Cyclic Branched Coverings of Lens Spaces

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Abstract

Somw infinite familly is constructed of orientable three-dimensional closed manifolds $M_n(p, q)$, where $n \geqslant 2$, $p \geqslant 3$, $0 < q < p$ and $(p, q) = 1$, such that $M_n(p, q)$ is an n-fold cyclic covering of the lens space $L(p, q)$ branched over a two-component link.

Keywords: three-dimensional manifold, branched covering, Heegaard diagram
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Introduction

In the present paper we consider the class of the orientable three-dimensional closed manifolds that are cyclic branched coverings of lens spaces. Recall that by the Alexander theorem [1] each orientable three-dimensional closed manifold is a branched covering of the 3-sphere S^3. The manifolds admitting presentations as cyclic coverings of S^3 branched over knots or links undergone intensive investigation. In particular, the following problem is actual: for a manifold, given in terms of the fundamental polyhedron, Dehn surgery, Heegaard diagram, or crystallization, decide whether this manifold is a cyclic branched covering of S^3 and describe the corresponding branching set.

Most studied is the case when the branching set is a two-bridge knot or a two-bridge link. Some examples of the kind are well known: the Weber – Seifert dodecahedral hyperbolic space [2], which is a strongly-cyclic five-fold covering of S^3 branched over the Whitehead link; the Fibonacci manifolds [3] which are cyclic coverings of S^3 branched over the figure-eight knot; the fractional Fibonacci manifolds [4] which are cyclic coverings of S^3 branched over the two-bridge $(2k + \frac{1}{2k})$– knots. Various descriptions are presented in [5] (in terms of fundamental polyhedra, Dehn surgeries, Heegaard splittings, crystallizations, or two-fold branched coverings of the sphere) of the three-dimensional manifolds that are cyclic coverings of S^3 branched over 2-bridge knots and links.

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Interest in various descriptions of three-dimensional manifolds with cyclic symmetries was connected, in particular, with Dunwoody's question from [6] where some infinite family was constructed of three-dimensional manifolds whose Heegaard diagrams admit cyclic symmetries. The orientable three-dimensional closed manifolds from this family are said to be *Dunwoody manifolds*. Dunwoody found representations of the fundamental groups of these manifolds as cyclically presented groups in the sense of [7]. It was observed that the polynomials, associated with cyclic presentations, coincide with the Alexander polynomials of some knots in S^3. The Dunwoody question was as follows: Are the manifolds under construction cyclic coverings of S^3 branched over knots with given Alexander polynomial? The affirmative answers to Dunwoody's question for many particular cases are given in [4, 8, 9, 10, 11, 12]. In the general case the result is as follows [13]: The Dunwoody manifolds are exactly the cyclic branched coverings of $(1, 1)$–knots, i.e. the knots that admit the 1-bridge presentation of genus 1. In other words, the Dunwoody manifolds are the cyclic coverings of manifolds admitting the Heegaard splittings of genus 1 branched over 1-bridge knots lying in these manifolds. Recall that the manifolds, admiting genus 1 Heegaard splitting, are S^3, $S^2 \times S^1$, and the lens spaces $L(p, q)$. The class of $(1, 1)$–knots contains all two-bridge knots and all torus knots in S^3.

The existence and uniqueness of cyclic branched coverings of $(1, 1)$-knots were studied in [14]; moreover, there was given some algorithm of finding the the fundamental group of a covering. Some approach based on the algorithm was realized in [15] for constructing the Alexander polynomial for a $(1, 1)$-knot. Some bounds for the complexity of Dunwoody manidolds were obtained in [16].

Thus, the cyclic branched coverings of two-bridge knots and links, as well as $(1, 1)$-knots, are enough studied. The cyclic branched coverings of lens spaces branched over links with two or more components are studied much less. We note only that the two-fold coverings of the lens spaces commensurable with Coxeter polyhedra was consideres in [17, 18].

The aim of the present paper is to develop a universal approach to constructing the fundamental polyhedra of these manifolds. More precisely, defining pairwise identifications of faces of simplicial complexes, we will construct an infinite family of orientable three-dimensional closed manifolds $M_n(p, q)$, where $n \geqslant 2$, $p \geqslant 3$, $0 < q < p$ and $(p, q) = 1$.

In Section 1 we exhibit an example of the three-dimensional manifold $M_3(3, 1)$, which is a three-fold cyclic branched covering of the lens space $L(3, 1)$. The fundamental polyhedron for this manifold is the regular hyperbolic $2\pi/3$-icosahedron. The generalization of the construction of $M_3(3, 1)$, investigated in [19, 20, 21], did admit to the construction of $M_n(3, 1)$ for $n \geqslant 2$. In Section 2 $M_n(3, 1)$ are generalized to the manifolds $M_n(p, 1)$, and in Section 3, to the manifolds $M_n(p, q)$. In Theorem 3.1 we prove that $M_n(p, q)$ is a manifold, and in Theorem 3.2 that $M_n(p, q)$ is an n-fold cyclic covering of the lens space $L(p, q)$ branched over a two-component link. Section 4

contains a detiled discussion of a typical particular case – the manifolds $M_3(5, q)$, $q = 1, 2, 3, 4$.

1. The Hyperbolic $2\pi/3$-Icosahedron and the Manifold $M_3(3, 1)$

It is well known that for constructing the first examples of orientable three-dimensional closed manifolds of constant negative curvature some regular polyhedra were used that were realized in the Lobachevskii space $\mathbb{H}^3$. Namely [2], the Weber — Seifert dodecahedral hyperbolic space was obtained by pairwise identification of faces of the regular $2\pi/5$-dodecahedron by isometries of $\mathbb{H}^3$. Other manifolds arising from the $2\pi/5$-dodecahedron were constructed in [22, 23]. The regular hyperbolic polyhedron, serving as a source of interesting examples, is the $2\pi/3$-icosahedron. Three hyperbolic manifolds, having the $2\pi/3$-icosahedron as the fundamental polyhedron, were constructed in [22].

The complete list of orientable three-dimensional closed hyperbolic manifolds whose fundamental polyhedron is the $2\pi/5$-dodecahedron or the $2\pi/3$-icosahedron was obtained by Richardson and Rubinstein in 1982 [24]. But the paper is still a preprint up to the author's knowledge. One of the sources, where the list of Richardson — Rubinstein manifolds can be found, is [25]. The list contains six manifolds for which $2\pi/3$-icosahedron is the fundamental polyhedron. In [25] they were denoted by M_{23}, M_{24}, M_{25}, M_{26}, M_{27}, and M_{28}. It can be seen from the construction that M_{24} and M_{25} have symmetries of order 3. Moreover, it was demonstrated in [19, 20] that both manifolds are three-fold cyclic branched coverings of the lens space $L(3, 1)$ (for more details about lens spaces see, for example, [26, §60]).

Here and below we will denote the manifold M_{24} from [25] by $M_3(3, 1)$. This will agree with the further notation of the paper. Let us recall the construction of $M_3(3, 1)$.

Consider the simplicial complex $\mathcal{P}_3(3)$, presented in Fig. 1, where the right and left sides, denoted by $P_1R_1S_1$, are assumed identified. $\mathcal{P}_3$ has 12 vertices, 30 edges, and 20 faces, and it coincides with the icosahedron combinatorially.

