1,2) \cong G_5(2,3)$.

Proof of the theorem. Consider an irreducible group such as

$$G_n(m,k) = \langle x_1, \dots, x_n \mid x_i x_{i+m} = x_{i+k}, \ \ i = 1, \dots, n \rangle.$$

Setting $c_i = x_i^{-1}$, $i = 1, \dots, n$, we obtain

$$G_n(m,k) = \langle c_1, \dots, c_n \mid c_{i+m} c_i = c_{i+k}, \ \ i = 1, \dots, n \rangle.$$

Put $j = i + m$. Then $i = j - m$, and the system of defining relations can be represented thus:

$$c_j c_{j-m} = c_{j-m+k}, \ \ j = 1, \dots, n.$$

Put $\ell = n/r$, where $r = (n, k-m)$, and divide all generators $c_1, \dots, c_n$ into r groups each containing ℓ elements. We obtain

$$\begin{gathered} A_1 = \{c_1, \ c_{1+(k-m)}, \ c_{1+2(k-m)}, \ \dots, \ c_{1+(\ell-1)(k-m)}\}, \\ A_2 = \{c_2, \ c_{2+(k-m)}, \ c_{2+2(k-m)}, \ \dots, \ c_{2+(\ell-1)(k-m)}\}, \\ \dots \\ A_r = \{c_r, \ c_{r+(k-m)}, \ c_{r+2(k-m)}, \ \dots, \ c_{r+(\ell-1)(k-m)}\}. \end{gathered}$$

Note that the division of generators into the classes $A_1, \dots, A_r$ necessitates the partition of relations into r classes $R_1, \dots, R_r$ each containing ℓ relations. Namely, we have

$$\begin{aligned}
R_1: \quad & c_1 c_{1-m} = c_{1+(k-m)},\ c_{1+(k-m)} c_{1+k-2m} = c_{1+2(k-m)},\ \dots,\\
& c_{1+(\ell-1)(k-m)} c_{1+(\ell-1)(k-m)-m} = c_{1+\ell(k-m)};\\
R_2: \quad & c_2 c_{2-m} = c_{2+(k-m)},\ c_{2+(k-m)} c_{2+k-2m} = c_{2+2(k-m)},\ \dots,\\
& c_{2+(\ell-1)(k-m)} c_{2+(\ell-1)(k-m)-m} = c_{2+\ell(k-m)};\\
& \dots\\
R_r: \quad & c_r c_{r-m} = c_{r+(k-m)},\ c_{r+(k-m)} c_{r+k-2m} = c_{r+2(k-m)},\ \dots,\\
& c_{r+(\ell-1)(k-m)} c_{r+(\ell-1)(k-m)-m} = c_{r+\ell(k-m)}.
\end{aligned}$$

Moreover, $c_{p+\ell(k-m)} = c_p$ for $p = 1, \dots, r$. Indeed, in view of $r = (n, k-m)$, we can put $n = rn_1$ and $k - m = rr_1$, where $n_1, r_1 \in \mathbf{N}$ and $(n_1, r_1) = 1$. It follows that $\ell(k-m) = (n/r)(k-m) = (n/r)rr_1 = nr_1 \equiv 0 \pmod n$, and hence $c_{p+\ell(k-m)} = c_p$. Therefore if, for each relation in the class R_p, we choose the first generator in the left part and the generator occurring in the right, we are faced up to generators sitting in the class A_p.

A presentation for the second group is as follows:

$$G_n(m', k') = \langle y_1, \dots, y_n \mid y_i y_{i+m'} = y_{i+k'},\ i = 1, \dots, n \rangle.$$

Dividing its generators into r classes with ℓ elements in each, we obtain

$$\begin{aligned}
B_1 &= \{y_1,\ y_{1+k'},\ y_{1+2k'},\ \dots,\ y_{1+(\ell-1)k'}\},\\
B_2 &= \{y_2,\ y_{2+k'},\ y_{2+2k'},\ \dots,\ y_{2+(\ell-1)k'}\},\\
&\dots\\
B_r &= \{y_r,\ y_{r+k'},\ y_{r+2k'},\ \dots,\ y_{r+(\ell-1)k'}\}.
\end{aligned}$$

Similarly to the previous case, dividing the defining relations for $G_n(m', k')$ into r classes with ℓ relations in each, we have

$$\begin{aligned}
Q_1: \quad & y_1 y_{1+m'} = y_{1+k'},\ y_{1+k'} y_{1+k'+m'} = y_{1+2k'},\ \dots,\\
& y_{1+(\ell-1)k'} y_{1+(\ell-1)k'+m'} = y_{1+\ell k'};\\
Q_2: \quad & y_2 y_{2+m'} = y_{2+k'},\ y_{2+k'} y_{2+k'+m'} = y_{2+2k'},\ \dots,\\
& y_{2+(\ell-1)k'} y_{2+(\ell-1)k'+m'} = y_{2+\ell k'};\\
& \dots\\
Q_r: \quad & y_r y_{r+m'} = y_{r+k'},\ y_{r+k'} y_{r+k'+m'} = y_{r+2k'},\ \dots,\\
& y_{r+(\ell-1)k'} y_{r+(\ell-1)k'+m'} = y_{r+\ell k'}.
\end{aligned}$$

Since r divides k', we see that $\ell k' = (n/r)k' \equiv 0 \pmod n$, and hence $y_{p+lk'} = y_p$, $p = 1, \dots, r$. Therefore if, for each defining relation in the class Q_p, we choose the first generator in the left part and the generator occurring in the right, we are faced up to generators sitting in the class B_p.

Now we define the map $\varphi : G_n(m, k) \to G_n(m', k')$ acting on generators by the rule

$$\varphi(c_{p+q(k-m)}) = y_{p+qk'},\ \ 1 \leqslant p \leqslant r,\ 0 \leqslant q \leqslant \ell - 1,$$

and check that each defining relation for $G_n(m, k)$ is mapped into one for $G_n(m', k')$.

Indeed, consider a relation $c_1 c_{1-m} = c_{1+k-m}$. By assumption, we can find positive integers i and j, where $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant (n/r) = l$, so that $i + j(k-m) \equiv 1 - m \pmod n$. Therefore, the relation in question can be written in the form $c_1 c_{i+j(k-m)} = c_{1+k-m}$. The image of this relation under φ is

$y_1 y_{i+jk'} = y_{1+k'}$. By assumption, $m'+1 \equiv i+jk' \,(\mathrm{mod}\, n)$, and so $y_1 y_{1+m'} = y_{1+k'}$, which is a defining relation for the irreducible group $G_n(m', k')$.

