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SURGERIES ON SMALL VOLUME HYPERBOLIC 3-ORBIFOLDS

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Introduction

The computer program SnapPea by J. Weeks is a powerful tool for calculating the volumes of hyperbolic 3-manifolds. The small volume hyperbolic 3-manifolds have been studied rather intensively. The smallest known manifold $\mathscr{M}_1$, of volume 0.942707, was found independently by A. T. Fomenko and S. V. Matveev, and by J. Weeks. The ten smallest volume manifolds were described in [1, 2] where they were obtained in particular by Dehn surgeries on small volume hyperbolic knots and links.

The structure of the set of volumes of hyperbolic 3-orbifolds is given in [3]. The computation of volumes of hyperbolic 3-orbifolds with singular sets obtainable by generalized surgeries on links in the three-dimensional sphere is possible due to SnapPea as well.

If the singular set of a 3-orbifold is a graph other than a link, such a general tool is unavailable and the computation of volumes becomes a difficult problem that needs an individual approach in each particular case. It seems thus natural to study 3-orbifolds obtainable by surgeries on hyperbolic 3-orbifolds.

The aim of this paper is to study closed hyperbolic 3-orbifolds obtainable by surgery on the smallest cusped hyperbolic 3-orbifolds and study coverings of these orbifolds by hyperbolic 3-manifolds obtainable by surgery on links. Recall [3, 4] that every closed hyperbolic 3-orbifold is obtainable by surgery on an orbifold with a *nonrigid* cusp (i.e., a cusp on which Dehn surgery, or Dehn filling, can be performed). Three smallest volume hyperbolic 3-orbifolds with nonrigid cusps were described by Adams [4]. Minimal regular coverings of these orbifolds are complements to the well-known links in the three-dimensional sphere. For example, the smallest orbifold with a nonrigid cusp is the *Picard orbifold* (the quotient of the hyperbolic three-dimensional space by the Picard group) which is covered by the complement of the Borromean rings; and the orbifolds obtainable by surgery on the Picard orbifold are covered by manifolds (generally, by cone-manifolds) obtainable by suitable surgeries on the Borromean rings.

In this paper we establish an exact correspondence between the surgery parameters on the Adams orbifolds and their covering manifolds. This makes it possible to use the computer program SnapPea for calculating the volumes of hyperbolic 3-orbifolds.

We only consider orientable 3-orbifolds, using the basic facts of the orbifold theory as in [3, 5].

Like the volumes of hyperbolic 3-manifolds, the volumes of hyperbolic 3-orbifolds form a well-ordered nondiscrete subset of order type ω^ω of the real axis $\mathbb{R}$, and each volume is realized only for finitely many orbifolds [3]. In particular, there is a hyperbolic orbifold of smallest volume (which is not known yet), as well as the smallest limit volume.

The singular set of the smallest known hyperbolic 3-orbifold is shown in Fig. 0.1 (the underlying space is the three-dimensional sphere; the edge labels 2 are omitted in the figure). Its volume is (approximately) 0.039050.

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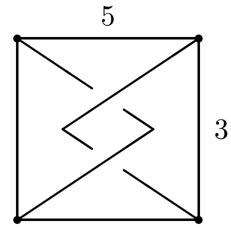

Fig. 0.1. The smallest volume orbifold.

By [4], the unique smallest 3-orbifold with a nonrigid cusp is the *Picard orbifold* defined as the quotient of hyperbolic three-dimensional space $\mathbb{H}^3$ by the action of the Picard group $PSL(2,\mathbb{Z}[i])$ and having the volume 0.305321. Consequently, this is the smallest limit volume. By the hyperbolic surgery theorem, the volume of a cusped orbifold is the exact upper limit of the volumes of the orbifolds obtainable by surgery on its cusp. Since all hyperbolic orbifolds of volume bounded by some constant are obtainable by surgery on one of finitely many cusped orbifolds [3], it follows that all but finitely many hyperbolic 3-orbifolds whose volumes are smaller than that of the Picard orbifold are in fact obtainable by surgery on the Picard orbifold.

The smallest cusped hyperbolic 3-orbifolds were found by Meyerhoff and Adams. All of them have exactly one rigid cusp; so their volumes cannot be limit volumes (various descriptions of these orbifolds are given in [6]).

The three smallest hyperbolic 3-orbifolds with nonrigid cusps were found by Adams [4]: their volumes are 0.305321, 0.444451, and 0.457982. We call these orbifolds the *Adams orbifolds* and denote them by $\mathscr{A}_1$, $\mathscr{A}_2$, and $\mathscr{A}_3$. In particular, $\mathscr{A}_1$ is the Picard orbifold $\mathbb{H}^3/PSL(2,\mathbb{Z}[i])$. The underlying space of $\mathscr{A}_1$ is the 3-sphere with one punctured point $S^3 \setminus \{\infty\}$, its singular set is shown in Fig. 0.2, the left picture. We denote by $\mathscr{A}_1(p,q)$ the orbifold in Fig. 0.2, the right picture, obtainable by (p,q)-surgery on the cusp of $\mathscr{A}_1$ (see Section 1 for definitions).

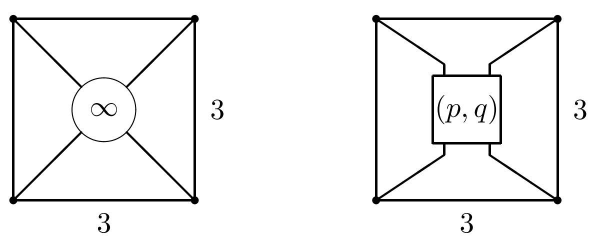

Fig. 0.2. The orbifolds $\mathscr{A}_1$ and $\mathscr{A}_1(p,q)$.

