ON THE YOSHINAGA POLYNOMIAL OF SPATIAL GRAPHS

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ABSTRACT. An explicit relation between two well-known polynomial invariants of spatial graphs is established. If g is a diagram of a spatial graph with p vertices and q edges, then $Y(g;\, t) = (-t - t^{-1})^{p-q}\, R(g';\, t)$, where $Y(g)$ is the Yoshinaga polynomial of g and $R(g')$ is the Yamada polynomial of its mirror image g'.

1. INTRODUCTION

In [1] Yoshinaga introduced an invariant of spatial trivalent graphs in $\mathbb{R}^3$ associated with the spin 1 representation of $U_q(sl(2, \mathbb{C}))$, the quantized universal enveloping algebra of $sl(2, \mathbb{C})$ in the sense of Drinfel'd and Jumbo. For a fixed Morse function f on a graph diagram g he associated a function $\langle g, f \rangle \in \mathbb{Z}[t^{1/2},\, t^{-1/2}]$ which is an ambient isotopy invariant up to $\pm t^i$, $i \in \mathbb{Z}$. Moreover, it was shown that $\langle g, f \rangle$ does not depend on the choice of a Morse function and coincides with the Laurent polynomial with integer coefficients. We will refer it to as the Yoshinaga polynomial and denote as $Y(g)$ below. The polynomial $Y(g)$ can be regarded as a modification of the Yamada polynomial $R(g)$ defined in [2]. Despite of small difference in their skein relations (see Section 2), it is not known an exact relation between them. We will demonstrate such a relation (see Theorem 1) and discuss some properties of the polynomial $Y(g)$ that follows from this relation (see Section 3). In Section 4 we will give an example of the computation of the Yoshinaga polynomial for an infinite family of spatial theta-graphs.

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2. Polynomials of spatial graphs

All graphs considered are finite, undirected ones with or without self-loops and multiple edges. A graph $G(V, E)$ has the vertex set $V = V(G)$ and the edge set $E = E(G)$, and $p = |V|$, $q = |E|$. We denote the number of components of G by $\omega(G)$. The Betti number of G is defined by $\beta(G) = q(G) - p(G) + \omega(G)$. The result of removing $F \subseteq E(G)$ from G will be denoted by $G - F$. An edge $e \in E(G)$ is said to be a *cut edge* if $\omega(G - e) > \omega(G)$.

Following [2], we define a 2-variable Laurent polynomial for a graph G by

$$h(G; x, y) \;=\; \sum_{F \subseteq E} (-x)^{-|F|}\, x^{\omega(G-F)}\, y^{\beta(G-F)}, \tag{1}$$

where F ranges over the family of subsets of $E = E(G)$, $|F|$ is the number of elements of F. In particular, $h(\emptyset) = 1$. It is easy to see that $h(G)$ is a specialization of the Negami polynomial of G defined in [3], and so, is an invariant of G. Note that if G_1 and G_2 are homeomorphic graphs (as 1-dimensional complexes) then $h(G_1; x, y) = h(G_2; x, y)$. If G has a cut edge then $h(G; x, y) = 0$. Using $h(G; x, y)$, we define two auxiliary 1-variable polynomials for an abstract graph G:

$$f_Y(G;\, t) \;=\; h\left(G; t + t^{-1}, \frac{t + 2 + t^{-1}}{t + t^{-1}}\right) \tag{2}$$

and

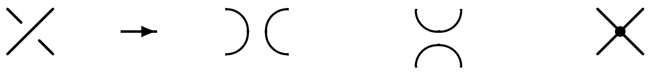
$$f_R(G; t) \;=\; h(G; -1, -t - 2 - t^{-1}). \tag{3}$$

Now consider *spatial* graphs, i.e. graphs which are piecewise linear embedded in $\mathbb{R}^3$. We suppose that spatial graphs are *flat vertex graphs*, i.e. for each vertex $v \in V(G)$ there exists a neighbourhood U_v such that $G \cap U_v$ belongs to a plane. Throughout the paper we assume that all spatial graphs are flat vertex graphs. Two graphs G_1 and G_2 are said to be *equivalent* if there exists a *flat deformation* from G_1 to G_2, i.e. an ambient isotopy $h_t : \mathbb{R}^3 \to \mathbb{R}^3$, $t \in [0, 1]$ such that $h_0 = id_{\mathbb{R}^3}$, $h_1(G_1) = G_2$ and $h_t(G_1)$ is a flat vertex graph for each $t \in [0, 1]$.

