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HYPERBOLIC VOLUMES OF FIBONACCI MANIFOLDS

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This article is devoted to the study of three-dimensional compact orientable hyperbolic manifolds connected with the Fibonacci groups. The Fibonacci groups

$$F(2,m) = \langle x_1, x_2, \dots, x_m : x_i x_{i+1} = x_{i+2}, i \bmod m\rangle$$

were introduced by J. Conway [1]. The first natural question connected with these groups was whether they are finite or not [1]. It is known from [2–6] that the group $F(2,m)$ is finite if and only if $m = 1, 2, 3, 4, 5, 7$. Some algebraic generalizations of the groups $F(2,m)$ were considered in [7].

A new stage in studying the Fibonacci groups began with [5], where it was shown that the group $F(2,2n)$, $n \geq 4$, is isomorphic to a discrete cocompact subgroup of $\mathrm{PSL}_2(\mathbb{C})$, the full group of orientation-preserving isometries of the Lobachevskiĭ space $\mathbb{H}^3$. Moreover, the group $F(2,6)$ is isomorphic to a three-dimensional affine group.

The hyperbolic manifolds $M_n = \mathbb{H}^3/F(2,2n)$, $n \geq 4$, uniformized by Fibonacci groups are referred to as the *Fibonacci manifolds.*

It was shown in [8] that the manifold M_n is the n-fold cyclic covering of the three-dimensional sphere $\mathbb{S}^3$ branched over the figure-eight knot. We note that M_n are isometric to the hyperbolic manifolds described in [9].

In the present article we continue studying the algebraic, topological, and arithmetic properties of the Fibonacci manifolds. We establish that the hyperbolic volumes of the manifolds M_n agree with the volumes of the noncompact hyperbolic manifolds arising from complementing some well-known knots and links. In consequence it is shown that there are arithmetic and nonarithmetic manifolds with the same hyperbolic volume.

§ 1. Hyperbolic Volumes. The Thurston-Jørgensen Theorem

In this section we recall some properties of the volumes of hyperbolic manifolds. An *n-dimensional hyperbolic manifold* is thought of as the quotient space $M^n = \mathbb{H}^n/\Gamma$, where Γ is a fixed-point-free discrete group of isometries of the Lobachevskiĭ space $\mathbb{H}^n$. The notions of hyperbolic area and hyperbolic volume in $\mathbb{H}^2$ and $\mathbb{H}^3$ are naturally carried over to M^2 and M^3. Further we consider the set $\mathcal{M}^n$, $n = 2, 3$, of all n-dimensional orientable hyperbolic manifolds of finite volume.

Consider the volume function $v_n : \mathcal{M}^n \to \mathbb{R}$, $n = 2, 3$, that associates the hyperbolic volume $vol(M^n)$ with each manifold $M^n \in \mathcal{M}^n$. It is worth observing that the volume functions v_2 and v_3 have essentially different properties.

The two-dimensional case is completely described by the Gauss-Bonnet theorem. If M^2 is a hyperbolic surface of genus g with k points removed, then

$$vol(M^2) = 2\pi\,(2g - 2 + k).$$

Therefore, the range of the function v_2 is a discrete set of the form $2\pi\mathbb{N}$, where $\mathbb{N}$ is the set of positive integers (see Fig. 1):

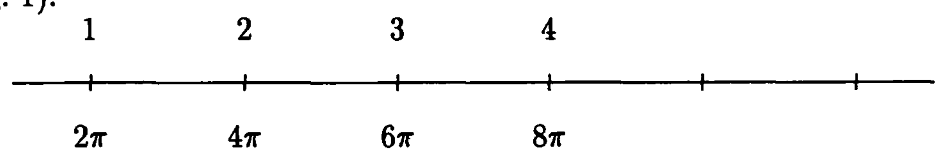

Fig. 1

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Given $v_0 = 2\pi n_0$, $n_0 \in \mathbb{N}$, there are only finitely many nonhomeomorphic surfaces M^2 with area $vol(M^2) = v_0$. All of them satisfy the equality

$$2g - 2 + k = n_0.$$

In particular, given an even $n_0 \in \mathbb{N}$, there are compact and noncompact surfaces with the same area $v_0 = 2\pi n_0$.

In the three-dimensional case the following remarkable theorem of Thurston and Jørgensen is valid: *the set of the volumes of three-dimensional hyperbolic manifolds is a well-ordered subset of type ω^ω in the real line.* This set is plotted schematically in Fig. 2, where some well-known values of the function v_3 are listed.