It is well known (see [4, 7] or [8]) that $\mathscr{A}_1$ is 24-fold covered by the complement of the Borromean rings which is the three-component link 6_2^3 in Rolfsen's notations [9]. In Section 2 we show that the manifold $6_2^3(p,q)$ (or, possibly, an orbifold or cone-manifold) obtainable by (p,q)-surgery on all three components of the Borromean rings is a regular 24-fold covering of the orbifold $\mathscr{A}_1(p-2q,p+2q)$ obtainable by $(p-2q,p+2q)$-surgery on the Picard orbifold. This makes it possible to compute the volumes of such orbifolds by using SnapPea (see [10]).

The second Adams orbifold can be obtained as the quotient-space of the form $\mathscr{A}_2 = \mathbb{H}^3/PGL(2,O_7)$, where for a square free positive integer d we denote by O_d the ring of integers of the field $\mathbb{Q}(\sqrt{-d})$. In particular, $\mathscr{A}_2$ is also arithmetic. The underlying space of $\mathscr{A}_2$ is $S^3 \setminus \{\infty\}$, and its singular set is shown in Fig. 0.3 (see [11], where a picture of $\mathbb{H}^3/PSL(2,O_7)$ is presented, or [12]).

It was remarked by Adams [4] that $\mathscr{A}_2$ is 12-fold covered by the complement of the three-component link 6_1^3 in S^3. Some link complements commensurable with $\mathscr{A}_2$ are given in [8, 13]. In Section 3, we establish an exact correspondence between surgery parameters for the orbifold $\mathscr{A}_2$ and the link 6_1^3. Finally, the orbifold $\mathscr{A}_3$ is 16-fold covered by the complement of the four-component link 8_2^4. The exact correspondence between surgery parameters on $\mathscr{A}_3$ and on 8_2^4 will be given in Section 4.

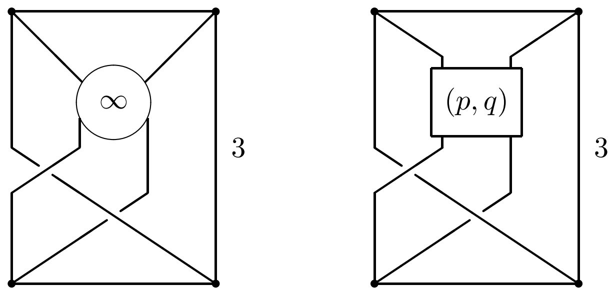

Fig. 0.3. The orbifolds $\mathscr{A}_2$ and $\mathscr{A}_2(p, q)$.

1. Preliminaries

Given a pair of rational numbers p and q, there is a unique presentation $(p, q) = d(p', q')$, where p' and q' are coprime integers and d is a positive rational number.

By definition, the result of (p, q)-surgery on a knot in S^3 is the *cone-manifold* whose underlying space is the three-dimensional manifold obtainable by the usual p'/q'-surgery on the knot [9] and whose singular set is the central curve of the glued solid torus, with a cone angle of $2\pi/d$ around it (see [14] or [15] for cone-manifolds). Note that this cone-manifold is a manifold (i.e., it has the empty singular set) or an orbifold exactly if α is equal to one or an integer, respectively.

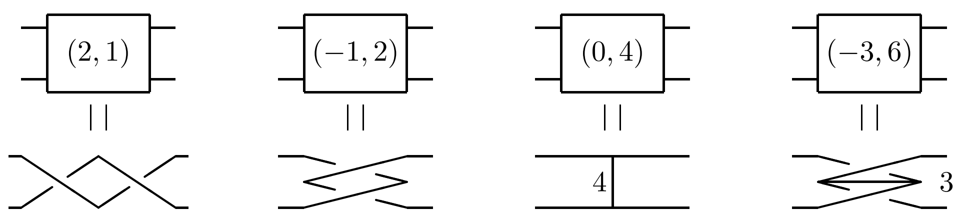

Fig. 1.1. Examples of tangles.

A *rational* (p, q)*-tangle* is a cone-manifold given by a rational tangle of slope p'/q' whose two arcs are labeled by 2, with an additional arc labelled by d for the cone-angle $2\pi/d$ (which for $d = 1$ is not present; see [16] for details and pictures). Some illustrating examples are shown in Fig. 1.1. The result of (p, q)-surgery on a nonrigid cusp of a 3-orbifold is the cone-manifold obtainable by gluing a (p, q)-tangle to the cusp as indicated in Fig. 0.2. The cone-manifold is an orbifold exactly if d is an integer.

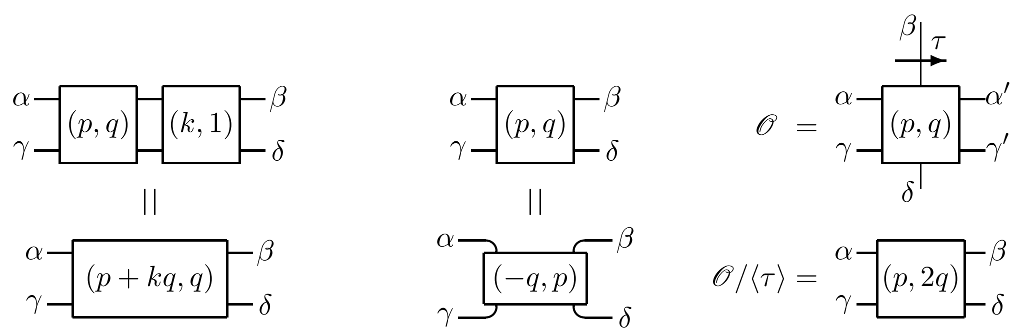

Fig. 1.2. Properties of tangles.