Consider a canonical projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ with $\pi(x, y, z) = (x, y)$. An image $\pi(G)$ is said to be a *regular* projection if it has only finite number of multiple points and each multiple point of $\pi|_G$ is a double point of two transversal edges of G. A double point of $\pi(G)$ is called a *crossing point*. The *crossing number* $c(G)$ of a spatial graph G is defined to be the minimal number of crossings among all regular projections of G. A regular projection $\pi(G)$ with over/under information at each crossing point is called a *diagram* of G and we denote it by $g = g(G)$.

For different kinds of isotopy equivalences between spatial graphs and local moves of diagrams, corresponding these isotopies we refer to [1, 2, 4, 5].

Let g be a diagram. For a crossing $z \in g$ we define spins s_+ ("positive spin"), s_- ("negative spin") and s_0 ("zero spin") as shown below



z s_+ s_- s_0

A *state* S of g is a plane graph obtained from g by replacing each crossing by one of the above spins. If S contains a hoop then we add a vertex and make it a loop. Thus a diagram with c crossings gives us 3^c states. Denote the set of all states obtained from g by $\mathcal{U}(g)$. Suppose that a state S has been obtained from a diagram g after applying k_+ positive spins, k_- negative spins and k_0 zero spins. Let us define

$$c_Y(g|S) = (t^{-1})^{k_+}\, t^{k_-} (-t - t^{-1})^{k_0} \tag{4}$$

and

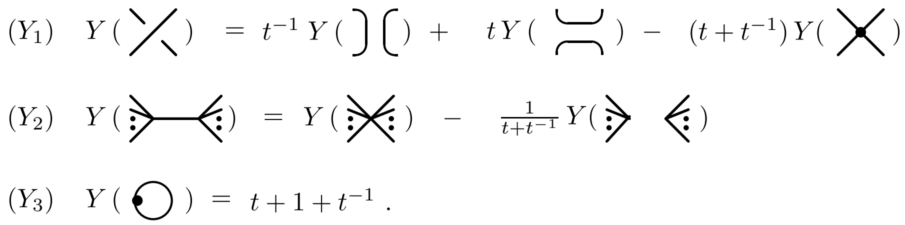
$$c_R(g|S) = t^{k_+ - k_-}. \tag{5}$$

Now we are ready to define polynomial invariants for spatial graphs. Yoshinaga demonstrated that his invariant $\langle g \rangle$ for trivalent spatial graphs is equal to the following polynomial [1].

The *Yoshinaga polynomial* $Y(g;t)$ of a diagram $g(G)$ is a Laurent polynomial defined by

$$Y(g) \;=\; Y(g;t) \;=\; \sum_{S \in \mathcal{U}(g)} c_Y(g|S)\, f_Y(S;t). \tag{6}$$

If a diagram $g(G)$ has no crossing points then $\mathcal{U}(g) = G$ and $Y(g) = f_Y(G)$. This polynomial is an ambient isotopy invariant up to $\pm t^i$, $i \in \mathbb{Z}$, and it satisfies the following skein relations [1]:



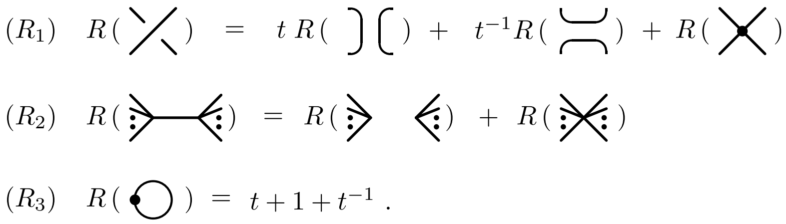
Recall that here and below in skein relations we consider diagrams which have differences in one crossing only, and these differences are pictured in the skein relations.