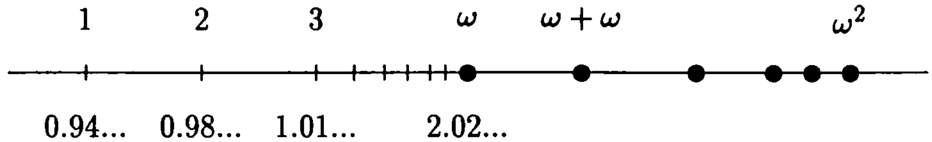

Fig. 2

In particular, it follows from the Thurston-Jørgensen theorem that there exists a three-dimensional hyperbolic manifold of the least volume. Some conjecture on the structure of the initial segment of the set of volumes was suggested in [10]. The manifold constructed independently by J. Weeks [11] and S. V. Matveev and A. T. Fomenko [10] has the least volume, $0.94\ldots$, among the manifolds known so far. The manifold obtained by W. Thurston [12] by the (5,1)-Dehn surgery on the figure-eight knot has the second known volume $0.98\ldots$. The third known value $1.01\ldots$ is equal to the volume of the Meyerhoff-Neumann manifold [13]. We point out that this value is not on the list of [10]. The minimal manifold among known noncompact hyperbolic manifolds is the complement of the figure-eight knot. Its volume equals $2.02\ldots$ and corresponds to the first limit ordinal number in the set of volumes.

In [12] W. Thurston constructed two noncompact manifolds with the different number of cusps, but with the same volume which corresponds to a limit ordinal of the set ω^ω. In the same article he posed the question of existence of a compact hyperbolic manifold whose volume corresponds to a limit ordinal. Below (see the theorem in § 5) we show that the compact Fibonacci manifolds enjoy this property.

§ 2. Fibonacci Manifolds as Branched Coverings

It was shown in [8] that the Fibonacci manifold M_n can be represented as the n-fold cyclic covering of the three-dimensional sphere $\mathbb{S}^3$ branched over the figure-eight knot (see Fig. 3). It means that M_n is the n-fold covering of the orbifold $\mathcal{O}(n)$ whose underlying space is $\mathbb{S}^3$ and whose singular set is the figure-eight knot with index n.

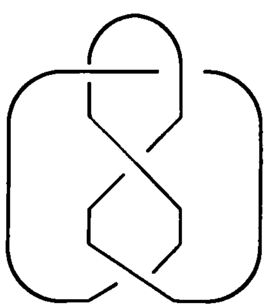

Fig. 3

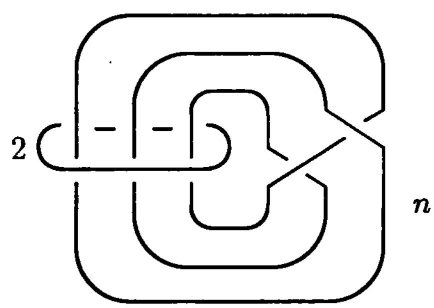

Fig. 4

The orbifold $\mathcal{O}(n)$ has a rotational symmetry of order 2 whose set of fixed points is disjoint from the singular set of the orbifold. After factorizing by this symmetry we obtain the orbifold $6_2^2(2, n)$

with underlying space $\mathbb{S}^3$ and singular set the link 6_2^2 (in notations of [14]) of two components with indices 2 and n (see Fig. 4).

The above implies that the following diagram of coverings holds for the Fibonacci manifolds M_n and the orbifolds $\mathcal{O}(n)$ and $6_2^2(2,n)$ (see Fig. 5):

$$\begin{array}{c} M_n \\ \downarrow n \\ \mathcal{O}(n) \\ \downarrow 2 \\ 6_2^2(2,n) \end{array}$$

Fig. 5

Therefore, the hyperbolic volumes satisfy the relation

$$vol(M_n) = nvol(\mathcal{O}(n)) = 2nvol(6_2^2(2,n)). \tag{1}$$

In the general case, denote by $6_2^2(m,n)$, $m,n \in \mathbb{N} \cup \{\infty\}$, the orbifold with underlying space $\mathbb{S}^3$ and singular set the link 6_2^2 of two components with indices m and n. Observe that the orbifold $6_2^2(m,n)$ can be obtained by the generalized Dehn surgery with parameters $(m,0)$ and $(n,0)$ on the two components of the link 6_2^2. The index ∞ indicates the removal of the corresponding component. In this case we deal with a noncompact orbifold.