Let $\varphi_3(3, 1)$ be a pairwise identification of faces of $\mathcal{P}_3(3)$, written in the notations of Fig. 1 as follows:

$$
\begin{aligned}
&a_i : \mathbf{A_i} \to \bar{\mathbf{A}}_\mathbf{i} \quad [P_iP_{i+1}Q_i \to R_{i+2}P_{i+2}Q_{i+1}], &\quad& b_i : \mathbf{B_i} \to \bar{\mathbf{B}}_\mathbf{i} \quad [R_iP_iQ_i \to S_iS_{i+1}R_{i+1}],\\
&c_i : \mathbf{C_i} \to \bar{\mathbf{C}}_\mathbf{i} \quad [Q_iR_iS_i \to S_iQ_iR_{i+1}], && d : \mathbf{D} \to \bar{\mathbf{D}} \quad [P_1P_2P_3 \to S_3S_1S_2],
\end{aligned}
$$

where $i = 1, 2, 3$, and all indices are taken mod 3. Denote by $M_3(3, 1)$ the quotient space $\mathcal{P}_3(3)/\varphi_3(3, 1)$. It is well known that the complex $\mathcal{P}_3(3)$ can be realized as the regular icosahedron in $\mathbb{H}^3$ with all dihedral angles $2\pi/3$. In this case the identification $\varphi_3(3, 1)$ can be realized by orientation-preserving isometries of the space $\mathbb{H}^3$, and the group, generated by these isometries, is discrete without elements of finite order. Thus, $M_3(3, 1)$ is an orientable three-dimensional closed hyperbolic manifold. Its geometric and topological properties were studied in [19, 20, 21]. In particular,

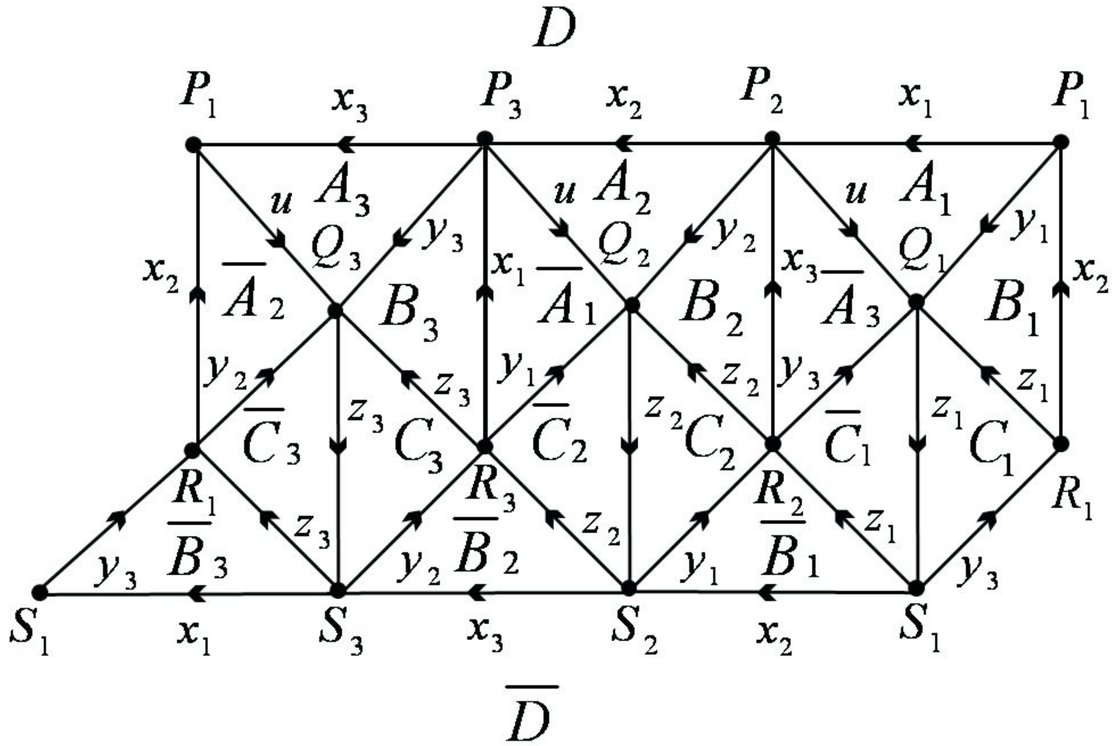

Figure 1: The complex $\mathcal{P}_3(3)$ for $M_3(3,1)$.

it was shown that the $M_3(3,1)$ is a three-fold cyclic covering of the lens space $L(3,1)$ branched over a two-component link. Generalizations of this example, the manifolds $M_n(3,1)$, which are n-fold cyclic branched coverings of $L(3,1)$ branched over 2-component links were investigated in [20, 21] (in these papers $M_n(3,1)$ were denoted by $M_{24}(n)$).

2. The Manifolds $M_n(p, 1)$

Let us construct some infinite family of three-dimensional manifolds that are cyclic coverings of the lens spaces $L(p,1)$ branched over two-component links. Consider the simplicial complex $\mathcal{P}_n(p)$, where $n \geqslant 2$, $p \geqslant 3$, presented in Fig. 2. It has $6n+2$ faces, $(7+p)n$ edges, and $(p+1)n$ vertices. The difference between $\mathcal{P}_3(p)$ and the complex $\mathcal{P}_3(3)$ presented in Fig. 1 is that faces $\mathbf{C_i}$ and $\mathbf{\bar{C}_i}$ are p-gons for the first instead ot triangles for the second. It was done by putting $s = p-3$ additional vertices $T_i^1, T_i^2, \dots T_i^s$ on every edge $S_iQ_i, i = 1, \dots, n$, where additional vertices are numerated in the direction from S_i to Q_i.

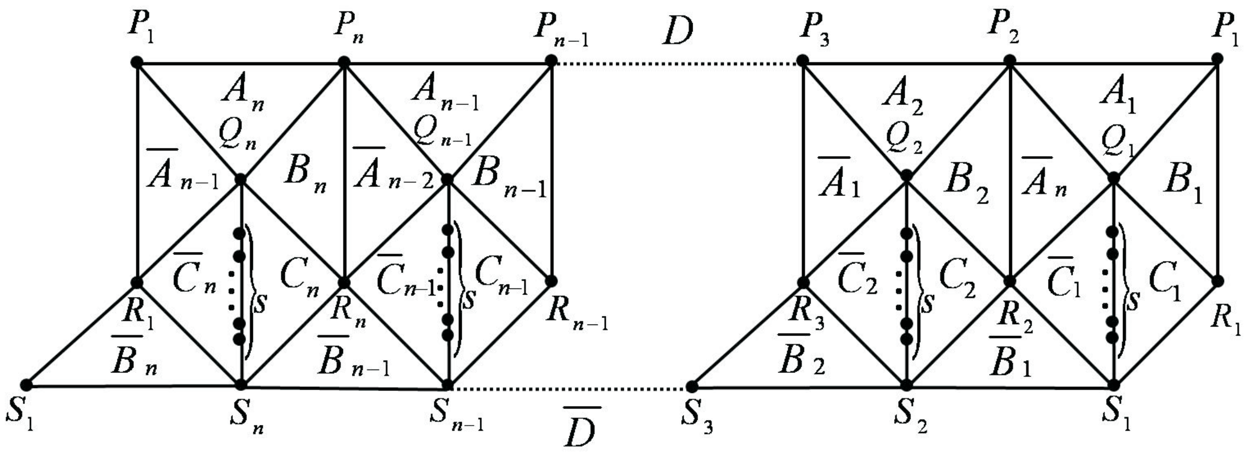

Figure 2: Construction of $M_n(p)$.