It remains to note that all defining relations for $G_n(m,k)$, and for $G_n(m',k')$, arise from the first relation under the cyclic permutation of indices of generators in it. Therefore φ is an homomorphism. It is easy to see that φ is invertible, and so it is an isomorphism. The theorem is proved.

As a particular case of Theorem 1.1 we obtain the following result of Gilbert and Howie in [29, Lemma 2.1].

COROLLARY 1.1. Let n and t be positive integers such that $n > t$ and $(n, t-1) = 1$ and take s so that $0 \leqslant s < n$ and $t \equiv (t-1)s \,(\mathrm{mod}\, n)$. Then groups $G_n(t,1)$ and $G_n(s,1)$ are isomorphic.

Proof. By Lemma 1.1(3), we have $G_n(t,1) \cong G_n(n-t, n-t+1)$ and $G_n(s,1) \cong G_n(n-s, n-s+1)$. Thus it suffices to prove that $G_n(n-t, n-t+1)$ and $G_n(n-s, n-s+1)$ are isomorphic. Setting $m = n-t$, $k = n-t+1$, $m' = n-s$, and $k' = n-s+1$, we see that $r = (n, k-m) = 1$, proving the condition of Theorem 1.1 saying that k' is divided by r. Putting $i = 1$ and $j = t$ in the system of conditions (2), we arrive at

$$\begin{cases} 1+t \equiv 1+t+n \;(\mathrm{mod}\, n), \\ n-s \equiv 1+t(n-s+1)-1 \;(\mathrm{mod}\, n). \end{cases}$$

The first condition is obviously satisfied, and the second is equivalent to $t \equiv s(t-1) \,(\mathrm{mod}\, n)$, which holds by assumption.

LEMMA 1.4. Let m, n, and ℓ be positive integers such that $(n,m) = 1$, $\ell \geqslant 2$, and $\ell m < n$. Then:

(1) groups $G_n(m, \ell m)$ and $G_n(1, \ell)$ are isomorphic;

(2) groups $G_n(\ell m, m)$ and $G_n(\ell, 1)$ are isomorphic.

Proof. (1) Consider groups

$$G = G_n(1,\ell) = \langle x_1, x_2, \ldots, x_n \mid x_i x_{i+1} = x_{i+\ell},\ i = 1, \ldots, n \rangle,$$

$$H = G_n(m, \ell m) = \langle y_1, y_2, \ldots, y_n \mid y_i y_{i+m} = y_{i+\ell m},\ i = 1, \ldots, n \rangle.$$

Define a map $\varphi : G \to H$ acting on generators as follows:

$$\varphi(x_j) = y_{1+m(j-1)},\ \ j = 1, 2, \ldots, n.$$

Obviously this is a map “onto.” We check that φ maps defining relations for G to those of H. Indeed,

$$\varphi\left(x_i\, x_{i+1}\, x_{i+\ell}^{-1}\right) = y_{1+m(i-1)}\, y_{1+mi}\, y_{1+m(i+\ell-1)}^{-1} = y_j\, y_{j+m}\, y_{j+\ell m}^{-1},$$

where $j = 1 + m(i-1)$. Therefore φ is an homomorphism. Since φ is invertible, it is an isomorphism, as required.

(2) Is proved similarly. The lemma is completed.

The properties pointed out above admit of reducing the study of some groups to the well-known Fibonacci groups and Sieradski groups.

COROLLARY 1.2. (1) $G_n(m, 2m)$ either is isomorphic to the Fibonacci group $F(2,n) \cong G_n(1,2)$ or is a free product.

(2) $G_n(2m, m)$ either is isomorphic to the Sieradski group $S(n) \cong G_n(2,1)$ or is a free product.

Proof. (1) If $(n,m) \neq 1$ then the group is a free product by item (2) of Lemma 1.2. If $(n,m) = 1$, by Lemma 1.1, we can suppose that $2m < n$, and then $G_n(m, 2m)$ will be isomorphic to $G_n(1,2)$ in view of Lemma 1.4.

(2) Is proved similarly.

2. ASPHERICITY OF GROUPS $G_n(m,k)$

In studying topological properties of groups, the question arises naturally as to whether their presentations are aspherical (cf. [33, 34]). Roughly speaking, asphericity of a presentation means that defining relations lack in non-trivial identity.

The property of being aspherical for $G_n(t,1)$ was taken up in [29] where, too, asphericity conditions are specified for values of parameters (n,t) other than $\{(8,3),(9,4),(9,7)\}$. Note also that groups $G_n(m,k)$ are particular cases of the following:

$$P(r,n,k,s,q) = \langle x_1,\dots,x_n \mid x_i x_{i+q}\cdots x_{i+q(r-1)} = x_{i+k}x_{i+k+q}\cdots x_{i+k+q(s-1)}\rangle,$$

where $i = 1,\dots,n$ and all indices are taken modulo n (cf. [25]). Obviously, we have $G_n(m,k) \cong P(2,n,k,1,m)$. In [25], asphericity and atoricity conditions are specified for groups $P(r,n,k,s,q)$ under the assumption that $r > 2s > 0$. The results obtained under this section fit in the case where $r = 2$, $s = 1$, and, in this sense, are complementary to those in [25]. The asphericity of generalized Fibonacci groups $F(r,2r+1)$, $F(r,2r)$, and of some others, not treated in [25], was investigated in [35].

In this section, we obtain the condition of being aspherical for presentations $G_n(m,k)$, thus generalizing a result in [29].

At the moment we recall some basic facts from [33] concerning aspherical groups and asphericity of relative presentations, which will be used below.

A *relative presentation* is a triple $\mathbf{P} = \langle H, X|R\rangle$, where H is a group, $X = \{x_1,x_2,\dots\}$ is a set, and R is a set of words of the form $r = y_1h_1y_2h_2\cdots y_nh_n$, where $y_i \in X\cup X^{-1}$, $h_i \in H$, in the alphabet $H\cup X\cup X^{-1}$. We suppose that words are cyclically reduced in the following sense: if $h_i = 1$ then $y_{i+1} \neq y_i^{-1}$, where all indices are taken modulo n. Elements of $X\cup X^{-1}$ are called X-*symbols*.

Words in R are elements of the free product $H * \langle X\rangle$. The presentation $\mathbf{P}$ defines a group which is the quotient of a group $H*\langle X\rangle$ w.r.t. the normal closure of the set R.

For a given subset $S \subseteq R$, S^* denotes the set of all cyclic permutations of the words in $S\cup S^{-1}$ beginning with X-symbols.