Now, consider noncompact manifolds connected with the link 6_2^2. Denote by Th_n, $n \geq 2$, the closed 3-strings braid $\left(\sigma_1\sigma_2^{-1}\right)^n$. Observe that the members of the family Th_n are well known. In particular, Th_2 is the figure-eight knot, Th_3 are the Borromean rings, Th_4 is the Turk's head knot 8_{18} and Th_5 is the knot 10_{123} in the notation of [14]. It was shown in [12] that the manifolds $\mathbb{S}^3 \setminus Th_n$, $n \geq 2$, are hyperbolic and can be represented as the n-fold cyclic coverings of the orbifold $6_2^2(n,\infty)$. In particular, for the hyperbolic volumes we have

$$vol\left(\mathbb{S}^3 \setminus Th_n\right) = nvol(6_2^2(n,\infty)). \tag{2}$$

The values of the volumes in (1) and (2) will be calculated in § 3 and § 4.

§ 3. Volumes of Compact Orbifolds and Manifolds

In this section, we calculate the volumes of the above-introduced compact hyperbolic orbifolds by means of the Lobachevskiĭ function.

We recall that an ideal tetrahedron T in $\mathbb{H}^3$ with four ideal vertices is described completely (up to isometry) by a single complex parameter z with $\operatorname{Im} z > 0$. In this case the dihedral angles of the tetrahedron $T = T_z$ equal $\arg z$, $\arg \frac{z-1}{z}$, and $\arg \frac{1}{1-z}$; and each value occurs twice for a pair of opposite edges.

It is well known [15, 16] that the volume of the ideal tetrahedron T_z is given by

$$vol(T_z) = \Lambda(\arg z) + \Lambda\left(\arg \frac{z-1}{z}\right) + \Lambda\left(\arg \frac{1}{1-z}\right), \tag{3}$$

where

$$\Lambda(x) = -\int\limits_0^x \ln|2\sin\zeta|\, d\zeta$$

is the Lobachevskiĭ function. We recall some properties of the function $\Lambda(x)$:

$$\Lambda(-x) = -\Lambda(x), \quad \Lambda(x+\pi) = \Lambda(x).$$

Below we express the hyperbolic volumes of the orbifolds $\mathcal{O}(n)$ and $6_2^2(2,n)$ and the manifolds M_n in terms of the Lobachevskiĭ function.

Lemma 1. *For $n \geq 4$ the hyperbolic volume of the orbifold $\mathcal{O}(n)$ is equal to*

$$vol(\mathcal{O}(n)) = 2(\Lambda(\beta + \delta) + \Lambda(\beta - \delta)),$$

where $\delta = \pi/n$ and $\beta = \frac{1}{2}\arccos(\cos(2\delta) - 1/2)$.

PROOF. Consider the orbifold $\mathcal{O}(n)$ as the result of performing the generalized $(n, 0)$-Dehn surgery to the complement of the figure-eight knot. By analogy to [17], the orbifold $\mathcal{O}(n)$ can be obtained by the completion of the noncomplete hyperbolic structure on the union of two ideal tetrahedra T_z and T_w whose complex parameters z and w satisfy the conditions

$$zw(z-1)(w-1) = 1, \quad (w(1-z))^n = 1, \quad \operatorname{Im} z > 0, \quad \operatorname{Im} w > 0. \tag{4}$$

From here we obtain the following equation in z:

$$z^2 + \left(2i \sin \frac{2\pi}{n} - 1\right) z + e^{-2\pi i/n} = 0.$$

It has the solution

$$z = \frac{1}{2} - i \sin\left(\frac{2\pi}{n}\right) \pm i \sqrt{1 - \left(\cos\left(\frac{2\pi}{n}\right) - \frac{1}{2}\right)^2}.$$

Setting $\varphi = 2\pi/n$, $n \geq 4$, we have $-1/2 \leq \cos\varphi - 1/2 < 1/2$. Choose ψ, $0 < \psi < \pi$, such that $\cos\psi = \cos\varphi - 1/2$. Then $z = 1/2 + i(\pm\sin\psi - \sin\varphi)$. By virtue of the condition $\operatorname{Im} z > 0$, we choose the solution with the plus sign:

$$z = \frac{1}{2} + i\,(\sin\psi - \sin\varphi)\,. \tag{5}$$

Therefore, from (4) we have

$$w = \frac{\cos\varphi + i\sin\varphi}{1/2 - i(\sin\psi - \sin\varphi)}. \tag{6}$$

For $n \geq 5$ expressions (5) and (6) satisfy conditions (4). In the case $n = 4$ we have $\operatorname{Im} z < 0$ and $vol(T_z) < 0$. It means that the volume of the orbifold equals the difference of the volumes of the tetrahedra T_w and T_z.