Define the pairwise identifications $\varphi_n(p,1)$ of faces of $\mathcal{P}_n(p)$ according to the following rule:

$$
\begin{array}{ll}
a_i : \mathbf{A_i} \to \mathbf{\bar{A}_i} & [P_iP_{i+1}Q_i \to R_{i+2}P_{i+2}Q_{i+1}],\\
b_i : \mathbf{B_i} \to \mathbf{\bar{B}_i} & [R_iP_iQ_i \to S_iS_{i+1}R_{i+1}],\\
c_i : \mathbf{C_i} \to \mathbf{\bar{C}_i} & [Q_iR_iS_iT_i^1 \ldots T_i^s \to T_i^sQ_iR_{i+1}S_iT_i^1 \ldots T_i^{s-1}],\\
d : \mathbf{D} \to \mathbf{\bar{D}} & [P_1P_2 \ldots P_{n-1}P_n \to S_3S_4 \ldots S_1S_2],
\end{array}
$$

where $i = 1, \ldots, n$, all indices are taken mod n, and the faces are identified in arcording to the given order of vertices. Obviously, $\varphi_n(p,1)$ defines some equivalences on the sets of faces, edges, and vertices of $\mathcal{P}_n(p)$.

Proposition 2.1. *The quotient space $M_n(p,1) = \mathcal{P}_n(p)/\varphi_n(p,1)$, where $n \geqslant 2$ and $p \geqslant 3$, is an orientable three-dimensional manifold.*

Proof. Recall that by the Seifert —Threlfall theorem [26], $M_n(p,1)$ is a manifold if and only if the Euler characteristic of $M_n(p,1)$ vanishes. Denote by σ_k for $k = 0,1,2,3$, the number of k-dimensional cells in $M_n(p,1)$. Obviously, $\sigma_3 = 1$. Since 2-cells are split to the cosets corresponding to a_i, b_i, c_i, and d, where $i = 1, \ldots, n$, we get $\sigma_2 = 3n + 1$. All 1-cells are split to the cosets of four types:

$$
\begin{array}{ll}
(x_i): & P_iP_{i+1} \xrightarrow{a_i} R_{i+2}P_{i+2} \xrightarrow{b_{i+2}} S_{i+2}S_{i+3} \xrightarrow{d^{-1}} P_iP_{i+1};\\
(y_i): & P_iQ_i \xrightarrow{a_i} R_{i+2}Q_{i+1} \xrightarrow{c_{i+1}^{-1}} S_{i+1}R_{i+1} \xrightarrow{b_i^{-1}} P_iQ_i;\\
(z_i): & R_iQ_i \xrightarrow{b_i} S_iR_{i+1} \xrightarrow{c_i^{-1}} T_i^1S_i \xrightarrow{c_i^{-1}} \ldots \xrightarrow{c_i^{-1}} Q_iT_i^s \xrightarrow{c_i^{-1}} R_iQ_i;\\
(u): & P_2Q_1 \xrightarrow{a_1} P_3Q_2 \xrightarrow{a_2} \ldots \xrightarrow{a_{n-1}} P_1Q_n \xrightarrow{a_n} P_2Q_1;
\end{array}
$$

where $i = 1, \ldots, n$. Therefore, $\sigma_1 = 3n + 1$. It is easy to see that all vertices lie in a sole coset and $\sigma_0 = 1$. Thus, $\chi(M_n(p, 1)) = 1 - (3n + 1) + (3n + 1) - 1 = 0$, and $M_n(p, 1)$ is an orientable closed 3-manifold. $\square$

Recall (for example, see [27]) that a representation of the fundamental group of a three-dimensional manifold is said to be *geometric* if it corresponds to the Heegaard diagram of the manifold.

Proposition 2.2. *The fundamental group of $M_n(p, 1)$, where $n \geqslant 2$ and $p \geqslant 3$, has the following geometric presentation:*

$$\pi_1(M_n(p, 1)) = \langle a_1, \ldots, a_n; b_1, \ldots, b_n; c_1, \ldots, c_n; d \quad | \quad a_1 a_2 \ldots a_n = 1,$$
$$a_i b_{i+2} d^{-1} = 1, \qquad a_i c_{i+1}^{-1} b_i^{-1} = 1, \qquad b_i c_i^{-(p-1)} = 1; \qquad i = 1, \ldots, n\rangle.$$

Proof. It is known [26, § 63] that if a three-dimensional manifold is given by a pairing of faces of the polyhedron, and the number of pairs of faces is h, then there exists a Heegaard splitting of genus h for the manifold. In our case $h = 3n + 1$. An open Heegaard diagram for $M_n(p, 1)$ looks as in Fig. 3.

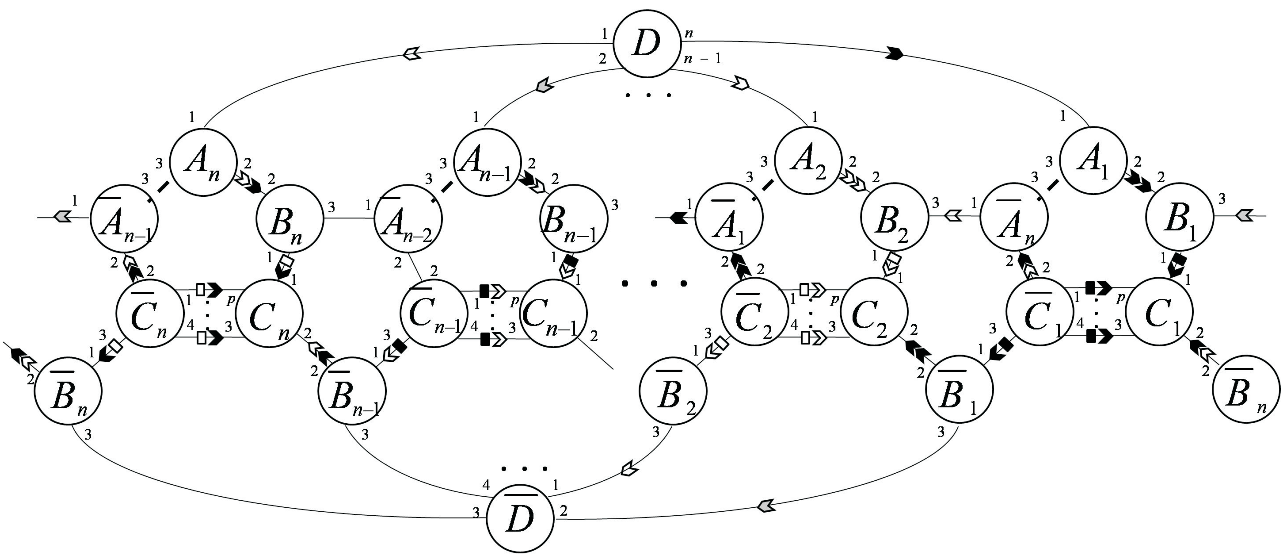

Figure 3: Heegaard diagram of $M_n(p, 1)$.