We define the following operator on R^*. Represent a word $r \in R^*$ as $r = sh$, where $h \in H$ and s begins with and ends in X-symbols. Put $\overline{r} = s^{-1}h^{-1}$. Note that $\overline{\overline{r}} = r$ and $\overline{r} \in R^*$.

A *picture* $\mathcal{P}$ is a finite collection of pairwise disjoint discs $\{\Delta_1,\dots,\Delta_d\}$ in the interior of a disc D^2, together with a finite collection of pairwise disjoint simple arcs $\{\alpha_1,\dots,\alpha_q\}$ properly embedded in the closure of $D^2 \setminus \bigcup\{\Delta_1,\dots,\Delta_d\}$. A *boundary* $\partial\mathcal{P}$ of the picture $\mathcal{P}$ is the boundary ∂D^2 of D^2. *Corners* of a disc Δ_i, $i = 1,\dots,d$, are closures of the connected components of $\partial\Delta_i \setminus \bigcup\limits_{j=1}^{q}\alpha_j$, where $\partial\Delta_i$ is the boundary of the disc Δ_i. *Regions* of $\mathcal{P}$ are closures of the connected components of $D^2 \setminus \left(\bigcup\limits_{i=1}^{d}\Delta_i \cup \bigcup\limits_{j=1}^{q}\alpha_j\right)$. A simply-connected region is said to be *interior* if it does not meet $\partial\mathcal{P}$. The picture $\mathcal{P}$ is *non-trivial* if $d \geqslant 1$; $\mathcal{P}$ is *connected* if so is $\bigcup\limits_{i=1}^{d}\Delta_i \cup \bigcup\limits_{j=1}^{q}\alpha_j$; and $\mathcal{P}$ is *spherical* if it is non-trivial and $\bigcup\limits_{j=1}^{q}\alpha_j \cap \partial\mathcal{P} = \varnothing$.

For a given relative presentation $\mathbf{P} = \langle H, X|R\rangle$, a picture $\mathcal{P}$ is said to be *labeled* if the following conditions are satisfied:

(i) each arc is equipped with a normal orientation, indicated by a short arrow meeting the arc transversely and labeled by an element of $X\cup X^{-1}$;

(ii) each corner of $\mathcal{P}$ is oriented anticlockwise and labeled by an element of H.

For a corner c of a disc Δ of a labeled picture $\mathcal{P}$, we denote by $\omega(c)$ the word that arises when reading anticlockwise the labels over arcs and corners meeting $\partial\Delta$, beginning with a label over the arc after the corner c. In this case the label t over an arc gives us t if orientation of the arc coincides with the direction in which we read, and gives t^{-1} otherwise.

We say that a labeled picture $\mathcal{P}$ is a *picture over the relative presentation* $\mathbf{P}$ if the following conditions are satisfied:

(1) if c is a corner of $\mathcal{P}$ then $\omega(c) \in R^*$;

(2) if $h_1, h_2, \ldots, h_m$ is a sequence of all labels of corners listed in the anticlockwise direction along the boundary of some interior disc, then $h_1 h_2 \cdots h_m = 1$ in H.

An ordinary presentation $\mathbf{Q} = \langle X|R\rangle$ can be conceived of as a particular case of the relative presentation for which $H = \{1\}$. In this instance every corner of a picture is labeled by 1, and condition (2) is satisfied. Ignoring these 1's, we are faced with an (ordinary) picture over an (ordinary) presentation $\mathbf{Q}$ (cf. [33]).

We come back to the relative presentation $\mathbf{P}$. A *dipole* in a picture over $\mathbf{P}$ is a pair of corners c and c' with an arc α connecting the beginning of one corner with the end of the other in such a way that c and c' belong to the same region of the picture, and $\omega(c') = \overline{\omega(c)}$.

A picture over $\mathbf{P}$ is *reduced* if it does not contain a dipole. A relative presentation $\mathbf{P}$ is *aspherical* if every connected spherical picture over $\mathbf{P}$ contains a dipole, that is, it is not reduced.

LEMMA 2.1. Let $G = A * B$ be a free product. A presentation G is aspherical if and only if at least one of the presentations A and B is aspherical.

Proof. Remark that the picture over G is decomposed into two parts, not connected with each other. One part corresponds to the picture over the presentation A and the other corresponds to the picture over B. Thus the picture over G contains a dipole if and only if at least one of the pictures, over A or over B, contains a dipole.

Thus below we discuss asphericity only for the cases where a group $G_n(m,k)$ is not a free product, or such does not reduce to groups $G_n(t,1)$, whose asphericity was treated in [29]. We say that a group $G_n(m,k)$ is *strongly irreducible* if its defining parameters satisfy the following conditions:

$$\begin{cases} 0 < m < k < n, \\ (n,m,k) = 1, \\ (n,k) > 1, \ \ (n,k-m) > 1. \end{cases} \tag{3}$$

The next theorem supplies us with a condition of being aspherical for strongly irreducible groups.

THEOREM 2.1. Let $G_n(m,k)$ be a strongly irreducible group. Then $G_n(m,k)$ is aspherical if none of the following conditions are satisfied:

(1) there exists an integer $\ell \geqslant 1$ such that n divides $\ell(2k-m)$ and $1/\ell + (n,k)/n + (n,k-m)/n > 1$;

(2) $n = k + m$;

(3) $n = 2(k-m)$ and $(n,k) \leqslant n/2$;

(4) $n = 2k$ and $(n,k-m) < n/2$.

Proof. Consider a split extension $Y_n(m,k)$ of the group $G_n(m,k)$ by a cyclic group of order n generated by an element s. Here, s acts by a cyclic permutation on generators $x_1, x_2, \ldots, x_n$ of $G_n(m,k)$, that is, $s^{-1}x_i s = x_{i+1}$ for $i = 1, \ldots, n$. Therefore $s^{-i}x_1 s^i = x_{i+1}$. In this case the first defining relation of $G_n(m,k)$ can be written in the form $x_{1+m}^{-1}x_1^{-1}x_{1+k} = 1$, which yields $s^{-m}xs^m xs^{-k}x^{-1}s^k = 1$, where $x = x_1^{-1}$. Thus the group $Y_n(m,k)$ (recall that $0 < m < k < n$) has the following presentation:

$$Y_n(m,k) = \langle x, s \mid s^n = 1, \, xs^m xs^{-k}x^{-1}s^{k-m} = 1\rangle.$$

Note that this presentation can be conceived of as a relative presentation if $H = \langle s \mid s^n = 1\rangle$.