For finding the volume of the ideal tetrahedron T_z with complex parameter z, we shall calculate the values of the following arguments of complex numbers:

$$\arg z, \quad \arg\frac{z-1}{z}, \quad \arg\frac{1}{1-z}.$$

Proposition 1. *With the above notation, the following equalities hold:*

$$\arg z = \arg\frac{1}{1-z} = \frac{\pi - \varphi - \psi}{2}, \quad \arg\frac{z-1}{z} = \varphi + \psi.$$

PROOF. By straightforward computation from (5) we have

$$\tan(\arg z) = \frac{\sin\psi - \sin\varphi}{1/2} = \frac{\sin\psi - \sin\varphi}{\cos\varphi - \cos\psi} = \cot\frac{\psi + \varphi}{2} = \tan\frac{\pi - \varphi - \psi}{2}.$$

Similarly, for the second complex parameter we obtain

$$\frac{1}{1-z} = \frac{1}{1/2 - i(\sin\psi - \sin\varphi)} = \frac{1/2 + i(\sin\psi - \sin\varphi)}{1/4 + (\sin\psi - \sin\varphi)^2},$$

$$\tan\left(\arg\frac{1}{1-z}\right) = \frac{\sin\psi - \sin\varphi}{1/2} = \tan\frac{\pi-\varphi-\psi}{2}.$$

Therefore,

$$\arg z = \arg\frac{1}{1-z} = \frac{\pi-\varphi-\psi}{2}.$$

To prove the remaining part of Proposition 1, observe that

$$\arg z + \arg\frac{z-1}{z} + \arg\frac{1}{1-z} = \pi.$$

Hence,

$$\arg\frac{z-1}{z} = \pi - (\pi-\varphi-\psi) = \varphi+\psi,$$

which completes the proof.

From Proposition 1 and formula (3) we infer that

$$vol(T_z) = \Lambda(\varphi+\psi) + 2\Lambda\left(\frac{\pi-\varphi-\psi}{2}\right).$$

Now, we turn to considering the tetrahedron T_w with complex parameter w.

Proposition 2. *With the above notation, the following equalities hold*:

$$\arg w = \arg\frac{1}{1-w} = \frac{\pi-\psi+\varphi}{2}, \quad \arg\frac{w-1}{w} = \psi-\varphi.$$

PROOF. Using Proposition 1, from (6) we obtain

$$\arg w = \arg\frac{e^{i\varphi}}{1-z} = \varphi + \frac{\pi-\varphi-\psi}{2} = \frac{\pi-\psi+\varphi}{2}.$$

Similarly,

$$\frac{w-1}{w} = 1-\frac{1}{w} = 1-\frac{1/2 - i\,(\sin\psi-\sin\varphi)}{\cos\varphi + i\sin\varphi} = \frac{\cos\varphi - 1/2 + i\sin\psi}{\cos\varphi+i\sin\varphi} = \frac{\cos\psi + i\sin\psi}{\cos\varphi + i\sin\varphi} = e^{i(\psi-\varphi)},$$

and therefore $\arg((w-1)/w) = \psi-\varphi$. Thus,

$$\arg\frac{1}{1-w} = \pi - \arg w - \arg\frac{w-1}{w} = \pi - \frac{\pi-\psi+\varphi}{2} - (\psi-\varphi) = \frac{\pi-\psi+\varphi}{2},$$

which completes the proof.

From Proposition 2 and formula (3) we infer that

$$vol(T_w) = \Lambda(\psi-\varphi) + 2\Lambda\left(\frac{\pi+\varphi-\psi}{2}\right).$$

The volume of the orbifold $\mathcal{O}(n)$ equals

$$\begin{aligned} vol(\mathcal{O}(n)) &= vol(T_z) + vol(T_w) \\ &= 2\Lambda\left(\frac{\pi-\psi-\varphi}{2}\right) + \Lambda(\varphi+\psi) + 2\Lambda\left(\frac{\pi-\psi+\varphi}{2}\right) + \Lambda(\psi-\varphi) \\ &= 2\left(\Lambda\left(\frac{\psi+\varphi}{2}\right) + \Lambda\left(\frac{\psi-\varphi}{2}\right)\right). \end{aligned}$$

In the last equality we used the following property of the Lobachevskiĭ function [16]:

$$2\Lambda(x) = \Lambda(2x) + 2\Lambda\left(\frac{\pi}{2}-x\right).$$

We return to the proof of Lemma 1. Assign $\delta = \varphi/2$ and $\beta = \psi/2$. Then $\delta = \pi/n$ and $\beta = {}^1\!/_2 \arccos(\cos(2\delta) - 1/2)$. Therefore, $vol(\mathcal{O}(n)) = 2(\Lambda(\beta+\delta) + \Lambda(\beta-\delta))$, which completes the proof of Lemma 1.