Let us identify the disks A_1 and $\overline{A}_1$ so as to garantee that labels 1, 2, 3 on the boundary of the first disk will be identified with the namesake labels on the second disk. Provide some similar identifications of the disks $A_2, \ldots, A_n$, $B_1, \ldots, B_n$, $C_1, \ldots, C_n$, D with $\overline{A}_2, \ldots, \overline{A}_n$, $\overline{B}_1, \ldots, \overline{B}_n$, $\overline{C}_1, \ldots, \overline{C}_n$, $\overline{D}$, respectively. In result, from the open Heegaard diagram we will get a Heegaard diagram of genus $3n + 1$ for $M_n(p, 1)$. The arcs of the open Heegaard diagram, having endpoints on

the boundaries of the disks, will be joined into $3n+1$ closed curves on the Heegaard surface. Each of these curves defines a relation in the representation of the manifold fundamental group which looks as the product of labels on the disks passing along a curve [28]. For example, if traveling along the curve we proceed from the disk A_i to the disk $\overline{A_i}$, then the label is a_i. □

Proposition 2.3. *The manifold $M_n(p,1)$, $n \geqslant 2$, $p \geqslant 3$ is an n-fold cyclic covering of the lens space $L(p,1)$ branched over a two-component link.*

Proof. Obviously, the Heegaard diagram in Fig. 3 has some cyclic symmetry of order n that transfers A_i to A_{i+1}, $\overline{A}_i$ to $\overline{A}_{i+1}$, B_i to B_{i+1}, $\overline{B}_i$ to $\overline{B}_{i+1}$, C_i to C_{i+1}, $\overline{C}_i$ to $\overline{C}_{i+1}$, $i = 1,2,\ldots,n$, and transfers the disk D and $\overline{D}$ to themselves. This symmetry induces the symmetry of $M_n(p,1)$, which we will denote by ρ_n. The quotient space $M_n(p,1)/\rho_n$ is a three-dimensional orbifold. This orbifold is characterised by its underlying space (a three-dimensional manifold), its singular set, and its order of singularity. The open Heegaard diagram of the underlying space of the orbifold $M_n(p,1)/\rho_n$ is represented on the upper part of Fig. 4. Two components $\mathcal{L}_1$ and $\mathcal{L}_2$ of the singular set $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ are presented in the figure by dashed lines. The component $\mathcal{L}_1$ goes from the disk D to $\overline{D}$ it is the image of the symmetry of order n. The component $\mathcal{L}_2$ goes from the disk A to $\bar{A}$ along the arc connecting the points that are labeled by 3 on the boundaries of these disks.

Let us demonstrate that the underlying space of the orbifold $M_n(p,1)/\rho_n$ is the lens space $L(p,1)$. To this end, we will apply the Singer moves [29] for transforming the Heegaard diagram of the underlying space to the canonical diagram of the lens space $L_{p,1}$. Describe the sequence of moves according to Fig 4.

We start with the diagram on the top left part in Fig. 4. The genus of this diagram is 4. Let us glue the disks A and $\overline{A}$ forming a 1-handle. The curve, connecting these disks, corresponds to a 2-handle. Gluing of the 2-handle will give a filling of the 1-handle. In result we will get a Heegaard diagram of genus 3. Herewith the component $\mathcal{L}_2$ of the singular set will form the trivial knot. Onwards we will not present this component on the diagram.

After gluing the two 1-handles corresponding to the pairs of the disks B and $\overline{B}$, and well as D and $\overline{D}$, the curve, that connect B with D and $\overline{B}$ with $\overline{D}$ will correspond to a 2-handle. This 2-handle will fill the two above-mentioned 1-handles into one. Cutting this handle, we will get a genus 2 diagram, where the disks B and D are jointed into B, and the disks $\overline{B}$ and $\overline{D}$ into $\overline{B}$.

Cut the diagram along a curve dividing G and $\overline{G}$ in parts. Herewith the two new disks G and $\overline{G}$ will appear. We attach $\overline{G}$ to the main part of the diagram by identifying C and $\overline{C}$.

Let us glue the 1-handles corresponding to the pairs of the disks B and $\overline{B}$ and G and $\overline{G}$. The curves, connecting these 1-handles, correspond to the 2-handles whose attaching will decrease the genus of the Heegaard diagram. By jointing the disks B and G into G, as well as jointing the disks $\overline{B}$ and $\overline{G}$ into $\overline{G}$, we will get the Heegaard diagram of genus 1. It is well known [28], that

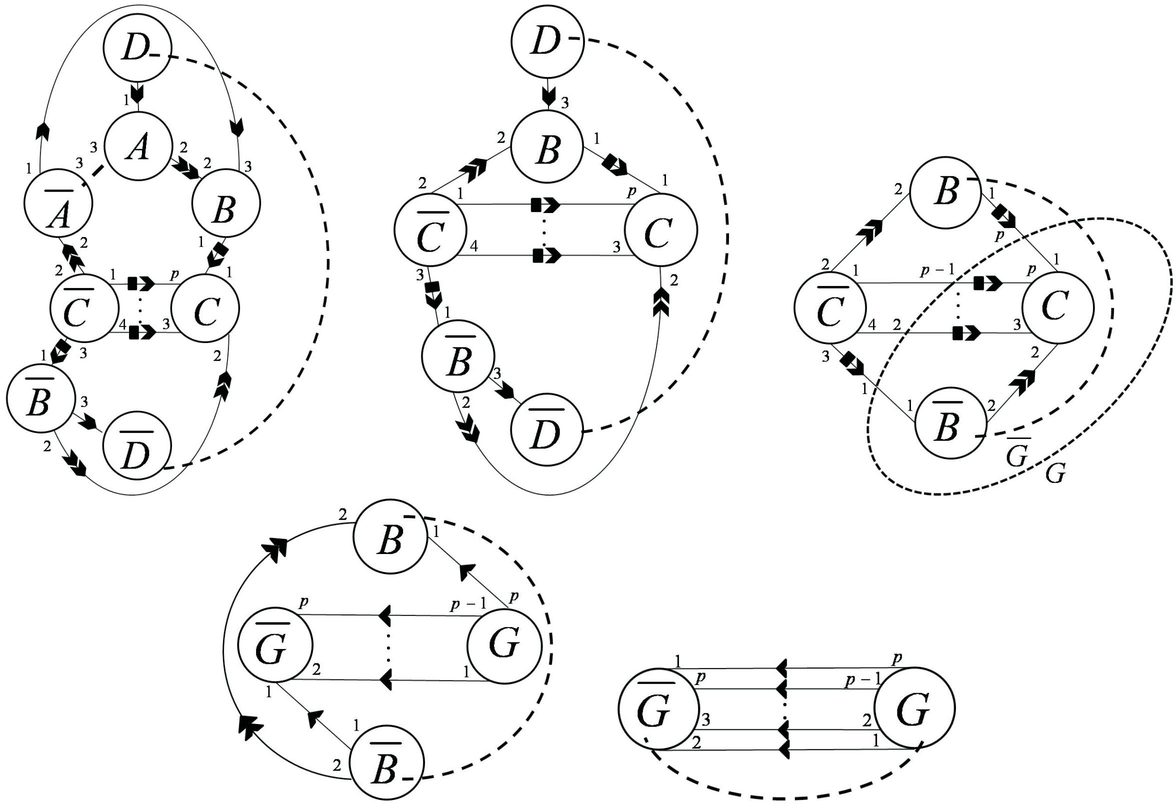

Figure 4: Transformations of the Heegaard diagram of the quotient space $M_n(p,1)/\rho_n$.