Similarly to [29, Lemma 3.1], we state the following:

LEMMA 2.2. If the presentation $Y_n(m,k)$ is aspherical, then the presentation $G_n(m,k)$ too is aspherical.

Proof. Let $\mathcal{P}$ be a picture over $G_n(m,k)$. Then $\mathcal{P}$ contains discs Δ_i corresponding to relations $x_{i+m}^{-1}x_i^{-1}x_{i+k} = 1$ (see Fig. 1).

We replace each disc Δ_i by a picture $\mathcal{Q}_i$ over $Y_n(m,k)$, treated as an ordinary (not relative) presentation. The picture $\mathcal{Q}_i$ is depicted in Fig. 2.

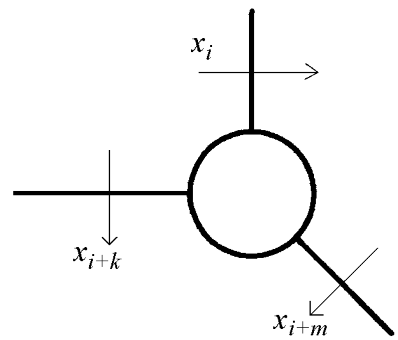

Fig. 1

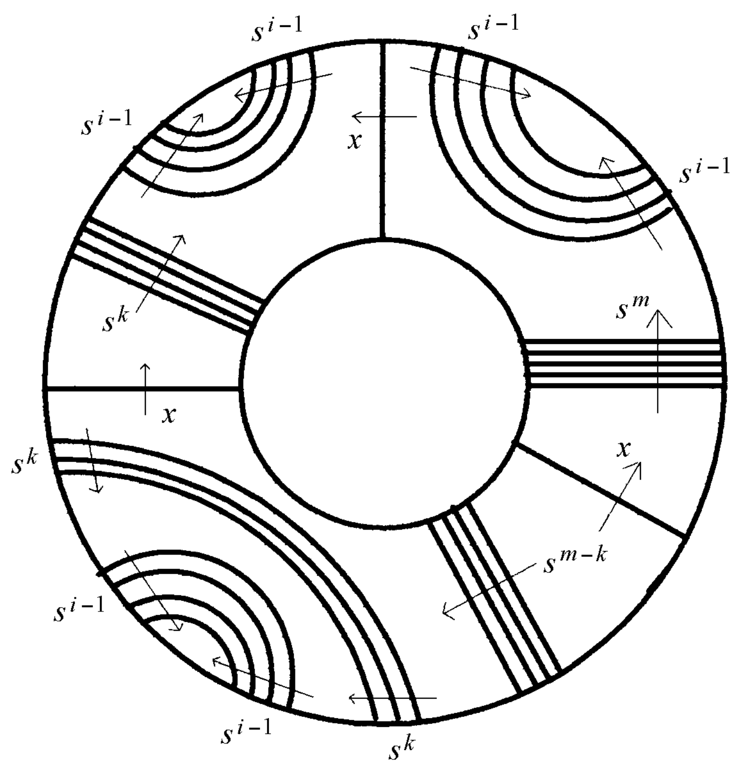

Fig. 2

Here, the arcs labeled x_i, x_{i+m}, and x_{i+k} are replaced by families of arcs using the following relations:

$$x_i = s^{-(i-1)}x^{-1}s^{i-1},$$

$$x_{i+m} = s^{-(i+m-1)}x^{-1}s^{i+m-1},$$

$$x_{i+k} = s^{-(i+k-1)}x^{-1}s^{i+k-1},$$

where $x = x_1^{-1}$. Along the boundary $\partial\mathcal{Q}_i$, we obtain the relation

$$(s^{-(i+m-1)}xs^{i+m-1})(s^{-(i-1)}xs^{i-1})(s^{-(i+k-1)}x^{-1}s^{i+k-1}) = 1,$$

which is equivalent to $x_{i+m}^{-1}x_i^{-1}x_{i+k} = 1$. Along the boundary of the interior disc, we arrive at

$$xs^m xs^{-k}x^{-1}s^{k-m} = 1,$$

a relation from the presentation $Y_n(m,k)$. The arcs in $\mathcal{Q}_i$ having both ends on $\partial\mathcal{Q}_i$ give us free circles, which can be removed from the picture. We replace all other arcs with labels s by labels in corners of the disc (see Fig. 3).

Repeating the same trick with each disc in the initial picture $\mathcal{P}$ over the presentation $G_n(m,k)$ yields a picture $\mathcal{Q}$ over the relative presentation $Y_n(m,k)$. Under the assumption that $Y_n(m,k)$ is aspherical, the picture $\mathcal{Q}$ contains a dipole, that is, a pair of contrarily oriented discs connected by an arc and defining pairwise inverse words (see Fig. 4).

It is not hard to note that by the construction of $\mathcal{Q}$, each such dipole in $\mathcal{Q}$ arises from a pair of equal but contrarily oriented discs in $\mathcal{P}$, connected by an arc with label x_i for some i. If $\mathcal{Q}$ has a pair of cancellable discs, then so does $\mathcal{P}$, and hence the initial picture $\mathcal{P}$ contains a dipole. Thus any non-empty picture over the presentation $G_n(m,k)$ is equivalent to a picture having two less discs, and so the presentation is aspherical. The lemma is proved.

To examine $Y_n(m,k)$ for asphericity, we use the following criterion due to Edjvet.

THEOREM 2.2 [31]. Let $G = \langle H, x \mid xaxbx^{-1}c\rangle$ for some group H. Suppose that the orders of elements b and c in H are equal to p and q, respectively, where $1 < q \leqslant p < \infty$ and $(p,q) \neq (8,4), (9,3)$. Then G is aspherical if and only if none of the following hold in H:

(a) $(a^{-1}bac^{-1})^{\ell} = 1$ for some integer ℓ such that $1/\ell + 1/p + 1/q > 1$;

(b) $a^{-1}bac = 1$;

(c) $a^{-1}b^2ac = 1$ or $a^{-1}bac^2 = 1$;

(d) $q = 2$ and $a^{-1}b^{-1}aca^{-1}bac = 1$;

(e) $q = 2$, $p = 3$, and $(a^{-1}bac)^2(a^{-1}b^{-1}ac)^2 = 1$;

(f) $p = q = 3$ and $a^{-1}baca^{-1}b^{-1}ac^{-1} = 1$;

(g) $q = 3$, $p = 6$, and $a^{-1}b^2ac^{-1} = 1$;

(h) $q = p = 7$ and either $a^{-1}b^2ac^{-1} = 1$ or $a^{-1}b^{-1}ac^2 = 1$;

(i) $q = p = 9$ and either $a^{-1}b^2ac^{-1} = 1$ or $a^{-1}b^{-1}ac^2 = 1$.