Consider the second polynomial invariant for spatial graphs introduced in [2].

The *Yamada polynomail* $R(g;t)$ of a diagram $g(G)$ is a Laurent polynomial defined by

$$R(g) \;=\; R(g;t) \;=\; \sum_{S \in \mathcal{U}(g)} c_R(g|S)\, f_R(S;t). \tag{7}$$

If a diagrams $g(G)$ has no crossing points then $R(g)$ coincides with $f_R(G)$. The Yamada polynomial is an ambient isotopy invariant and it satisfies the following skein relation [2]:



Considering links as 2-regular graphs (with at least one vertex for each connected component) we see that $R(g)$ and $Y(g)$ are link invariants. Relations of $R(g)$ and $Y(g)$ with some famous polynomial invariants of knots and links were discussed in [2, 1].

3. Relation between polynomials Y and R

An explicit relation between the considered polynomials is given by the following theorem.

Theorem 1. *Let g be a diagram of a spatial graph with P vertices and Q edges. Then*

$$Y(g;t) = (-t - t^{-1})^{P-Q} R(g; t^{-1}).$$

Proof. By the definition of the Yamada polynomial, we have:

$$\begin{aligned} R(g;t) &= \sum_{S \in \mathcal{U}(g)} t^{k_+ - k_-} \sum_{F \subseteq E(S)} (-1)^{\omega(S-F)} (-t - 2 - t^{-1})^{\beta(S-F)} \\ &= \sum_{S} t^{k_+ - k_-} \sum_{F} (-1)^{\beta+\omega}\, t^{-\beta}\, (t+1)^{2\beta}, \end{aligned}$$

where $\omega = \omega(S-F)$ is the number of components in S after deleting the edge set F, and $\beta = \beta(S-F)$. Since $(-1)^{\beta+\omega} = (-1)^{\beta-\omega}$ and $\beta(S-F) \geq 0$ for any F, we can write

$$R(g;t) = \sum_{S} \sum_{F} (-1)^{\beta-\omega}\, t^{k_+ - k_-}\, t^{-\beta} \sum_{m=0}^{2\beta} \binom{2\beta}{m} t^m. \tag{8}$$

Now consider the Yoshinaga polynomial. By the definition of $Y(g)$, we have

$$\begin{aligned} Y(g;t) &= \sum_{S \in \mathcal{U}(g)} t^{k_- - k_+} (-t - t^{-1})^{k_0} \,\cdot \\ &\quad \cdot \sum_{F \subseteq E(S)} (-t - t^{-1})^{-|F|}\, (t + t^{-1})^{\omega(S-F)} \left(\frac{t + 2 + t^{-1}}{t + t^{-1}} \right)^{\beta(S-F)} \\ &= \sum_{S} (-1)^{k_0}\, t^{k_- - k_+}\, (t + t^{-1})^{k_0 + p - q} \sum_{F} (-1)^{-j}\, t^{-\beta} \sum_{m=0}^{2\beta} \binom{2\beta}{m} t^m, \end{aligned} \tag{9}$$

where $p = |V(S)|$, $q = |E(S)|$, $j = |F|$, $\omega = \omega(S-F)$ and $\beta = \beta(S-F) = q - j - p + \omega$.

The number k_0 can be expressed through the numbers of vertices and edges of G and the state S. Indeed, k_0 double points have been replaced by vertices under constructing S. Since $q = Q + 2k_0$ and $p = P + k_0$, we have $k_0 = (P - Q) - (p - q)$. Substituting this quantity to (9), one can obtain

$$Y(g;t) = (-t - t^{-1})^{(P-Q)} \sum_{S} \sum_{F} (-1)^{\beta-\omega}\, t^{-(k_+ - k_-)}\, t^{-\beta} \sum_{m=0}^{2\beta} \binom{2\beta}{m} t^m. \tag{10}$$