For convenience, a cone-manifold obtainable by (p, q)-surgery on an orbifold will be referred to as an orbifold too. Some well known properties of rational tangles, to be used below, are presented in Fig. 1.2 (see also [17]).

2. The Borromean Rings, the Picard Orbifold, and the Wolcott θ-Graph

In this section we consider the class of closed three-dimensional cone-manifolds obtainable by surgery on the Borromean rings 6_2^3 shown in Fig. 2.1.

Recall that 6_2^3 has symmetries pairwise interchanging every two of its components. We denote by $6_2^3((p_1,q_1),(p_2,q_2),(p_3,q_3))$ the cone-manifold obtainable by surgery on the three components of the Borromean rings 6_2^3, with surgery parameters (p_1,q_1), (p_2,q_2), and (p_3,q_3). We note that all manifolds obtainable by surgery on the Whitehead link 5_1^2 and the figure-eight knot 4_1 belong to this class. More exactly, $6_2^3((p_1,q_1),(p_2,q_2),(-1,1)) = 5_1^2((p_1,q_1),(p_2,q_2))$ and respectively $6_2^3((p_1,q_1),(-1,1),(1,1)) = 4_1(p_1,q_1)$. As the Borromean rings 6_2^3 is a hyperbolic link [5], most of the manifolds obtainable by surgery are also hyperbolic.

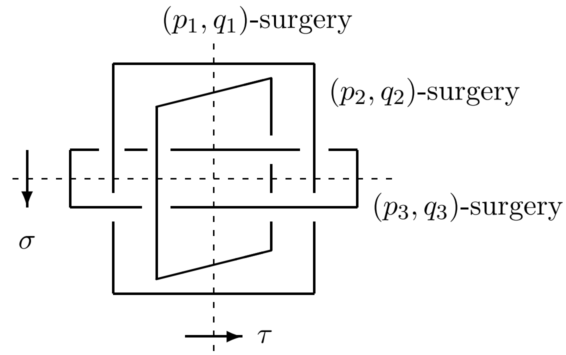

Fig. 2.1. The Borromean rings 6_2^3.

The orientation-preserving symmetry group of 6_2^3 is isomorphic to the symmetric group $\mathbb{S}_4$ of order 24 (see [18] or [19]), and this symmetry group can be realized by an orthogonal action of $\mathbb{S}_4$ on the 3-sphere leaving the link invariant. We will start from the consideration of the case of $\mathbb{S}_4$-equivariant surgery on 6_2^3, i.e., the surgery in which the parameters for all three components of 6_2^3 are the same and equal, say, to (p,q). We denote the corresponding cone-manifold by $6_2^3(p,q)$ in this case. The action of $\mathbb{S}_4$ on the complement of 6_2^3 induces an action of $\mathbb{S}_4$ on $6_2^3(p,q)$.

Theorem 2.1. *For arbitrary rational numbers p and q, the cone-manifold $6_2^3(p,q)$ obtainable by $\mathbb{S}_4$-equivariant (p,q)-surgery on the Borromean rings is a regular $\mathbb{S}_4$-covering of the orbifold $\mathscr{A}_1(p-2q,p+2q)$ obtainable by $(p-2q,p+2q)$-surgery on the Picard orbifold.*

PROOF. We first consider the cone-manifold $6_2^3((p_1,q_1),(p_2,q_2),(p_3,q_3))$ obtainable by surgery on the Borromean rings (see Fig. 2.1). The axes of involutions τ and σ in the symmetry group $\mathbb{S}_4$ are pictured by dashed lines.

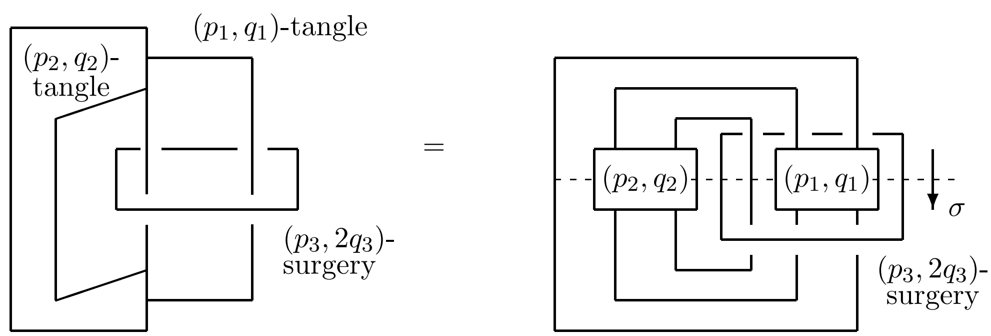

Fig. 2.2. The orbifold $6_2^3((p_1,q_1),(p_2,q_2),(p_3,q_3))/\langle\tau\rangle$.