From the diagram of coverings (Fig. 5) and Lemma 1 we obtain

Corollary 1. *For $n \geq 4$ the hyperbolic volume of the Fibonacci manifold M_n is equal to*

$$vol(M_n) = 2n(\Lambda(\beta+\delta) + \Lambda(\beta-\delta)),$$

where $\delta = \pi/n$ and $\beta = {}^1\!/_2 \arccos(\cos(2\delta) - 1/2)$.

Corollary 2. *For $n \geq 4$ the orbifold $6_2^2(2,n)$ is hyperbolic and*

$$vol(6_2^2(2,n)) = \Lambda(\beta+\delta) + \Lambda(\beta-\delta),$$

where $\delta = \pi/n$ and $\beta = {}^1\!/_2 \arccos(\cos(2\delta) - 1/2)$.

For some values of n the arguments of the Lobachevskiĭ function in Lemma 1 admit simpler expressions.

Corollary 3. *For $n = 4$ the following equality holds:*

$$vol(\mathcal{O}(4)) = \frac{3}{2}\Lambda\left(\frac{\pi}{3}\right).$$

PROOF. For $n = 4$ we have $\delta = \pi/4$, $\beta = \pi/3$. In this case Lemma 1 implies

$$vol(\mathcal{O}(4)) = 2\left(\Lambda\left(\frac{\pi}{3}+\frac{\pi}{4}\right) + \Lambda\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\right) = 2\left(\Lambda\left(\frac{7\pi}{12}\right) + \Lambda\left(\frac{\pi}{12}\right)\right).$$

Recall that the Lobachevskiĭ function has the following property [15]:

$$\Lambda\left(m\theta\right) = m\sum_{k=0}^{m-1}\Lambda\left(\theta+\frac{k\pi}{m}\right). \tag{7}$$

For $m = 4$ and $\theta = \pi/12$, from (7) we obtain $\Lambda(\pi/4) = 4(\Lambda(\pi/12) + \Lambda(\pi/3) + \Lambda(7\pi/12) - \Lambda(\pi/6))$ by straightforward computation. For $m = 2$ and $\theta = \pi/6$, from (7) we have $2\Lambda(\pi/6) = 3\Lambda(\pi/3)$; hence, $3\Lambda(\pi/3) = 4(\Lambda(7\pi/12) + \Lambda(\pi/12))$, which completes the proof of the corollary.

Corollary 4. *For $n = 6$ the following equality holds:*

$$vol(\mathcal{O}(6)) = \frac{8}{3}\Lambda\left(\frac{\pi}{4}\right).$$

PROOF. For $n = 6$ we have $\delta = \pi/6$ and $\beta = \pi/4$. By Lemma 1,

$$vol(\mathcal{O}(6)) = 2\left(\Lambda\left(\frac{\pi}{4}+\frac{\pi}{6}\right) + \Lambda\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\right) = 2\left(\Lambda\left(\frac{5\pi}{12}\right) + \Lambda\left(\frac{\pi}{12}\right)\right).$$

For $m = 3$ and $\theta = \pi/12$, from (7) we obtain $4\Lambda(\pi/4) = 3(\Lambda(5\pi/12) + \Lambda(\pi/12))$ by straightforward computation, and the corollary follows.

A similar argument yields

Corollary 5. *For $n = 10$ the following equality holds:*

$$vol(\mathcal{O}(10)) = 2\left(\Lambda\left(\frac{3\pi}{10}\right) + \Lambda\left(\frac{\pi}{10}\right)\right).$$

§ 4. Volumes of Noncompact Orbifolds and Manifolds

To calculate the volume of the manifold $\mathbb{S}^3 \setminus Th_n$, we need the following

Lemma 2. *For $n \geq 2$ the orbifold $6_2^2(n,\infty)$ is hyperbolic and*

$$vol(6_2^2(n,\infty)) = 4(\Lambda(\alpha+\gamma)+\Lambda(\alpha-\gamma)),$$

where $\gamma = \pi/2n$ and $\alpha = {}^1\!/\!_2 \arccos(\cos(2\gamma)-1/2)$.