this diagram is a Heegaard diagram of the lens space $L_{p,1}$. Therefore, the underlying space of the orbifold $M_n(p,1)/\rho_n$ is the lens space $L(p,1)$. The component $\mathcal{L}_1$ of the singular set is presented in the figure by the dashed line, and the component $\mathcal{L}_2$ is the trivial knot. Thus, the manifold $M_n(p,1)$ is an n-fold cyclic covering of $L(p,1)$ branched over a two-component link. □

3. The Manifolds $M_n(p,q)$

Let us again consider the simplicial complex $\mathcal{P}_n(p)$, $n \geqslant 2$, $p \geqslant 3$, presented in Fig. 2. In Section 2, we have defined the identifications $\varphi_n(p,1)$ that yield to the manifolds $M_n(p,1) = \mathcal{P}_n(p)/\varphi_n(p,1)$. Define on $\mathcal{P}_n(p,q)$ the identifications $\varphi_n(p,q)$, $0 < q < p$, $(p,q) = 1$, such that

$\varphi_n(p,1)$ is a particular case. Assume that $\varphi_n(p,q)$ identifies the faces of $\mathcal{P}_n(p,q)$ as follows:

$$
\begin{array}{ll}
a_i : \mathbf{A_i} \to \bar{\mathbf{A}}_\mathbf{i} & [P_iP_{i+1}Q_i \to R_{i+2}P_{i+2}Q_{i+1}],\\
b_i : \mathbf{B_i} \to \bar{\mathbf{B}}_\mathbf{i} & [R_iP_iQ_i \to S_iS_{i+1}R_{i+1}],\\
c_i : \mathbf{C_i} \to \bar{\mathbf{C}}_\mathbf{i} & [Q_iR_iS_iT_i^1 \dots T_i^s \to T_i^sQ_iR_{i+1}S_iT_i^1 \dots T_i^{s-1}], \quad \text{å } q = 1;\\
 & [Q_iR_iS_iT_i^1 \dots T_i^s \to T_i^{s-1}T_i^sQ_iR_{i+1}S_iT_i^1 \dots T_i^{s-2}], \quad \text{å } q = 2;\\
 & \vdots\\
 & [Q_iR_iS_iT_i^1 \dots T_i^s \to T_i^1 \dots T_i^sQ_iR_{i+1}S_i], \quad \text{å } q = p-3;\\
 & [Q_iR_iS_iT_i^1 \dots T_i^s \to S_iT_i^1 \dots T_i^sQ_iR_{i+1}], \quad \text{å } q = p-2;\\
 & [Q_iR_iS_iT_i^1 \dots T_i^s \to R_{i+1}S_iT_i^1 \dots T_i^sQ_i], \quad \text{å } q = p-1;\\
d : \mathbf{D} \to \bar{\mathbf{D}} & [P_1P_2 \dots P_{n-1}P_n \to S_3S_4 \dots S_1S_2].
\end{array}
$$

It is clear from the definition that for the identification $\varphi_n(p,q)$ the actions of a_i, b_i, and d are exactly the same as for the identification $\varphi_n(p,1)$ from the preceeding section and they do not depend on q. But the action of c_i depends on q.

Theorem 3.1. *The quotient space $M_n(p,q) = \mathcal{P}_n(p)/\varphi_n(p,q)$, where $n \geqslant 2$ and $p \geqslant 3$, $0 < q < p$, $(p,q) = 1$, is an orientable three-dimensional manifold.*

Proof. We now demonstrate that the Euler characterisitc $\chi(M_n(p,q))$ vanishes. Let σ_k be a number of k-dimensional cells in $M_n(p,q)$. By analogy to the proof of Proposition 2.1, $\sigma_3 = 1$ and $\sigma_2 = 3n+1$. All 1-cells are split into the four types of cosets: the classes classes x_i, $i = 1,\dots,n$, and u do not depend on q:

$$
\begin{array}{ll}
(x_i): & P_iP_{i+1} \xrightarrow{a_i} R_{i+2}P_{i+2} \xrightarrow{b_{i+2}} S_{i+2}S_{i+3} \xrightarrow{d^{-1}} P_iP_{i+1};\\
(u): & P_2Q_1 \xrightarrow{a_1} P_3Q_2 \xrightarrow{a_2} \dots \xrightarrow{a_{n-1}} P_1Q_n \xrightarrow{a_n} P_2Q_1,
\end{array}
$$

and the classes y_i and z_i, where $i = 1,\dots,n$, depend on q. In accord to the values of q, the relations y_i look as follows:

$$
\begin{array}{rl}
q = 1: & P_iQ_i \xrightarrow{a_i} R_{i+2}Q_{i+1} \xrightarrow{c_{i+1}^{-1}} S_{i+1}R_{i+1} \xrightarrow{b_i^{-1}} P_iQ_i;\\
q = 2: & P_iQ_i \xrightarrow{a_i} R_{i+2}Q_{i+1} \xrightarrow{c_{i+1}^{-1}} T_{i+1}^1S_{i+1} \xrightarrow{c_{i+1}^{-1}} T_{i+1}^3T_{i+1}^2 \xrightarrow{c_{i+1}^{-1}} \dots \xrightarrow{c_{i+1}^{-1}} S_{i+1}R_{i+1} \xrightarrow{b_i^{-1}} P_iQ_i;\\
 & \vdots\\
q = p-3: & P_iQ_i \xrightarrow{a_i} R_{i+2}Q_{i+1} \xrightarrow{c_{i+1}^{-1}} T_{i+1}^{s-1}T_{i+1}^{s-2} \xrightarrow{c_{i+1}^{-1}} T_{i+1}^{s-4}T_{i+1}^{s-5} \xrightarrow{c_{i+1}^{-1}} \dots \xrightarrow{c_{i+1}^{-1}} S_{i+1}R_{i+1} \xrightarrow{b_i^{-1}} P_iQ_i;\\
q = p-2: & P_iQ_i \xrightarrow{a_i} R_{i+2}Q_{i+1} \xrightarrow{c_{i+1}^{-1}} T_{i+1}^sT_{i+1}^{s-1} \xrightarrow{c_{i+1}^{-1}} T_{i+1}^{s-2}T_{i+1}^{s-3} \xrightarrow{c_{i+1}^{-1}} \dots \xrightarrow{c_{i+1}^{-1}} S_{i+1}R_{i+1} \xrightarrow{b_i^{-1}} P_iQ_i;\\
q = p-1: & P_iQ_i \xrightarrow{a_i} R_{i+2}Q_{i+1} \xrightarrow{c_{i+1}^{-1}} Q_{i+1}T_{i+1}^s \xrightarrow{c_{i+1}^{-1}} T_{i+1}^sT_{i+1}^{s-1} \xrightarrow{c_{i+1}^{-1}} \dots \xrightarrow{c_{i+1}^{-1}} S_{i+1}R_{i+1} \xrightarrow{b_i^{-1}} P_iQ_i.
\end{array}
$$