The involution τ is a strong inversion for two components of 6_2^3. The singular set of the quotient-orbifold $6_2^3((p_1,q_1),(p_2,q_2),(p_3,q_3))/\langle\tau\rangle$ is presented in Fig. 2.2. The involution σ of 6_2^3 induces an involu-

tion of the quotient-orbifold, and we denote it by σ too. The singular set of the orbifold $6_2^3((p_1,q_1),(p_2,q_2),(p_3,q_3))/\langle\tau,\sigma\rangle$ is presented in Fig. 2.3. This singular set is a generalization of a spatial θ-graph in the sense of Wolcott [20]. In particular, if $i=2q_1/p_1$, $j=2q_2/p_2$, and $k=2q_3/p_3$ are integers then the corresponding singular set coincides with the Wolcott θ-graph $\mathscr{W}(i,j,k)$.

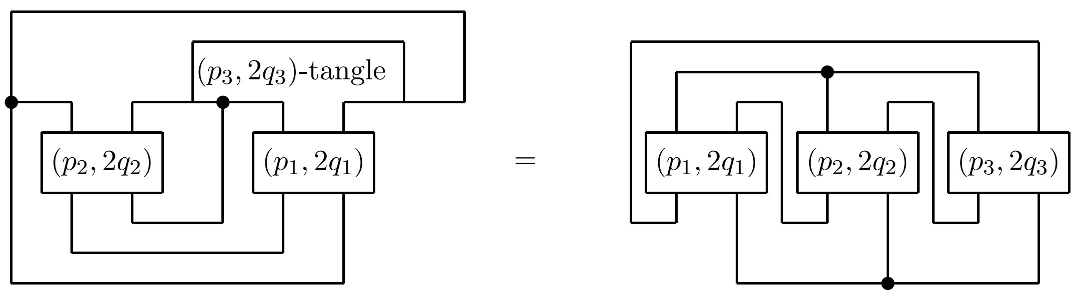

Fig. 2.3. The orbifold $6_2^3((p_1,q_1),(p_2,q_2),(p_3,q_3))/\langle\tau,\sigma\rangle$.

From now on, we consider $\mathbb{S}_4$-equivariant surgery; i.e., we suppose that $(p_1,q_1)=(p_2,q_2)=(p_3,q_3)=(p,q)$. Then, obviously, the singular set in Fig. 2.3 has a symmetry ρ of order 3 that interchanges tangles and leaves the vertices of the graph fixed. The singular set of the quotient-orbifold $(6_2^3(p,q)/\langle\tau,\sigma\rangle)/\langle\rho\rangle$ is shown in Fig. 2.4. In the sequence of equivalent transformations we used the properties of tangles which are presented in Fig. 1.2.

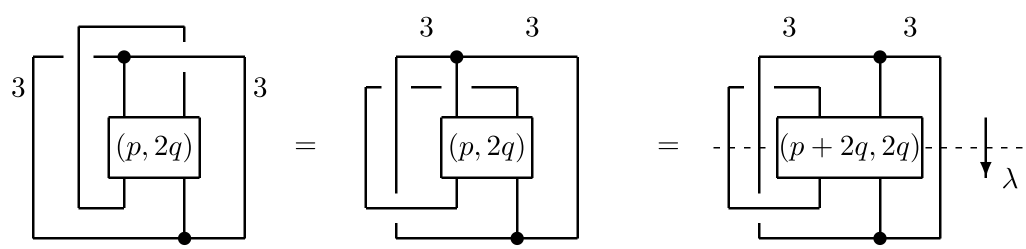

Fig. 2.4. The orbifold $(6_2^3(p,q)/\langle\tau,\sigma\rangle)/\langle\rho\rangle$.

Obviously, the singular set in Fig. 2.4. has a symmetry λ of order 2. The singular set of the quotient-orbifold $((6_2^3(p,q)/\langle\tau,\sigma\rangle)/\langle\rho\rangle)/\langle\lambda\rangle$ is presented in Fig. 2.5.

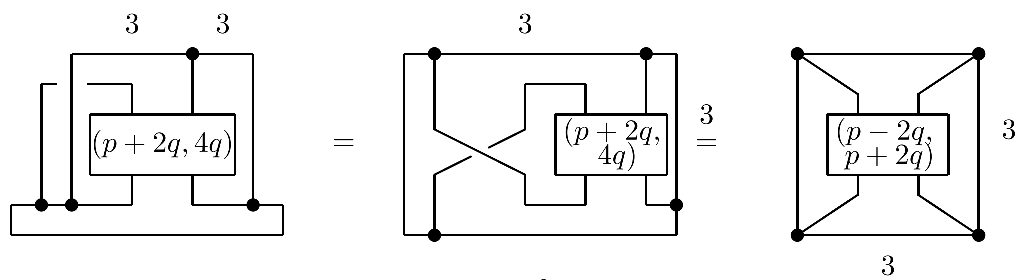

Fig. 2.5. The orbifold $((6_2^3(p,q)/\langle\tau,\sigma\rangle)/\langle\rho\rangle)/\langle\lambda\rangle$.

As we see, this quotient-orbifold coincides with the orbifold $\mathscr{A}_1(p-2q,p+2q)$ obtainable by $(p-2q,p+2q)$-filling on the cusp of the Picard orbifold $\mathscr{A}_1$. This finishes the proof of Theorem 2.1.