PROOF. Choose generators a and τ of the fundamental group $\pi_1(\mathbb{S}^3 \setminus 6_2^2)$ in the manner indicated in Fig. 6.

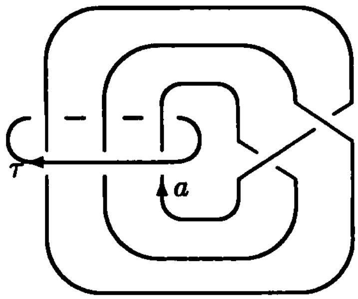

Fig. 6

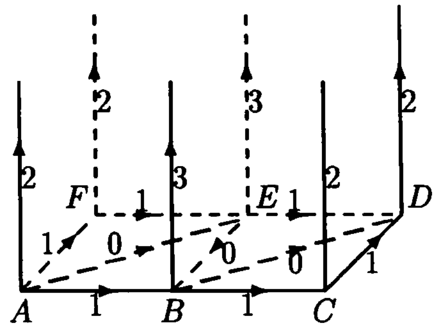

Fig. 7

Using the Wirtinger algorithm [18], we obtain the following presentation for $\pi_1(\mathbb{S}^3 \setminus 6_2^2)$:

$$\langle a,\tau \mid (\tau a^{-1}\tau a\tau^{-1}a\tau^{-1})(\tau^2 a^{-1}\tau a\tau^{-1}a\tau^{-2})(\tau a^{-1}\tau a\tau^{-1}a\tau^{-1})^{-1} = a\rangle.$$

With new generators x and y such that $a = x^{-1}y^{-1}$ and $\tau = y^{-1}$, the group has presentation

$$\begin{aligned}\pi_1(\mathbb{S}^3 \setminus 6_2^2) &= \langle x,y \mid (xy^{-1}x^{-2})(y^{-1}xy^{-1}x^{-2}y)(xy^{-1}x^{-2})^{-1} = x^{-1}y^{-1}\rangle \\ &= \langle x,y \mid y^{-1}(x^2yx^{-1}yx^2)^{-1}y(x^2yx^{-1}yx^2) = 1\rangle \\ &= \langle x,y \mid (x^2yx^{-1}yx^2)y(x^2yx^{-1}yx^2)^{-1}y^{-1} = 1\rangle. \end{aligned} \qquad (8)$$

Demonstrate that this group is isomorphic to a discrete group of isometries of the Lobachevskiĭ space. Consider some polyhedron in $\mathbb{H}^3$ composed of four ideal regular tetrahedra (see Fig. 7). Denote the ideal vertices of the polyhedron by A, B, C, D, E, F, and ∞. Let u, v, t, and r be isometries of the hyperbolic space $\mathbb{H}^3$ which identify the following faces of the polyhedron pairwise:

$$\begin{aligned} u:&\quad ABE &\to&\quad EDB,\\ v:&\quad AEF &\to&\quad BDC,\\ t:&\quad AF\infty &\to&\quad CD\infty,\\ r:&\quad ABC\infty &\to&\quad FED\infty.\end{aligned}$$

Let Γ be the group generated by u, v, t, and r. By the Poincaré theorem, the complete list of relations for Γ is as follows:

$$\begin{aligned} 0:&\quad u^2 = v,\\ 1:&\quad u = rvt^{-1}vr,\\ 2:&\quad trt^{-1}r^{-1} = 1,\\ 3:&\quad rr^{-1} = 1.\end{aligned}$$

Moreover, the ideal vertices of the polyhedron fall into two equivalence classes whose link diagrams are shown in Fig. 8.

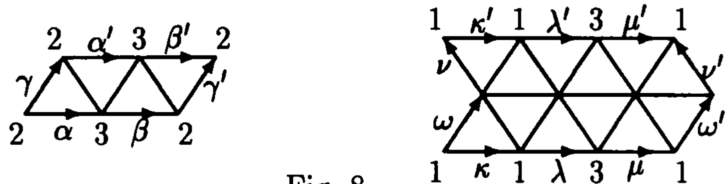

Fig. 8.