According to the values of q, the relations z_i look as follows:

$$
\begin{aligned}
q = 1: &\quad R_iQ_i \xrightarrow{b_i} S_iR_{i+1} \xrightarrow{c_i^{-1}} T_i^1 S_i \xrightarrow{c_i^{-1}} T_i^2 T_i^1 \xrightarrow{c_i^{-1}} \ldots \xrightarrow{c_i^{-1}} Q_i T_i^s \xrightarrow{c_i^{-1}} R_iQ_i;\\
q = 2: &\quad R_iQ_i \xrightarrow{b_i} S_iR_{i+1} \xrightarrow{c_i^{-1}} T_i^2 T_i^1 \xrightarrow{c_i^{-1}} T_i^4 T_i^3 \xrightarrow{c_i^{-1}} \ldots \xrightarrow{c_i^{-1}} T_i^s T_i^{s-1} \xrightarrow{c_i^{-1}} R_iQ_i;\\
&\quad \vdots\\
q = p-3: &\quad R_iQ_i \xrightarrow{b_i} S_iR_{i+1} \xrightarrow{c_i^{-1}} T_i^s T_i^{s-1} \xrightarrow{c_i^{-1}} T_i^{s-3} T_i^{s-4} \xrightarrow{c_i^{-1}} \ldots \xrightarrow{c_i^{-1}} T_i^2 T_i^1 \xrightarrow{c_i^{-1}} R_iQ_i;\\
q = p-2: &\quad R_iQ_i \xrightarrow{b_i} S_iR_{i+1} \xrightarrow{c_i^{-1}} Q_i T_i^s \xrightarrow{c_i^{-1}} T_i^{s-1} T_i^{s-2} \xrightarrow{c_i^{-1}} \ldots \xrightarrow{c_i^{-1}} T_i^1 S_i \xrightarrow{c_i^{-1}} R_iQ_i;\\
q = p-1: &\quad R_iQ_i \xrightarrow{b_i} S_iR_{i+1} \xrightarrow{c_i^{-1}} R_iQ_i.
\end{aligned}
$$

It is easy to see that for a given q for each i the element c_i^{-1} appears in the relations y_{i-1} and z_i p times in total. Therefore, $\sigma_1 = 3n + 1$. Since all vertices belong to the same coset, $\sigma_0 = 1$. Thus, $\chi(\mathcal{M}_n(p, q)) = 0$, and by the Seifert-Threllfal theorem [26, § 63], $M_n(p, q)$ is a closed 3-manifold. □

Theorem 3.2. *The manifold $M_n(p, q)$, $n \geqslant 2$, $p \geqslant 3$, $0 < q < p$, $(p, q) = 1$, is an n-fold cyclic covering of the lens space $L(p, q)$ branched over a two-component link.*

Proof. By analogy to the proof of Proposition 2.3, denote by ρ_n the rotational symmetry of the complex $\mathcal{P}_n(p)$ inducing the cyclic symmetry of the quotient space $M_n(p, q) = \mathcal{P}_n(p)/\varphi_n(p, q)$. Denote this symmetry also by ρ_n. The quotient space $M_n(p, q)/\rho_n$ is an three-dimensional orbifold. By the Heegaard diagram of the orbifold $M_n(p, q)/\rho_n$ we will mean the Heegaard diagram of its underlying space equipped with information about the singular set (Fig. 5). The singular set $\mathcal{L}$ has two components: $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$. The component $\mathcal{L}_1$ corresponds to the axis of the rotation ρ_n and the component $\mathcal{L}_2$, to the class of edges $P_{i+1}Q_i$, $i = 1, \ldots, n$. By a sequence of Singer moves in Fig. 5, which are analogous to the moves in Fig. 4, the Heegaard diagrams of the underlying space of the quotient space $M_n(p, q)/\rho_n$ transforms to the canonical diagram of the lens space $L_{p,q}$. The labels $q, q + 1, \ldots, q + (p - 1)$ in the diagram are taken mod p. □

4. Example: The Manifolds $M_3(5, q)$

Consider the manifolds $M_3(5, q)$ where $q = 1, 2, 3, 4$, in detail. Each manifold $M_3(5, q)$ is obtained by the pairing $\varphi_3(5, q)$ of the faces of the complex $P_3(5)$ which is presented in Fig. 6. For every q the identifications a_i, b_i and d are the same:

$$
\begin{aligned}
a_i &: \mathbf{A_i} \to \bar{\mathbf{A}}_\mathbf{i} \quad [P_iP_{i+1}Q_i \to R_{i+2}P_{i+2}Q_{i+1}],\\
b_i &: \mathbf{B_i} \to \bar{\mathbf{B}}_\mathbf{i} \quad [R_iP_iQ_i \to S_iS_{i+1}R_{i+1}],\\
d &: \mathbf{D} \to \bar{\mathbf{D}} \quad [P_1P_2P_3 \to S_3S_1S_2],
\end{aligned}
$$