Denote by $\mathbb{D}_2\cong\mathbb{Z}_2\times\mathbb{Z}_2$ the normal subgroup of $\mathbb{S}_4$ that is generated by the involutions τ and σ in the proof of Theorem 2.1. It was remarked above that the singular sets of the orbifolds $6_2^3((1,i),(1,j),(1,k))/\mathbb{D}_2$ are the Wolcott θ-graphs $\mathscr{W}(i,j,k)$ of [20]. They have the following property. Each graph $\mathscr{W}(i,j,k)$ is nonplanar (i.e., not homeomorphic in S^3 to a planar θ-graph), and all three constituent knots formed by any two of the three edges of the θ-graph are trivial. By [21], the unique $\mathbb{D}_2$-covering

of such θ-graph is a homology 3-sphere. In this case, the 2-fold branched covering of S^3 along any of these three constituent knots is S^3 again, and the preimage of the third edge is a knot in S^3 whose 2-fold branched covering is the $\mathbb{D}_2$-covering of the θ-graph. Note that the manifold $6_2^3((p_1,q_1),(p_2,q_2),(p_3,q_3))$ is a homology 3-sphere if and only if it is of the form $6_2^3((1,i),(1,j),(1,k))$.

Given some integers i,j,k, consider the knots $\mathscr{K}(i,j,k)$ introduced in [22] and presented in Fig. 2.6, where j and k denote numbers of half-twists on two strings, and i denotes the number of full twists on three strings.

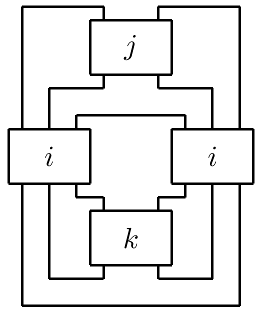

Fig. 2.6. The knot $\mathscr{K}(i,j,k)$.

In general, the three knots $\mathscr{K}(i,j,k)$, $\mathscr{K}(j,k,i)$ and $\mathscr{K}(k,i,j)$ are pairwise nonequivalent (see [22]).

Theorem 2.2. *The homology 3-sphere $6_2^3((1,i),(1,j),(1,k))$ is the regular branched $\mathbb{D}_2$-covering of the Wolcott θ-graph $\mathscr{W}(i,j,k)$, and also the 2-fold branched covering of each of the three knots $\mathscr{K}(i,j,k)$, $\mathscr{K}(j,k,i)$, and $\mathscr{K}(k,i,j)$.*

Proof. The first statement follows from the first part of the proof of Theorem 2.1. The second statement follows from [22], where it was shown that for the 2-fold branched covering of any of the three constituent knots of the θ-graph $\mathscr{W}(i,j,k)$, the preimage of the remaining edge is one of the three knots $\mathscr{K}(i,j,k)$, $\mathscr{K}(j,k,i)$, and $\mathscr{K}(k,i,j)$, respectively. This finishes the proof.

Another interesting situation occurs when the surgery parameters (p_i,q_i) on the components of the Borromean rings are of the form $(1,n)$ or $(2,n)$, where n is odd in the second case. In this situation, the singular set of the quotient orbifold $6_2^3((p_1,q_1),(p_2,q_2),(p_3,q_3))/\mathbb{D}_2$ can acquire some additional arcs of branching index two (see Fig. 2.2, 2.3). Again, all three quotients of the manifold $6_2^3((p_1,q_1),(p_2,q_2),(p_3,q_3))$ by involutions in $\mathbb{D}_2$ are the 3-spheres but now the branch sets are links in general (see also [23]).

Remarks.

(A) Hyperbolic volume. By Theorem 2.1, the orbifold $\mathscr{A}_1(x,y)$ obtainable by surgery on the Picard orbifold has a 24-fold covering by the cone-manifold $6_2^3((y+x)/2,(y-x)/4)$ obtainable by surgery on the Borromean rings. Applying the computer program SnapPea to the Borromean rings, we obtain the volumes of these cone-manifolds (if hyperbolic) and, dividing by 24, the volumes of the corresponding orbifolds. The smallest volumes we found are 0.040890, i.e., the volume of $\mathscr{A}_1(4,1)$, covered by $6_2^3(5/2,-3/4)$, and 0.052654, i.e., the volume of $\mathscr{A}_1(3,2)$ covered by $6_2^3(5/2,-1/4)$. These seem to be the second and third smallest known volumes of hyperbolic 3-orbifolds, after the volume 0.039050 of the orbifold shown in Fig. 0.1.

We remark that independent computations of the volumes of orbifolds obtainable by surgeries on the Picard orbifold have recently been accomplished by V. Petrov [24] by explicitly constructing orbifold fundamental polyhedra.

(B) Manifolds of equal volumes. By Theorem 2.1, the cone-manifolds $6_2^3(2p,q)$ and $6_2^3(2q,p)$ are $\mathbb{S}_4$-coverings of the orbifolds $\mathscr{A}_1(2p-2q,2p+2q)$ and $\mathscr{A}_1(2q-2p,2p+2q)$ which are homeomorphic to one another (up to reflection). Therefore, $6_2^3(2p,q)$ and $6_2^3(2q,p)$, if hyperbolic, have the same volume.

(C) Nonhyperbolic manifolds. The manifold $6_2^3(1,1)$, which is the Poincaré homology 3-sphere, has a spherical structure; so the orbifold $6_2^3(1,1)/\mathbb{S}_4 = \mathscr{A}_1(-1,3)$ is also spherical.

The two-fold branched covering of the Borromean rings or, equivalently, of the orbifold $6^3_2(2,0)$ is the Hantzsche–Wendt manifold M which has an Euclidean structure. Thus, $6^3_2(2,0)/\mathbb{S}_4 = \mathscr{A}_1(2,2) = M/(\mathbb{Z}_2 \times \mathbb{S}_4)$ is also Euclidean. Here $\mathbb{Z}_2 \times \mathbb{S}_4$ is the orientation-preserving isometry group of the Hantzsche–Wendt manifold M (see [19]).