We continue to prove Theorem 2.1.

Note that the defining relation $xs^m xs^{-k}x^{-1}s^{k-m} = 1$ in the presentation $Y_n(m,k)$ has the same form as in the Edjvet theorem, where $H = \langle s \mid s^n = 1\rangle$, $a = s^m$, $b = s^{-k}$, and $c = s^{k-m}$. In this case the order of $b = s^{-k}$ is equal to $n/(n,k)$ and the one of $c = s^{k-m}$ is equal to $n/(n,k-m)$. Put $p = n/(n,k)$ and $q = n/(n,k-m)$, with $q \leqslant p$. By the assumption of Theorem 2.1, $p > 1$ and $q > 1$.

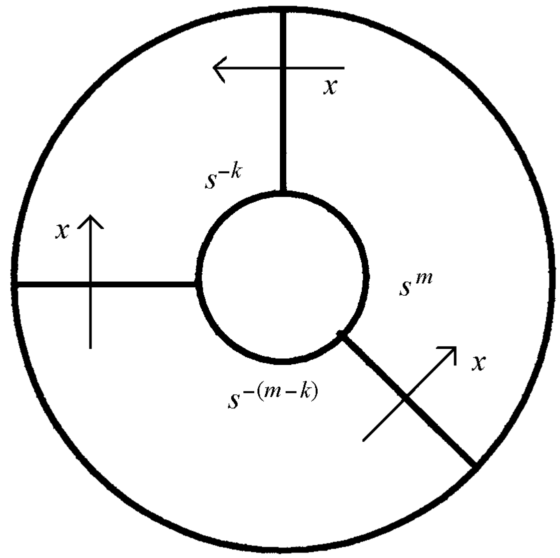

Fig. 3

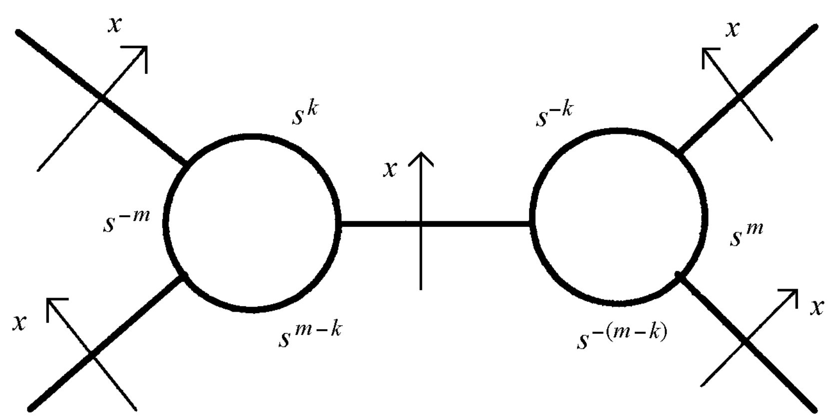

Fig. 4

In case (a) we have $a^{-1}bac^{-1} = s^{m-2k}$. The order of s^{2k-m} is equal to $n/(n, 2k-m)$. If $s^{\ell(2k-m)} = 1$ for some positive integer ℓ, then $n \mid \ell(2k-m)$. Therefore condition (1) of Theorem 2.1 holds.

In case (b) we have $a^{-1}bac = s^{-m}$. Since $0 < m < k < n$, the equality $s^{-m} = 1$ fails.

In case (c) we have two possibilities. For $a^{-1}b^2ac = s^{-k-m} = 1$ to be satisfied, it is necessary that $n \mid (k+m)$. Since $0 < m < k < n$, this is possible only if $n = k+m$. For $a^{-1}bac^2 = s^{k-2m} = 1$ to be satisfied, it is necessary that $n \mid (k-2m)$. Since $0 < k < m < n$, this is possible only if $k = 2m$, that is, the group is $G_n(m, 2m)$. If $(n, m) \neq 1$, then the condition $(n, m, k) = (n, m, 2m) = 1$ is not satisfied. And if $(n, m) = 1$ then $(n, k-m) = (n, 2m-m) = (n, m) > 1$ is not. In both cases the group in question does not satisfy the assumption of Theorem 2.1. Thus condition (2) of Theorem 2.1 is met.

We pass to case (d). For $a^{-1}b^{-1}aca^{-1}bac = s^{2(k-m)} = 1$ to be satisfied, it is necessary that $n \mid 2(k-m)$. Since $0 < m < k < n$, this is possible only if $n = 2(k-m)$. Obviously, the condition that $q = 2$ holds also, in which case $q \leqslant p$ becomes $(n, k) \leqslant n/2$. Thus condition (3) of Theorem 2.1 is met.

Consider case (e). Since $p = 3$ and $q = 2$, we obtain $n = 3(n, k) = 2(n, k-m)$. Hence, for some integer

$r \geqslant 1$, we have $n = 6r$, $(n,k) = 2r$, and $(n, k-m) = 3r$. In virtue of the condition that $(n,m,k) = 1$, this is possible only if $r = 1$. Since $0 < m < k < n$, we conclude that $n = 6$, $k = 4$, and $m = 1$, which is a particular case of condition (3) in Theorem 2.1.

The remaining cases (f)-(i) of Theorem 2.2 cannot be realized in our situation. Indeed, if $p = uq$ for some integer $u \geqslant 1$, then $u(n,k) = (n, k-m)$. Put $v = (n,k)$; then $v \mid n$ and $v \mid k$, where $v > 1$ by the assumption of Theorem 2.1. Since $v \mid (n, k-m)$, $v \mid m$, which is a contradiction with $(n,m,k) = 1$.

The above cases, note, fit with the assumption that $q \leqslant p$, that is, $n/(n,k) \leqslant n/(n,k-m)$.

Now assume, to the contrary, that $n/(n,k) > n/(n,k-m)$. The relation $xs^m xs^{-k}x^{-1}s^{k-m} = 1$ in $Y_n(m,k)$ is equivalent to a relation $zs^{-m}zs^{m-k}z^{-1}s^k = 1$, where $z = x^{-1}$, which is shaped as in the Edjvet theorem, with $a = s^{-m}$, $b = s^{m-k}$, and $c = s^k$. In this case the order of b is equal to $n/(n,k-m)$, and then c has order $n/(n,k)$. Letting $p = n/(n,k)$ and $q = n/(n,k-m)$, we obtain $q < p$, and thereby the conditions of the Edjvet theorem are satisfied.