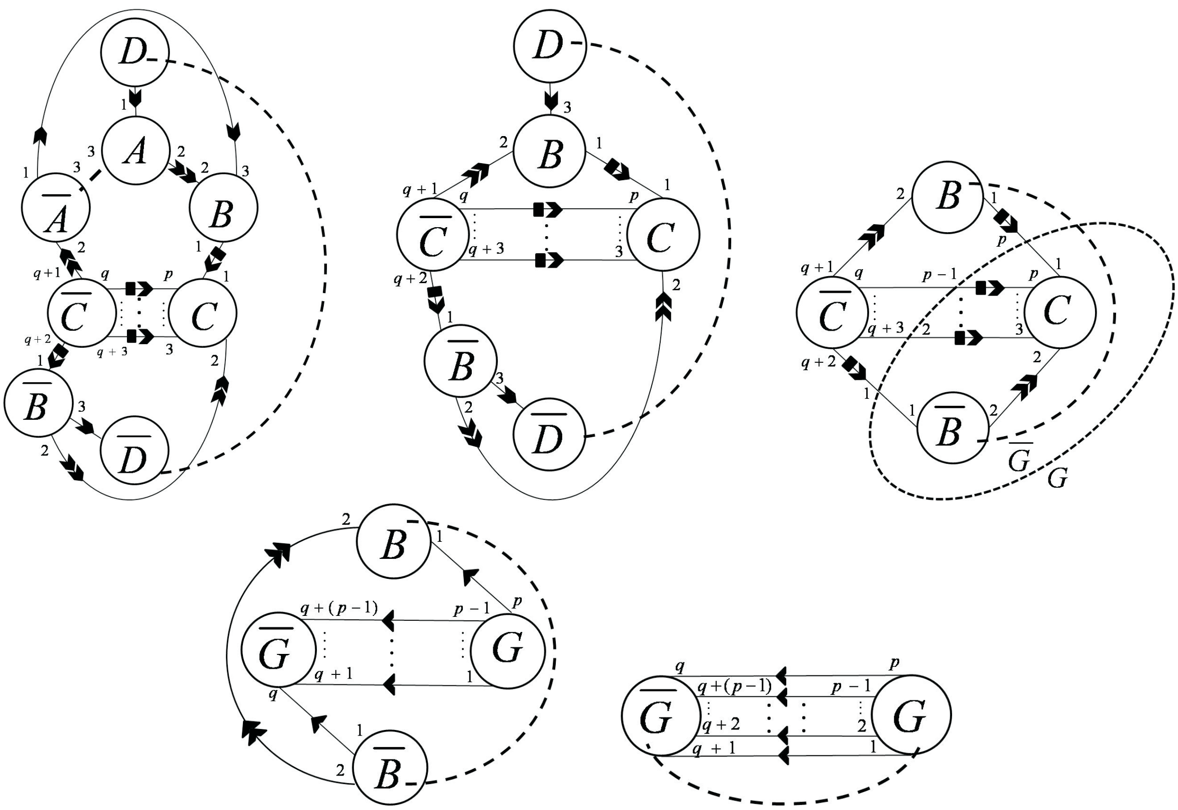

Figure 5: Transformations of the Heegaard diagram of the quotient space $M_n(p,q)/\rho_n$.

where $i = 1, 2, 3$ and all indices are taken mod 3. The identifications c_i are depend on q as follows:

$$\begin{array}{ll}
\text{for } \varphi_3(5,1): & c_i : \mathbf{C_i} \to \bar{\mathbf{C}}_\mathbf{i} \quad [Q_iR_iS_iT_i^1T_i^2 \to T_i^2Q_iR_{i+1}S_iT_i^1]; \\
\text{for } \varphi_3(5,2): & c_i : \mathbf{C_i} \to \bar{\mathbf{C}}_\mathbf{i} \quad [Q_iR_iS_iT_i^1T_i^2 \to T_i^1T_i^2Q_iR_{i+1}S_i]; \\
\text{for } \varphi_3(5,3): & c_i : \mathbf{C_i} \to \bar{\mathbf{C}}_\mathbf{i} \quad [Q_iR_iS_iT_i^1T_i^2 \to S_iT_i^1T_i^2Q_iR_{i+1}]; \\
\text{for } \varphi_3(5,4): & c_i : \mathbf{C_i} \to \bar{\mathbf{C}}_\mathbf{i} \quad [Q_iR_iS_iT_i^1T_i^2 \to R_{i+1}S_iT_i^1T_i^2Q_i].
\end{array}$$

For every q under the identification $\varphi_3(5,q)$ all faces of the complex $\mathcal{P}_3(5)$ split to 10 cosets, and all vertices are in a sole coset. All edges are splitting to 10 cosets, where x_i, $i = 1, 2, 3$, are u does not depend on q:

$$\begin{array}{ll}
(x_i): & P_iP_{i+1} \xrightarrow{a_i} R_{i+2}P_{i+2} \xrightarrow{b_{i+2}} S_{i+2}S_{i+3} \xrightarrow{d^{-1}} P_iP_{i+1}; \\
(u): & P_2Q_1 \xrightarrow{a_1} P_3Q_2 \xrightarrow{a_2} P_1Q_3 \xrightarrow{a_3} P_2Q_1;
\end{array}$$

and the classes y_i and z_i, $i = 1, 2, 3$, depend on q as follows: For $\varphi_3(5,1)$

$$\begin{array}{ll}
(y_i): & P_iQ_i \xrightarrow{a_i} R_{i+2}Q_{i+1} \xrightarrow{c_{i+1}^{-1}} S_{i+1}R_{i+1} \xrightarrow{b_i^{-1}} P_iQ_i; \\
(z_i): & R_iQ_i \xrightarrow{b_i} S_iR_{i+1} \xrightarrow{c_i^{-1}} T_i^1S_i \xrightarrow{c_i^{-1}} T_i^2T_i^1 \xrightarrow{c_i^{-1}} Q_iT_i^2 \xrightarrow{c_i^{-1}} R_iQ_i;
\end{array}$$

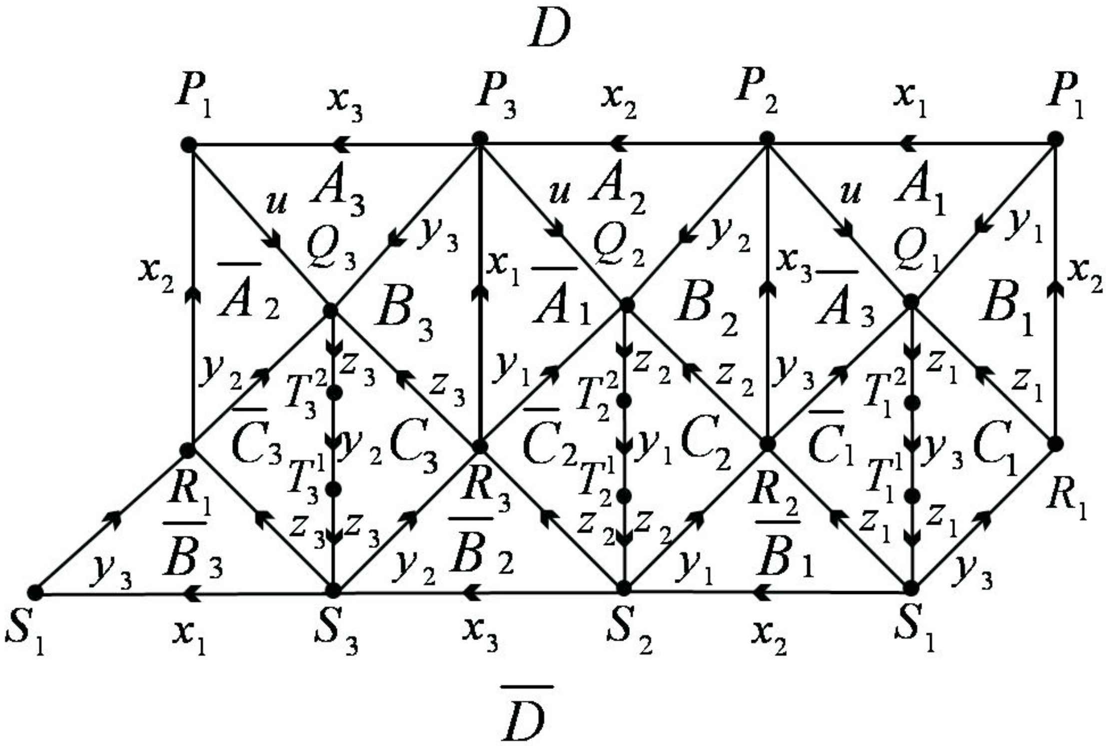

Figure 6: The construction of manifold $M_3(5,3)$.