The manifold $6^3_2(0,1)$ is the three-dimensional torus; so the quotient space $6^3_2(0,1)/\mathbb{S}_4 = \mathscr{A}_1(-2,2)$ is an Euclidean orbifold.

(D) Heegaard genus. Since the Borromean rings are a three-bridge link, every 3-manifold obtainable by surgery on the Borromean rings has Heegaard genus at most three. For integer co-prime surgery parameters with $p > 1$, the manifolds $6^3_2(p,q)$ have Heegaard genus three because their first homology is the 3-generator group $(\mathbb{Z}_p)^3$. On the other hand, all manifolds obtainable by surgery on the Whitehead link and the figure-eight knot (which are two-bridge links) is also obtainable by surgery on the Borromean rings and have Heegaard genus at most two. Recall that the ten hyperbolic 3-manifolds of smallest known volumes are also among these manifolds (see [2]).

By Theorem 2.2, the manifolds $6^3_2((1,i),(1,j),(1,k))$ are 2-fold branched coverings of the three-bridge knots $\mathscr{K}(i,j,k)$ (see Fig. 2.6); so the Heegaard genus of the homology 3-spheres $6^3_2((1,i),(1,j),(1,k))$ is at most two.

(E) Maximally symmetric manifolds and equivariant Heegaard genus. Some of the manifolds $6^3_2(p,q)$ are maximally symmetric $\mathbb{S}_4$-manifolds [25]: they admit a Heegaard splitting of genus three invariant under the action of the group $\mathbb{S}_4$ which realizes the maximal order $12(g-1)$ for finite group actions on handlebodies of genus g.

By Theorem 2.1, the manifolds $6^3_2(1+2q,q)$ and $6^3_2(1-2q,q)$ cover the orbifolds $\mathscr{A}_1(1,1+4q)$ and $\mathscr{A}_1(1-4q,1)$, respectively. These orbifolds admit a decomposition along an embedded 2-sphere into two handlebody-orbifolds (see [25]). The preimage of this 2-sphere gives a Heegaard splitting of genus three of the corresponding 3-manifold such that the covering group maps each handlebody of the Heegaard splitting to itself. So the manifolds $6^3_2(1 \pm 2q, q)$ are maximally symmetric $\mathbb{S}_4$-manifolds of genus three. Note also that the usual Heegaard genus of these manifolds is three if q is different from ∓ 1.

Using SnapPea, we found that the smallest volume for hyperbolic 3-manifolds of type $6^3_2(p,q)$ is 2.468232 that is the volume of $6^3_2(3,1)$. This manifold is a maximally symmetric $\mathbb{S}_4$-manifold of (both equivariant and usual) Heegaard genus three, and there is some evidence that this might be the smallest volume of any hyperbolic 3-manifold admitting an $\mathbb{S}_4$-action, and maybe also of Heegaard genus three. We note also that the Fomenko–Matveev–Weeks manifold $\mathscr{M}_1 = 6^3_2((-5,1),(-5,2),(-1,1))$ is a maximally symmetric $\mathbb{D}_6$-manifold and its Heegaard genus (both equivariant and usual) is two (see [2] and also the next section). Here $\mathbb{D}_6$ denotes the dihedral group of order 12. Apart from manifolds of Heegaard genus two and three, the only other maximally symmetric hyperbolic 3-manifold for which the equivariant and the usual Heegaard genus coincide is the manifold in [26], of genus 11, with an $(\mathbb{A}_5 \times \mathbb{Z}_2)$-invariant Heegaard decomposition of genus 11.

3. The Link 6^3_1, the Orbifold $\mathscr{A}_2$, and the Takahashi Manifolds

We consider the n-component alternating links L_n defined as in Fig. 3.1, where links for $n = 3$ and $n = 4$ are presented; in particular, $L_3 = 6^3_1$ and $L_4 = 8^4_1$.

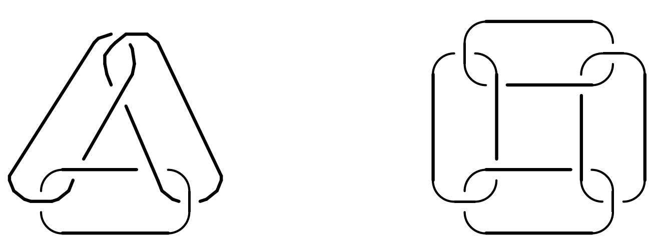

Fig. 3.1. The links $L_3 = 6^3_1$ and $L_4 = 8^4_1$.

Recall that by [8] the group $\pi_1(S^3 \setminus 6_1^3)$ is commensurable with $PGL(2, O_7)$ and $\pi_1(S^3 \setminus 8_1^4)$ is commensurable with $PGL(2, O_3)$.

Let $L_n(p, q) = L_n((p, q), \ldots, (p, q))$ be the cone-manifold obtainable by surgery with parameters (p, q) for each component of L_n; so, in particular, $6_1^3(p, q) = L_3(p, q)$ and $8_1^4(p, q) = L_4(p, q)$. It was shown in [5] that for $n \geq 3$ the manifold $S^3 \setminus L_n$ is hyperbolic; so almost all of the manifolds $L_n(p/q)$ are also hyperbolic.

Denote by $\mathscr{A}_2^n(p, q)$ the three-dimensional orbifold with underlying space S^3 and singular set the spatial graph in Fig. 3.2.