As above, for cases (a), (b), and (c), conditions (1) and (2) of Theorem 2.1 are met.

We handle case (d). For $a^{-1}b^{-1}aca^{-1}bac = s^{2k} = 1$ to be satisfied, it is necessary that $n \mid 2k$. Since $1 < k < n$, this is possible only if $n = 2k$. Obviously, the condition $q = 2$, too, is satisfied. And $q \leqslant p$ assumes the form $(n, k-m) < n/2$, yielding condition (4) of Theorem 2.1.

Consider case (e). Since $p = 3$ and $q = 2$, $n = 3(n,k-m) = 2(n,k)$. Therefore, for some integer $r \geqslant 1$, we have $n = 6r$, $(n,k) = 3r$, and $(n,k-m) = 2r$. In virtue of the condition that $(n,m,k) = 1$, this is possible only if $r = 1$. Since $0 < m < k < n$, we obtain $n = 6$, $k = 3$, and $m = 1$, which is a particular case of condition (4) in Theorem 2.1.

The remaining cases (f)-(i) of Theorem 2.2 cannot be realized in the present situation by the same arguments as in the previous.

Summing up the cases treated above and applying Lemma 2.2, we arrive at the statement of Theorem 2.1.

3. GROUPS WITH AN ODD NUMBER OF GENERATORS

In [1], the following question was posed: For which values of n, m, and k are $G_n(m,k)$ fundamental groups of 3-manifolds? As noted in Sec. 1, among $G_n(m,k)$ are Fibonacci and Sieradski groups, whose corresponding 3-manifolds are described in [21] and [27], respectively.

The next result, which is a generalization of the result in [24] for Fibonacci groups, shows that in many cases the groups under consideration cannot be fundamental groups of hyperbolic 3-manifolds. (For geometric structures on 3-manifolds, see, e.g., [37].)

THEOREM 3.1. Let n be odd, $k - m$ even, and $(m - 2k, n) = 1$. Then $G_n(m,k)$ cannot be a group of an hyperbolic 3-orbifold (in particular, 3-manifold) of finite volume.

Proof. The idea behind our proof is the same as in [24, 36]. Assume, on the contrary, that $G_n(m,k)$ is a group of an hyperbolic 3-orbifold of finite volume. Therefore $G_n(m,k)$ can be realized as a group of isometries of an hyperbolic 3-space $\mathbf{H}^3$, and is a crystallographic group of motions (cf. [39]). Consider a split extension $W_n(m,k)$ of $G_n(m,k)$ by a cyclic automorphism s such that $s : x_i \to x_{i+1}$ for $i = 1, 2, \ldots, n$. Denote its order by n_1. Obviously, $n_1 \mid n$. Recall that any isomorphism of crystallographic groups of motions of $\mathbf{H}^3$ is induced by a conjugation in $\mathrm{Isom}(\mathbf{H}^3)$ (cf. [39, Chap. 7]). Denote this conjugation also by s. Suppose

$$W_n(m,k) = \langle G_n(m,k), s\rangle = \langle x_1, \ldots, x_n, s \mid s^{n_1} = 1,\ x_1 s^{-m} x_1 s^m = s^{-k} x_1 s^k,\ x_{i+1} = s^{-1} x_i s,\ i = 1, \ldots, n\rangle.$$

Using Tits transformations, we obtain

$$W_n(m,k) = \langle x, s \mid s^{n_1} = 1,\, xs^{-m}xs^m = s^{-k}xs^k \rangle,$$

where $x = x_1$. Consider a verbal subgroup W^2 of $W = W_n(m,k)$ generated by squares of elements, that is, $W^2 = \langle w^2 \mid w \in W \rangle$. From the second defining relation for W, we have

$$s^k xs^{-m} = xs^{k-m}x^{-1}.$$

Since $k-m$ is even, $s^{k-m} \in W^2$. Therefore $xs^{k-m}x^{-1} \in W^2$, and hence $s^k xs^{-m} \in W^2$, that is, $x \in W^2$. Since n_1 is odd, we have $s \in W^2$. Thus the group W, which is a group of an hyperbolic 3-orbifold by assumption, contains only orientation preserving isometries of $\mathbf{H}^3$. Recall that a full group of orientation preserving isometries of $\mathbf{H}^3$ is isomorphic to $PSL_2(\mathbf{C})$ (cf. [37]). Consequently, W is a subgroup of $PSL_2(\mathbf{C})$.

Since W is a subgroup of $PSL_2(\mathbf{C})$, there exists a subgroup $\widetilde{W}$ of $SL_2(\mathbf{C})$, which is the preimage of W with respect to the canonical projection. Suppose that, for a generator x of W, the corresponding element in $\widetilde{W}$ is presented by the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbf{C})$. Since W uniformizes an hyperbolic 3-orbifold of finite volume, it is not elementary (in the sense of Kleinian group theory; cf. [38]), and so $\beta \neq 0$ and $\gamma \neq 0$.

For a generator $s \in W$, the corresponding matrix in $\widetilde{W}$ is $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \in SL_2(\mathbf{C})$, where ζ is a primitive root of 1 of degree $2n_1$. Then the relation

$$xs^{-m}xs^m = s^{-k}xs^k$$

induces the following matrix relation:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \zeta^{-m} & 0 \\ 0 & \zeta^m \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \zeta^m & 0 \\ 0 & \zeta^{-m} \end{pmatrix} =$$

$$\begin{pmatrix} \zeta^{-k} & 0 \\ 0 & \zeta^k \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \zeta^k & 0 \\ 0 & \zeta^{-k} \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix},$$

where $\varepsilon = \pm 1$. Multiplying the matrices on the left- and right-hand sides yields

$$\begin{pmatrix} \alpha^2 + \beta\gamma\zeta^{2m} & \beta(\alpha\zeta^{-2m} + \delta) \\ \gamma(\alpha + \delta\zeta^{2m}) & \beta\gamma\zeta^{-2m} + \delta^2 \end{pmatrix} = \begin{pmatrix} \varepsilon\alpha & \varepsilon\beta\zeta^{-2k} \\ \varepsilon\gamma\zeta^{2k} & \varepsilon\delta \end{pmatrix}.$$

Hence, using $\beta \neq 0$ and $\gamma \neq 0$, we have

$$\begin{cases} \alpha + \delta\zeta^{2m} = \varepsilon\zeta^{2k}, \\ \alpha\zeta^{-2m} + \delta = \varepsilon\zeta^{-2k}. \end{cases}$$

Multiplying the second equation by ζ^{2m} yields the equivalent system

$$\begin{cases} \alpha + \delta\zeta^{2m} = \varepsilon\zeta^{2k}, \\ \alpha + \delta\zeta^{2m} = \varepsilon\zeta^{2(m-k)}, \end{cases}$$

which gives us $\zeta^{2(m-2k)} = 1$. At the same time $\zeta^{2n_1} = 1$. Since we have assumed that $(n, m-2k) = 1$, we arrive at a contradiction. Therefore $G_n(m,k)$ cannot be a group of an hyperbolic 3-orbifold of finite volume.