for $\varphi_3(5,2)$

$$
\begin{aligned}
(y_i):&\quad P_iQ_i \xrightarrow{a_i} R_{i+2}Q_{i+1} \xrightarrow{c_{i+1}^{-1}} T_{i+1}^1S_{i+1} \xrightarrow{c_{i+1}^{-1}} Q_{i+1}T_{i+1}^2 \xrightarrow{c_{i+1}^{-1}} S_{i+1}R_{i+1} \xrightarrow{b_i^{-1}} P_iQ_i;\\
(z_i):&\quad R_iQ_i \xrightarrow{b_i} S_iR_{i+1} \xrightarrow{c_i^{-1}} T_i^2T_i^1 \xrightarrow{c_i^{-1}} R_iQ_i;
\end{aligned}
$$

for $\varphi_3(5,3)$

$$
\begin{aligned}
(y_i):&\quad P_iQ_i \xrightarrow{a_i} R_{i+2}Q_{i+1} \xrightarrow{c_{i+1}^{-1}} T_{i+1}^2T_{i+1}^1 \xrightarrow{c_{i+1}^{-1}} S_{i+1}R_{i+1} \xrightarrow{b_i^{-1}} P_iQ_i;\\
(z_i):&\quad R_iQ_i \xrightarrow{b_i} S_iR_{i+1} \xrightarrow{c_i^{-1}} T_i^1S_i \xrightarrow{c_i^{-1}} Q_iT_i^2 \xrightarrow{c_i^{-1}} R_iQ_i;
\end{aligned}
$$

for $\varphi_3(5,4)$

$$
\begin{aligned}
(y_i):&\quad P_iQ_i \xrightarrow{a_i} R_{i+2}Q_{i+1} \xrightarrow{c_{i+1}^{-1}} Q_{i+1}T_{i+1}^2 \xrightarrow{c_{i+1}^{-1}} T_{i+1}^2T_{i+1}^1 \xrightarrow{c_{i+1}^{-1}} T_{i+1}^1S_{i+1} \xrightarrow{c_{i+1}^{-1}} S_{i+1}R_{i+1} \xrightarrow{b_i^{-1}} P_iQ_i;\\
(z_i):&\quad R_iQ_i \xrightarrow{b_i} S_iR_{i+1} \xrightarrow{c_i^{-1}} R_iQ_i.
\end{aligned}
$$

The separation of the edges of the faces $\mathbf{C_i}$ and $\bar{\mathbf{C}}_i$ into cosets y_{i-1} and z_i is presented in Fig. 7.

The arising quotient spaces $M_3(5,q) = \mathcal{P}_3(5)/\varphi_3(5,q)$ are orientable three-manifolds with the

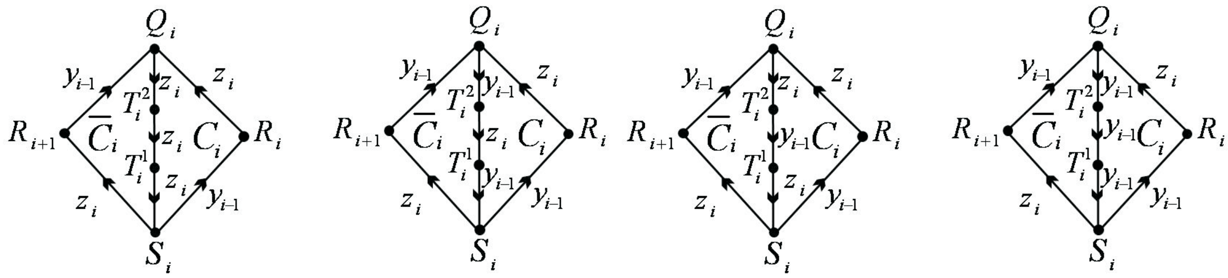

Figure 7: Cases for gluing the faces $\mathbf{C_i}$ and $\bar{\mathbf{C}}_i$.

fundamental groups

$$\pi_1(M_3(5,q)) = \langle a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3; d \quad | \quad a_1a_2a_3 = 1, \quad a_ib_{i+2}d^{-1} = 1,$$
$$R_i(q) = 1, \quad S_i(q) = 1, \quad i = 1,2,3\rangle,$$

where the words $R_i(q)$ and $S_i(q)$ are defined by q as follows:

$$\begin{array}{llll} R_i(1) = a_ic_{i+1}^{-1}b_i^{-1}, & R_i(2) = a_ic_{i+1}^{-3}b_i^{-1}, & R_i(3) = a_ic_{i+1}^{-2}b_i^{-1}, & R_i(4) = a_ic_{i+1}^{-4}b_i^{-1}, \\ S_i(1) = b_ic_i^{-4}, & S_i(2) = b_ic_i^{-2}, & S_i(3) = b_ic_i^{-3}, & S_i(4) = b_ic_i^{-1}. \end{array}$$

For every q manifold $M_3(5,q)$ is a three-fold cyclic covering of the lens space $L(5,q)$ branched over a two-component link. This covering corresponds to the order 3 cyclic symmetry ρ_3 of $M_3(5,q)$ induced by the cyclic symmetry of $\mathcal{P}_3(5)$, since the gluing rule $\varphi_3(5,q)$ is symmetric. The Heegaard diagram for the underlying space of the orbifold $M_3(5,q)/\rho_3$ is presented in Fig. 8.

The dashed lines in the diagram correspond to the two components of the singular set. After applying Singer moves, this diagram transforms to the standard Heegaard diagram for the lens space $L(5,q)$ presented in 8, where the dashed line denotes one of the components of the singular set.

Describing the manifolds $M_3(5,q)$ by fundamental polyhedra admits us to go to their triangulations, and to use the computer program Recognizer [30] for finding topological and geometric invariants. The results of computer calculations of the hyperbolic volumes and homology groups for $M_3(5,q)$ are given in the table:

manifold	volume	homology group
$M_3(5,1)$	6.882614782119...	$\mathbb{Z}_3 \oplus \mathbb{Z}_{45}$
$M_3(5,2)$	6.332666642499...	$\mathbb{Z}_{15}$
$M_3(5,3)$	6.602288090425...	$\mathbb{Z}_{15}$
$M_3(5,4)$	6.424381941185...	$\mathbb{Z}_3 \oplus \mathbb{Z}_{45}$

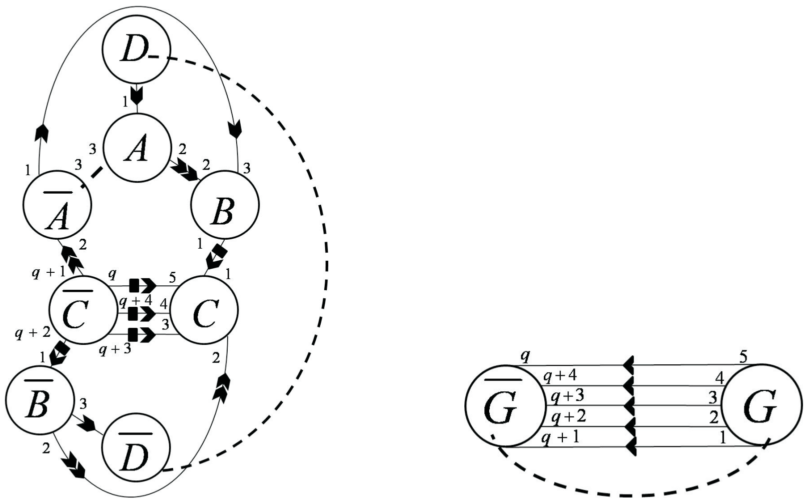

Figure 8: Heegaard diagrams of the support of the orbifold $M_3(5, q)/\rho_3$ and of the lens space $L(5, q)$.

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