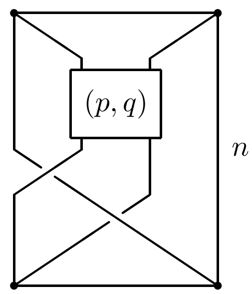

Fig. 3.2. The singular set of $\mathscr{A}_2^n(p, q)$.

One of its edges has singularity index n and all other edges have singularity index 2. For $n = 3$ we have the orbifolds $\mathscr{A}_2^3(p, q) = \mathscr{A}_2(p, q)$ obtainable by surgery on the second Adams orbifold.

Theorem 3.1. *Given arbitrary rational numbers p and q, the cone-manifold $L_n(p, q)$ obtainable by (p, q)-surgery on the link L_n is a regular $4n$-fold covering of the orbifold $\mathscr{A}_2^n(p, p + 2q)$ obtainable by $(p, p + 2q)$-surgery on the orbifold $\mathscr{A}_2^n$.*

Proof. By the Kirby calculus [9], the manifolds $L_n(p/q)$ are also obtainable by surgery on the $2n$-component links $\mathscr{L}_{2n}$ which are the natural generalizations of the link $\mathscr{L}_6$ in Fig. 3.3. More precisely, the manifold $L_n(p, q) = L_n((p, q), \ldots, (p, q))$ coincides with $\mathscr{L}_{2n}((1, 1), (p + 2q, q), \ldots, (1, 1), (p + 2q, q))$.

The manifolds obtainable by surgery on $\mathscr{L}_{2n}$ are usually referred to as *Takahashi manifolds,* since the nice presentations of their fundamental groups were found by Takahashi in [27].

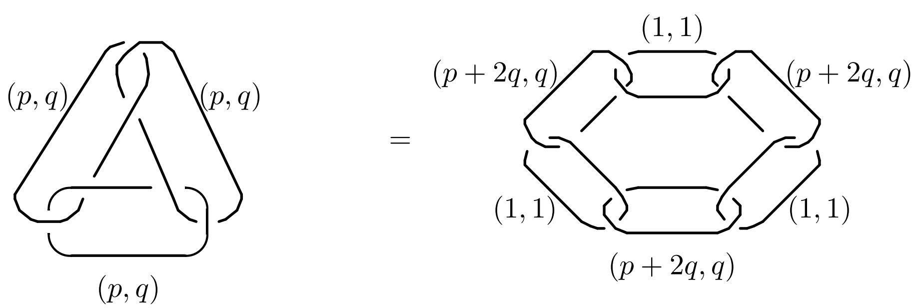

Fig. 3.3. The links L_3 and $\mathscr{L}_6$.

The link $\mathscr{L}_{2n}$ has a strong inversion (involution) τ whose axis intersects each component in two points. This involution induces an involution (denoted also by τ) of the manifold $L_n(p, q)$. The singular set of the quotient-orbifold $\mathscr{O}(n, p, q) = L_n(p, q)/\langle\tau\rangle$ is shown in Fig. 3.4, where we use $\alpha = (p + 2q, q)$ to simplify notation (also cf. [28, 29]).

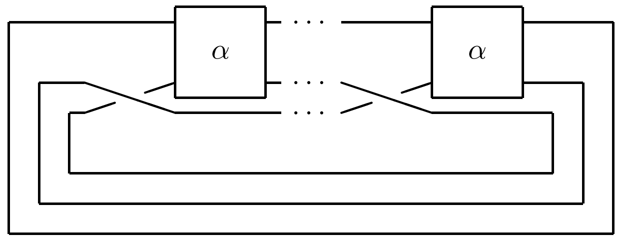

Fig. 3.4. The singular set of $\mathscr{O}(n, p, q)$.

In the notation of [28, 29], the singular set of $\mathscr{O}(n, p, q)$ is the closure of the rational 3-strings braid $(\sigma_1^{p/q+2}\sigma_2)^n$. For example, if $p/q = -3/2$ then the closure of $(\sigma_1^{1/2}\sigma_2)^3$ is the knot 9_{49} (see [30, p. 265]). The orbifold $\mathscr{O}(n, p, q)$ (and its singular set) has an obvious cyclic symmetry ρ of order n permuting the tangles. The singular set of the quotient-orbifold $\mathscr{O}(n, p, q)/\langle\rho\rangle$ is shown in Fig. 3.5.

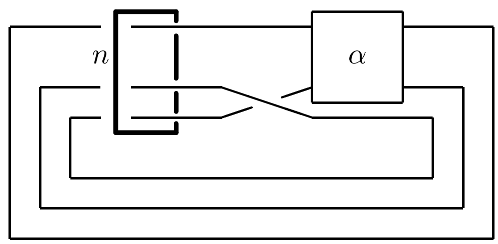

Fig. 3.5. The singular set of $\mathscr{O}(n, p, q)/\langle\rho\rangle$.

Obviously, the singular set in Fig. 3.5 is equivalent to the set in Fig. 3.6.

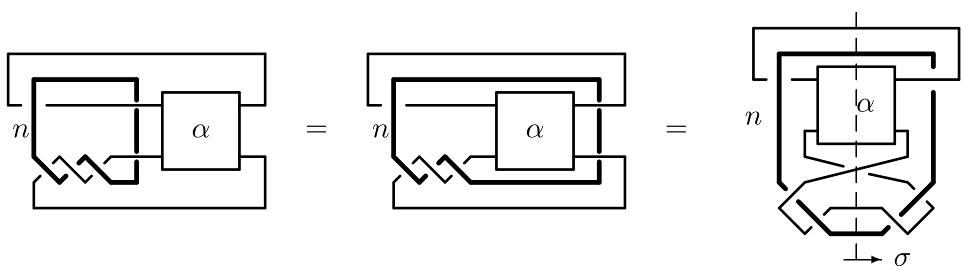

Fig. 3.6. The singular set of $\mathscr{O}(n, p, q)/\langle\rho\rangle$.