Compare formulas (8) and (10). Except the factor $(-t-t^{-1})^{P-Q}$, these formulas differ only in the sign of $(k_+ - k_-)$. Expressions (8) and (10) can be regarded as the sum of polynomials with the fixed S and F. We show that the coefficient at t^m in every such polynomial computed for Y coincides with the coefficient at t^{-m} in the corresponding polynomial calculated for R. Fix S and F. The binomial coefficients at t of the last sum of (8) and (10) are symmetrical. After multiplying by $t^{-\beta}$, all degrees at t become symmetrical with respect to zero degree. Next, the factors $t^{\pm(k_+-k_-)}$ shift degrees of t in the opposite directions in the symmetrical polynomials. □

Corollary 1. *(i) Let g be a diagram of a spatial graph with P vertices and Q edges, and g' its mirror image. Then $Y(g;t) = (-t-t^{-1})^{P-Q}R(g';t)$. (ii) Let g be a diagram of a link and g' its mirror image. Then $Y(g;t) = R(g';t) = Y(g';t^{-1})$.*

Proof. It was shown in [2] that $R(g;t) = R(g';t^{-1})$. Note that a link can be represented by a 2-regular spatial graph for which $P = Q$. Then $Y(g;t) = R(g';t) = R(g;t^{-1}) = Y(g';t^{-1})$. □

The obtained relation between polynomials shows that $Y(g)$ is an invariant under some types of spatial deformations. This follows from properties of $R(g)$ (see [2] for details).

Corollary 2. *(i) $Y(g)$ is a regular deformation invariant of g; (ii) $Y(g)$ is a flat deformation invariant of a diagram g up to a multiplying $(-t)^i$ for some integer i; (iii) If g is a diagram of a graph whose maximal degree is less than 4, then $Y(g)$ is a pliable deformation invariant of g up to multiplying $(-t)^i$ for some integer i.*

The notion of the span of a polynomial invariant was shown to be important for knots and links. In particular, the span of the Jones polynomial of a knot is bounded by $4c$, where c is a number of crossings in a knot diagram (see, for example, [6]).

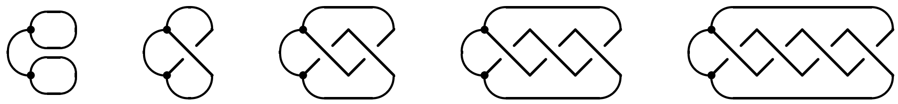
By the *span* of a Laurent polynomial $f = \sum_{i=m}^{n} \lambda_i\, t^i$ we mean a number *span* $f = |n-m|$, where n and m are the maximal and the minimal degrees of f, respectively.

Corollary 3. *Let G be a spatial graph with p vertices and q edges, and g be its diagram with the crossing number $c(g)$. Then* $\operatorname{span} Y(g) \leq 3c(g) + 2\beta(G) + 2(q-p)$.

The proof immediately follows from Theorem 1 and the estimation of *span* for the Yamada polynomial from [7].

4. Example

Consider embeddings of spatial graphs Θ_n, $n \geq 0$, depicted below for $n \leq 4$:



Θ_0 Θ_1 Θ_2 Θ_3 Θ_4

These diagrams can be obtained from canonical diagrams of $(2,n)$-torus knots (if n is odd) or links (if n is even) by adding an edge in the exterior region. Note that this edge is a tunnel of a knot according to the knot theory terminology. Obviously, Θ_n is the so-called theta-graph for odd n and the handcuff-graph for even n. Using Theorem 1 and the Yamada polynomial for this family from [8], we have

Proposition 1. *The Yoshinaga polynomial of spatial graph Θ_n, $n \geq 0$, has the form*

$$Y(\Theta_n;\, t) = t^{-2n} - t^n - ((-1)^n t^{-n} - t^n)(t^2 + 1 + t^{-2})(t + t^{-1})^{-1}.$$

For initial graphs, $Y(\Theta_0;\, t) = 0$, $Y(\Theta_1;\, t) = -t^2 - t - 1$, $Y(\Theta_2;\, t) = -t^3 - t^2 + t^{-3} + t^{-4}$, $Y(\Theta_3;\, t) = -t^4 - t^3 - 1 - t^{-4} + t^{-6}$ and $Y(\Theta_4;\, t) = -t^5 - t^4 - t + t^{-1} + t^{-5} + t^{-8}$.

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