The polygons in Fig. 8 consist of regular Euclidean triangles. Their edges are pairwise identified by Euclidean isometries. Consequently, the two cusps have a complete hyperbolic structure and the group

$$\begin{aligned}\Gamma &= \langle u, v, t, r \mid u^2 = v,\ u = rvt^{-1}vr,\ trt^{-1}r^{-1} = 1\rangle \\ &= \langle u, t, r \mid u = ru^2t^{-1}u^2r,\ trt^{-1}r^{-1} = 1\rangle \\ &= \langle u, r \mid (u^2ru^{-1}ru^2)r(u^2ru^{-1}ru^2)^{-1}r^{-1} = 1\rangle\end{aligned}$$

has the polyhedron in Fig. 7 as a fundamental set in $\mathbb{H}^3$. As is easily seen from (8), the correspondence $x \to u$, $y \to r$ determines an isomorphism between the groups $\pi_1(\mathbb{S}^3 \setminus 6_2^2)$ and Γ.

Now we turn to studying the orbifold $6_2^2(n,\infty)$ which results from applying the generalized $(n,0)$-Dehn surgery to one of two cusps of the hyperbolic manifold $\mathbb{S}^3 \setminus 6_2^2$. It means that the orbifold $6_2^2(n,\infty)$ can be obtained by completing the noncomplete hyperbolic structure on the union of four ideal tetrahedra (see Fig. 7) whose complex parameters z_1, z_2, z_3, and z_4 satisfy some system of algebraic equations. For finding these equations, consider the link diagrams of the two cusps of the manifold $\mathbb{S}^3 \setminus 6_2^2$ (see Fig. 9 and Fig. 10, where $z' = (z-1)/z$ and $z'' = 1/(1-z)$).

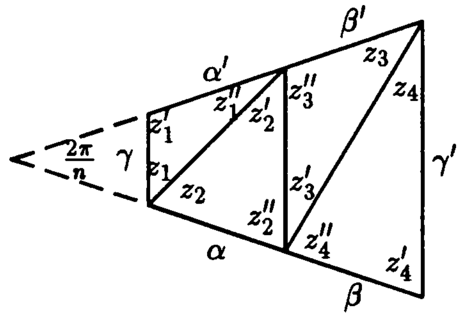

Fig. 9. The generalized $(n,0)$-surgery on the cusp of the manifold $\mathbb{S}^3 \setminus 6_2^2$.

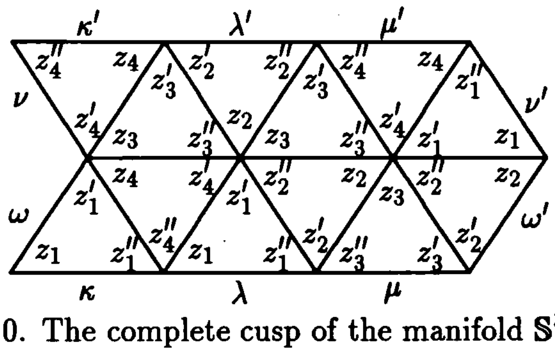

Fig. 10. The complete cusp of the manifold $\mathbb{S}^3 \setminus 6_2^2$.

Looking at Fig. 9 and Fig. 10, we obtain the following system of equations:

$$\begin{cases} (z_1-1)z_2z_3(z_4-1)=1, \\ z_1(z_2-1)(z_3-1)z_4=1, \\ (z_2(1-z_1))^n=1, \\ z_3(1-z_1)=1, \\ \operatorname{Im} z_i>0, \quad i=1,2,3,4. \end{cases} \tag{9}$$

Denoting $\zeta = 1/(1-z_1)$, from (9) we have

$$z_1=\frac{\zeta-1}{\zeta}, \quad z_2=e^{2\pi i/n}\zeta, \quad z_3=\zeta, \quad z_4=1-\frac{1}{e^{2\pi i/n}\zeta}. \tag{10}$$

Furthermore, system (9) reduces to the equation

$$\left(e^{\pi i/n}\zeta+\frac{1}{e^{\pi i/n}\zeta}-(e^{\nu i}+e^{-\nu i})\right)^2=1,$$

where $\nu=\pi/n$. Choose θ such that $e^{\pi i/n}\zeta=e^{\theta i}$. Then $(2\cos\theta-2\cos\nu)^2=1$, and hence $\cos\theta = \cos\nu \pm 1/2$. Since $\cos\theta \le 1$, we choose the solution with the minus sign: $\cos\theta = \cos\nu - 1/2$. Substituting $\zeta = e^{i(\theta-\nu)}$ into (10), we arrive at

$$z_1=1-1/e^{i(\theta-\nu)}, \quad z_2=e^{i(\theta+\nu)}, \quad z_3=e^{i(\theta-\nu)}, \quad z_4=1-1/e^{i(\theta+\nu)}.$$