Recall that groups $G_n(1,2)$ with an even number of generators $n \geqslant 8$ are fundamental groups of hyperbolic 3-manifolds, and so do not contain elements of finite order. Below is a result demonstrating that for n odd the situation is different.

Proposition 3.1. A group $G_n(m, 2m)$ with an odd number of generators has torsion.

Proof. By Corollary 1.2, $G_n(m, 2m)$ either is isomorphic to a Fibonacci group $G_n(1,2)$ or factors into a free product of Fibonacci groups with a lesser number of generators, which is likewise odd. Thus we need only examine groups $G_n(1,2)$.

Consider an element $w = \prod\limits_{i=0}^{2n-1} x_{1+i}$ of $G_n(1,2)$, where all indices are taken modulo n. We claim that this element is of order two. Indeed, if we represent w as $w = uv$, where

$$u = \prod_{i=0}^{n-1} x_{1+im} = \prod_{i=0}^{2n_1} x_{1+im},$$

$$v = \prod_{i=n}^{2n-1} x_{1+i} = \prod_{i=0}^{n-1} x_{1+(n+i)} = \prod_{i=0}^{2n_1} x_{1+n+i} = \prod_{i=0}^{2n_1} x_{1+i} = u,$$

since all indices are taken modulo n, we obtain $w = u^2$.

In virtue of the relation $x_i x_{i+1} = x_{i+2}$, which can be written in the form $x_{i+1} = x_i^{-1} x_{i+2}$, we have

$$w = \prod_{i=0}^{2n-1} x_{1+i} = \prod_{i=0}^{4n_1+1} x_{1+i} = \prod_{i=0}^{2n_1} (x_{1+2i} x_{2+2i}) = \prod_{i=0}^{2n_1} x_{3+2i} = \prod_{i=0}^{2n_1} \left(x_{2+2i)}^{-1} x_{4+2i} \right) =$$

$$x_2^{-1} x_{4+4n_1} = x_2^{-1} x_{2+2(2n_1+1)} = x_2^{-1} x_{2+2n} = x_2^{-1} x_2 = 1.$$

Therefore $u^2 = 1$. In essence, the idea of treating elements of such a form goes back to Johnson (cf. [6]).

Now we show that u is trivial if and only if generators x_j and x_{j-1} commute for all $j = 1, \ldots, n$. Indeed, the condition that $u = 1$ means that for each $j = 1, \ldots, n$ the following relation holds:

$$x_j \, x_{j+1} \, x_{j+2} \, \cdots \, x_{j+(2n_1-2)} \, x_{j+(2n_1-1)} \, x_{j+2n_1} = 1.$$

Using $x_i x_{i+1} = x_{i+2}$ with different indices yields

$$x_{j+2}^2 \, x_{j+3} \, \cdots \, x_{j+(2n_1-2)} \, x_{j+(2n_1-1)} \, x_{j+2n_1} = 1,$$

$$x_{j+2} \, x_{j+4}^2 \, \cdots \, x_{j+(2n_1-2)} \, x_{j+(2n_1-1)} \, x_{j+2n_1} = 1,$$

$$x_{j+2} \, x_{j+4} \, \cdots \, x_{j+(2n_1-2)} \, x_{j+2n_1}^2 = 1.$$

Hence

$$\left(x_{j+1}^{-1} x_{j+3}\right) \left(x_{j+3}^{-1} x_{j+5}\right) \cdots \left(x_{j+(2n_1-1)}^{-1} x_{j+(2n_1+1)}\right) x_{j+2n_1} = 1,$$

Consequently,

$$x_{j+1}^{-1} \, x_{j+2n_1+1} \, x_{j+2n_1} = 1.$$

All indices are taken modulo $n = 2n_1 + 1$, and so $x_j \, x_{j-1} = x_{j+1}$. Keeping in mind that $x_{j-1} \, x_j = x_{j+1}$, we see that x_j and x_{j-1} commute for all $j = 1, \ldots, n$.

We claim that every two generators, say, x_i and x_{i+p}, commute. This has been stated for $p = 1$. Suppose that x_i and x_{i+q} commute for $q < p$. Then

$$x_i x_{i+p} = x_i x_{i+p-2} x_{i+p-1} = x_{i+p-2} x_i x_{i+p-1} = x_{i+p-2} x_{i+p-1} x_i = x_{i+p} x_i.$$

Thus u is trivial if and only if $G_n(1,2)$ is Abelian. Recall that if n is odd, then the Fibonacci group $F(2,n) = G_n(1,2)$ is finite for $n = 1,3,5,7$ and so has torsion, and is infinite for $n \geqslant 9$ (see, e.g., [35]). The group cannot be Abelian since its Abelianizer is finite (cf. [6]). Thus u is a non-trivial element of order two. The proposition is proved.

Since the property of a group being commutative played an important part in the above argument, in view of a result due to Reidemeister, recall, the full list of Abelian groups which may appear as fundamental groups in 3-manifolds is exhausted by the following: $\mathbf{Z}_n$, $\mathbf{Z}$, $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$, $\mathbf{Z} \oplus \mathbf{Z}_2$ (see, e.g., [3, Chap. 5]).

4. FINITENESS OF $G_n(m,k)$

The study of Fibonacci groups $F(2,n) \cong G_n(1,2)$ was launched by the question of Conway in [5] asking whether the group $F(2,5)$ is finite. A survey of the results on the Fibonacci groups $F(2,n)$ is presented in [9]; in particular, these groups are finite if and only if $n = 1,2,3,4,5,7$. Moreover, $F(2,1) \cong F(2,2) \cong \langle \mathbf{1} \rangle$, $F(2,3) \cong Q_8$ (a quaternion group of order 8), $F(2,4) \cong \mathbf{Z}_5$, $F(2,5) \cong \mathbf{Z}_{11}$, and $F(2,7) \cong \mathbf{Z}_{29}$. A proof that $F(2,n)$ are infinite for $n \geqslant 6$ even can be found in [8, 21], and for $n \geqslant 9$ odd — in [35].