From Fig. 3.6 we see that the orbifold $\mathscr{O}(n, p, q)/\langle\rho\rangle$ admits an involution σ whose axis is drawn by the dashed line. The singular set of the quotient-orbifold $(\mathscr{O}(n, p, q)/\langle\rho\rangle)/\langle\sigma\rangle$ is shown in Fig. 3.7.

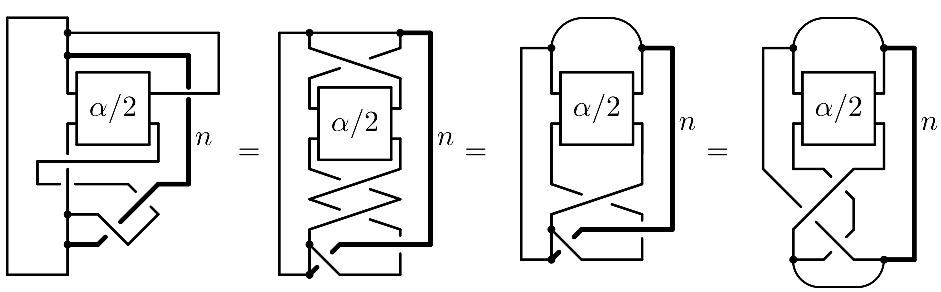

Fig. 3.7. The singular set of $(\mathscr{O}(n, p, q)/\langle\rho\rangle/)\langle\sigma\rangle$.

Using tangle calculations as in Fig. 1.2, it is easy to see that the singular set is equivalent to that of the orbifold $\mathscr{A}_2^n(p, p+2q)$ (see Fig. 3.2). This finishes the proof of Theorem 3.1.

The theorem admits us to compute the volumes of the orbifolds $\mathscr{A}_2(x, y)$ by surgery on 6_1^3. The smallest volume we found in this family is 0.065965 for the orbifold $\mathscr{A}_2(3, 2)$ which is covered by $6_1^3(-3, 1/2)$. It seems to be the fourth smallest volume known for hyperbolic 3-orbifolds. The second smallest volume we found is 0.078559 of the orbifold $\mathscr{A}_2(-3, 1)$ which is covered by the Fomenko–Matveev–Weeks manifold $\mathscr{M}_1 = 6_1^3(-3, 2)$.

Like in Section 2, the manifolds $6_1^3(p, (\pm 1 - p)/2)$, where p is odd, are maximally symmetric $\mathbb{D}_6$-manifolds of Heegaard genus two. In particular, the manifold $\mathscr{M}_1$ has this property.

We remark that the Fibonacci manifolds $\mathbb{H}^3/F(2, 2n)$ uniformized by the Fibonacci groups $F(2, 2n)$ can be obtained as $L_n(-3, 1)$, and are coverings of the orbifolds $\mathscr{A}_2^n(3, 1)$.

4. The Link 8_2^4 and the Orbifold $\mathscr{A}_3$

Denote by $8_2^4(p, q)$ the cone-manifold obtainable by (p, q)-surgery on all four components of the link 8_2^4 (see Fig. 4.1), and by $\mathscr{A}_3(p, q)$ the orbifold whose singular set is as in Fig. 4.1.

Fig. 4.1. The link 8_2^4 and the orbifold $\mathscr{A}_3(p, q)$.

By the Kirby calculus [9], the cone-manifold $8_2^4(p, q)$ is obtainable by surgery on the link $\mathscr{L}_8$ that belongs to the series of links considered in the previous section; in fact,

$$8_2^4(p, q) = \mathscr{L}_8((p+2q, q), (1, 1), (p+2q, q), (1, 1), (p, q), (-1, 1), (p, q), (1, 1))$$

is a Takahashi manifold.

Using the strong inversion (involution) of $\mathscr{L}_8$ considered in the previous section, we obtain $8_2^4(p, q)$ as the 2-fold branched covering of the closure of a generalized three-strings braid.

After three further steps of involutions, we come to the following result.

Theorem 4.1. *The cone-manifold $8_2^4(p, q)$ obtainable by (p, q)-surgery on the components of the link 8_2^4 is a regular 16-fold covering of the orbifold $\mathscr{A}_3(p, 2q)$ obtainable by $(p, 2q)$-surgery on the third Adams orbifold $\mathscr{A}_3$.*

The smallest volume of an orbifold of type $\mathscr{A}_3(x, y)$ that we found by SnapPea is 0.117838 of $\mathscr{A}_3(3, 2)$, covered by the manifold $8_2^4(3, 1)$. We note that $\mathscr{A}_3(3, 2)$ is a π-orbifold; i.e., all singularity indices are equal to two. It is the smallest π-orbifold that we know. The smallest known π-orbifold whose singular set is a knot or link is the π-orbifold whose singular set is the knot 9_{49} of volume 0.471354. Its 2-fold branched covering is the Fomenko–Matveev–Weeks manifold $\mathscr{M}_1$.

We remark that the volume of $\mathscr{M}_1$ is eight times the volume of the orbifold $\mathscr{A}_3(3, 2)$. But the orbifold $\mathscr{A}_3(3, 2)$, whose singular set is a spatial handcuff (or pince-nez) graph, has no regular covering of order less than or equal to eight by a manifold.

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