Straightforward computation yields the following result:

Proposition 3. *With the above notation, the following equalities hold:*

(i) $\arg z_1=\arg\frac{z_1-1}{z_1}=\frac{\pi-\theta+\nu}{2}$, $\arg\frac{1}{1-z_1}=\theta-\nu$;
(ii) $\arg z_2=\theta+\nu$, $\arg\frac{z_2-1}{z_2}=\arg\frac{1}{1-z_2}=\frac{\pi-\theta-\nu}{2}$;
(iii) $\arg z_3=\theta-\nu$, $\arg\frac{z_3-1}{z_3}=\arg\frac{1}{1-z_3}=\frac{\pi-\theta+\nu}{2}$;
(iv) $\arg z_4=\arg\frac{z_4-1}{z_4}=\frac{\pi-\theta-\nu}{2}$, $\arg\frac{1}{1-z_4}=\theta+\nu$.

Since a tetrahedron in $\mathbb{H}^3$ is determined uniquely from its dihedral angles, we see that $T_{z_1}=T_{z_3}$ and $T_{z_2}=T_{z_4}$. Therefore, using (3) we conclude:

$$\begin{aligned} vol(6_2^2(n,\infty)) &= 2\left(\Lambda(\theta+\nu)+\Lambda(\theta-\nu)+2\Lambda\left(\frac{\pi-\theta-\nu}{2}\right)+2\Lambda\left(\frac{\pi-\theta+\nu}{2}\right)\right) \\ &= 4\left(\Lambda\left(\frac{\theta+\nu}{2}\right)+\Lambda\left(\frac{\theta-\nu}{2}\right)\right). \end{aligned}$$

To complete the proof of Lemma 2, we assign $\gamma=\nu/2$ and $\alpha=\theta/2$. Then $\gamma=\pi/2n$ and $\alpha = {}^1\!/\!_2 \arccos(\cos(2\gamma)-1/2)$. Therefore, the expression for the volume of the orbifold takes the form

$$vol(6_2^2(n,\infty)) = 4(\Lambda(\alpha+\gamma)+\Lambda(\alpha-\gamma)).$$

The proof of Lemma 2 is complete.

In view of (2), we arrive at

Corollary 6. *For $n\ge 2$ the volume of the noncompact hyperbolic manifold $\mathbb{S}^3\setminus Th_n$ equals*

$$vol(\mathbb{S}^3\setminus Th_n)=4n(\Lambda(\alpha+\gamma)+\Lambda(\alpha-\gamma)),$$

where $\gamma=\pi/2n$ and $\alpha={}^1\!/\!_2\arccos(\cos(2\gamma)-1/2)$.

§ 5. Volumes of Fibonacci Manifolds

The principal result of the present article is:

Theorem 1. *For $n \geq 2$ the following equality holds*

$$vol(M_{2n}) = vol(\mathbb{S}^3 \setminus Th_n).$$

PROOF. The claim is a consequence of Lemmas 1 and 2. Namely, we achieve the assertion by applying Corollary 1 to the manifold M_{2n}, $n \geq 2$, and Corollary 6 to the manifold $\mathbb{S}^3 \setminus Th_n$, $n \geq 2$.

Thus, the volumes of the compact Fibonacci manifolds M_{2n} correspond to limit ordinals in the Thurston-Jørgensen theorem. In particular, the following assertions hold:

Corollary 7. *The volume of the manifold M_4 is equal to the volume of the complement of the figure-eight knot.*

Corollary 8. *The volume of the manifold M_6 is equal to the volume of the complement of the Borromean rings.*

Many properties of hyperbolic manifolds are determined by arithmeticity or nonarithmeticity of their fundamental groups [19]. As shown in [5, 8], the manifold M_n is arithmetic for $n = 4, 5, 6, 8, 12$ and nonarithmetic for the other values of n. It is proven in [20] that the figure-eight knot Th_2 is the only arithmetic knot. Furthermore, it is known [21] that the link Th_3 of Borromean rings is arithmetic too.

Corollary 9. *Manifolds with the same volume can be both arithmetic and nonarithmetic:*

n	M_{2n}	$S^3 \setminus Th_n$
2	*arithmetic*	*arithmetic*
3	*arithmetic*	*arithmetic*
4	*arithmetic*	*nonarithmetic*
5	*nonarithmetic*	*nonarithmetic*

We remark that, while discussing Corollary 9, A. Reid kindly informed the authors about the possibility of a number-theoretic approach to the construction of compact and noncompact arithmetic manifolds with the same volume.

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