In treating the Cavicchioli–Hegenbarth–Repovš groups, it is natural to ask the following:

Question 1. For which values of defining parameters n, m, k, subject to the natural restrictions that $0 < m < k < n$ and $(n,m,k) = 1$, are groups $G_n(m,k)$ finite?

We used the computer program *GAP* in [32] to find orders of $G_n(m,k)$ for small values of n.

For $n = 3$, the only possibility is $G_3(1,2) \cong F(2,3) \cong Q_8$. For $n = 4$, we have $G_4(1,3) \cong G_4(1,2) \cong F(2,4) \cong \mathbf{Z}_5$ (cf. Example 1.1), $|G_4(2,3)| = 24$, $|G_4^{\mathrm{ab}}(2,3)| = 3$, and $G_4(2,3)$ is solvable of index 3. Other results of computations are presented in Tables 1-4 below.

TABLE 1

$n=5$	$m \setminus k$	2	3	4
	1	11	120	11
	2	—	11	11
	3	—	—	120

TABLE 2

$n=6$	$m \setminus k$	2	3	4	5
	1	∞	7	7	∞
	2	—	9	∞	9
	3	—	—	56	56
	4	—	—	—	∞

TABLE 3

$n=7$	$m \setminus k$	2	3	4	5	6
	1	29	?	?	?	29
	2	—	?	29	29	?
	3	—	—	29	?	29
	4	—	—	—	?	?
	5	—	—	—	—	?

TABLE 4

$n=8$	$m \setminus k$	2	3	4	5	6	7
	1	∞	295245	17	17	295245	?
	2	—	?	∞	?	∞	?
	3	—	—	17	?	∞	17
	4	—	—	—	8	∞	?
	5	—	—	—	—	295245	295245
	6	—	—	—	—	—	?

In these tables, the intersection of an mth row and a kth column gives orders of $G_n(m,k)$. The sign "—" means that parameters m and k do not satisfy the condition that $m < k$; the sign "?" means that order is unknown.

$G_6(1,2)$ is the Fibonacci group $F(2,6)$, which is infinite by [21]. $G_6(2,4)$ is isomorphic to $G_3(1,2) * G_3(1,2)$, and is also infinite. Next, $G_8(1,2) \cong F(2,8) \cong G_8(3,6)$, and so $G_8(1,2)$ is infinite, $G_8(2,4) \cong G_4(1,2) * G_4(1,2)$, $G_8(2,6) \cong G_4(1,3) * G_4(1,3)$, and $G_8(4,6) \cong G_4(2,3) * G_4(2,3)$. $G = G_6(3,4)$ is metabelian of order 56, that is, its second derived group G'' is trivial, $|G'| = 8$, and $G/G' \cong \mathbf{Z}_7$. $G_5(1,3)$ coincides with its derived group. $G_6(4,5)$ is isomorphic to $G_6(2,1)$, has an infinite Abelianizer by [30], and so is infinite.

We pass to the problem of group isomorphisms.

Question 2. Is it possible to compute the function $f(n)$ which, given an integer $n \geqslant 3$, yields the number of pairwise non-isomorphic groups $G_n(m,k)$, where $0 < m < k < n$.

From the above cases, it follows that $f(3) = 1$, $f(4) = 2$, and $f(5) = 2$.

For $n = 6$, from Theorem 1.1 and the table of Abelianizers (cf. Table 5), we see that there are 6 classes of pairwise isomorphic groups: $G_6(1,2) \cong G_6(1,5)$, $G_6(1,3) \cong G_6(1,4)$, $G_6(2,3) \cong G_6(2,5)$, $G_6(2,4)$, $G_6(3,4) \cong G_6(3,5)$, and $G_6(4,5)$. Hence $f(6) = 6$.

For $n = 7$, from Theorem 1.1 and the table of Abelianizers (cf. Table 6), we see that there are 3 classes of pairwise isomorphic groups:

(1) $\{G_7(1,2), G_7(1,6), G_7(2,4), G_7(2,5), G_7(3,4), G_7(3,6)\}$;

(2) $\{G_7(1,3), G_7(1,5), G_7(2,3), G_7(2,6), G_7(4,5), G_7(4,6)\}$;

(3) $\{G_7(1,4), G_7(3,5), G_7(5,6)\}$.

Therefore $f(7) = 3$.

5. SOME OPEN QUESTIONS

In conclusion we formulate some open questions bearing on Cavicchioli–Hegenbarth–Repovš groups.

Question 3. What is the rank of a group $G_n(m,k)$, $0 \leqslant m < k < n$, equal to?

Question 4. For which values of n, m, k are groups $G_n(m,k)$ linear?

Question 5. Can groups $G_n(m,k)$ and $G_{n'}(m',k')$ be isomorphic for $n \neq n'$?

It is known from [6] that an Abelianizer of the Fibonacci group $F(2,n)$ is finite and its order is equal to $f_n - 1 - (-1)^n$, where f_n is a Fibonacci number.

Question 6. Does there exist a similar formula which gives the order of an Abelianizer of $G_n(m,k)$ if the Abelianizer is finite? Can such a formula be given in terms of numbers generalizing the Fibonacci numbers?

TABLE 5

$n=6$	$m \setminus k$	2	3	4	5
	1	$\mathbf{Z}_4^2$	$\mathbf{Z}_7$	$\mathbf{Z}_7$	$\mathbf{Z}_4^2$
	2	—	$\mathbf{Z}_9$	$\mathbf{Z}_2^4$	$\mathbf{Z}_9$
	3	—	—	$\mathbf{Z}_7$	$\mathbf{Z}_7$
	4	—	—	—	$\mathbf{Z}^2$

TABLE 6

$n=7$	$m \setminus k$	2	3	4	5	6
	1	$\mathbf{Z}_{29}$	$\mathbf{Z}_2^3$	1	$\mathbf{Z}_2^3$	$\mathbf{Z}_{29}$
	2	—	$\mathbf{Z}_2^3$	$\mathbf{Z}_{29}$	$\mathbf{Z}_{29}$	$\mathbf{Z}_2^3$
	3	—	—	$\mathbf{Z}_{29}$	1	$\mathbf{Z}_{29}$
	4	—	—	—	$\mathbf{Z}_2^3$	$\mathbf{Z}_2^3$
	5	—	—	—	—	1

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