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ISOMETRIES OF HYPERBOLIC FIBONACCI MANIFOLDS[†)]

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§ 1. Introduction

In the present paper we study the isometry groups of three-dimensional closed orientable hyperbolic manifolds uniformized by the Fibonacci groups. The *Fibonacci groups* $F(2,m)$, $m \geq 3$, were introduced by J. Conway [1] and are defined by the following presentation:

$$F(2,m) = \langle x_1,\dots,x_m \mid x_i x_{i+1} = x_{i+2},\ i = 1,\dots,m\rangle,$$

with all subscripts reduced mod m. Some survey of the results about finiteness, arithmeticity, and generalizations of the Fibonacci groups is given in [2]. Application of topological and geometric methods to studying the Fibonacci groups stems from the paper [3] by H. Helling, A. C. Kim and J. Mennicke wherein they constructed a family of three-dimensional manifolds M_n, $n \geq 2$, such that $\pi_1(M_n) \cong F(2,2n)$. Moreover, it was shown in [3] that these manifolds can be equipped with geometric structures of constant curvature. More exactly, the group $F(2,4) \cong \mathbb{Z}_5$ acts by isometries on the spherical space $\mathbb{S}^3$. In this case the manifold M_2 is the lens space $L(5,2)$. The group $F(2,6)$ is isomorphic to the Euclidean crystallographic group acting on $\mathbb{E}^3$. In this case the manifold M_3 in the Euclidean Hantzsche–Wendt manifold [4]. For $n \geq 4$ the group $F(2,2n)$ is isomorphic to a discrete cocompact subgroup of $\mathrm{PSL}(2,\mathbb{C})$ acting by isometries on the Lobachevskiĭ space $\mathbb{H}^3$ without fixed points. For $n \geq 4$ the manifold M_n is *hyperbolic*; i.e. it has a metric of constant negative curvature. The manifolds M_n of [3] are called the *Fibonacci manifolds*.

We recall some geometric and topological properties of the manifolds M_n. Exact formulae in terms of the Lobachevskiĭ function for the volumes of hyperbolic Fibonacci manifolds were obtained in [5]. It was shown in [6] that for $n \geq 2$ the manifold M_n is the n-fold cyclic covering of the three-dimensional sphere, branched over the figure-eight knot. Moreover [7], for $n \geq 2$ the manifold M_n is the two-fold covering of the three-dimensional sphere, branched over the closed three-strings braid $(\sigma_1\sigma_2^{-1})^n$. In particular, M_2 is two-fold branched over the figure-eight knot, and M_3 is two-fold branched over the three-component link named Borromean rings. In [8] the Fibonacci manifolds were obtained by 1/0-Dehn surgery on the manifold $\mathcal{M}(\phi)$ that is a punctured torus bundle. In this case the homeomorphism ϕ is induced by the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2$.

In the present paper we develop a geometric approach to studding the Fibonacci groups and describe the isometry groups of the hyperbolic Fibonacci manifolds. We recall that, by the rigidity theorem [9], the group $\mathrm{Isom}(M_n)$ of all isometries of the hyperbolic manifold M_n coincides with the group $\mathrm{Out}(F(2,2n))$ of the outer automorphisms of the fundamental group of M_n:

$$F(2,2n) = \langle x_1,\dots,x_{2n} \mid x_i x_{i+1} = x_{i+2},\ i = 1,\dots,2n\rangle.$$

The obvious automorphism $x_i \longrightarrow x_{i+1}$, $i = 1,\dots,2n$, generates a cyclic subgroup $\mathbb{Z}_{2n}$ of the isometry group $\mathrm{Isom}(M_n)$. In [10] N. Kuiper asked whether the group $\mathrm{Isom}(M_n)$ is a cyclic group of order $2n$. The negative answer follows from [7], where it was shown that M_n admits an involution that does

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not belong to the group $\mathbb{Z}_{2n}$. Moreover [8], all foliation-preserving homeomorphisms of $\mathcal{M}(\phi)$ induce homeomorphisms of the Fibonacci manifold. The group of foliation-preserving homeomorphisms of the manifold $\mathcal{M}(\phi)$ can be found in [11] and is of order $8n$.

The main aim of the present paper is to describe the group $\mathrm{Isom}(M_n)$. Precisely, we show in Theorem 3.2 that for $n \geq 4$ the group $\mathrm{Isom}(M_n)$ consists of $8n$ elements and has the presentation

$$\mathrm{Isom}(M_n) = \langle x, y | x^{2n} = y^4 = (yx)^2 = (y^{-1}x)^2 = 1 \rangle.$$

We recall [3, 6] that for $n = 4, 5, 6, 8, 12$ the Fibonacci manifold M_n is arithmetic (in the sense of [12]). For these values of n the theorem was proven in [8] by comparing the volumes of the Fibonacci manifolds, calculated in [5], and the estimates of [12] for the volumes of arithmetic orbifolds over a given field. In the present paper the proof of the theorem is given for all $n \geq 6$. We use the geometric approach that is based on the results of [13, 14] together with the ideas that appeared in [15–17]. Observe that the group $\mathrm{Isom}(M_n)$ is the group $(4, 2n|2, 2)$ in the notations of [18, p. 159]. This group can also be presented as the semidirect product $\mathbb{Z}_n \leftthreetimes \mathbb{D}_8$, where $\mathbb{Z}_n$ is the cyclic group of order n and $\mathbb{D}_8$ is the dihedral group of order 8.

The structure of the paper is as follows. In Section 2 we recall the classical results by M. Dehn [19] and W. Magnus [20] on the symmetries of the figure-eight knot and show that these symmetries induce isometries of the Fibonacci manifolds (Theorem 2.1). In Section 3 we study discrete extensions of the group $F(2, 2n)$ and compute the normalizer of this group in the full group of isometries of the Lobachevskiĭ space (Theorem 3.1). This in particular enables us to describe the group $\mathrm{Isom}(M_n)$ (Theorem 3.2). In Section 4 we consider the orbifolds resulting from the action of isometries on the Fibonacci manifolds.

§ 2. Symmetries of the Figure-Eight Knot

From the presentation of the Fibonacci group

$$F(2, 2n) = \langle x_1, \dots, x_{2n} \mid x_i x_{i+1} = x_{i+2},\ i = 1, \dots, 2n \rangle$$

we easily find an automorphism ρ of order n:

$$\rho : x_i \longrightarrow x_{i+2}, \quad i = 1, \dots, 2n,$$

where all subscripts are reduced mod $2n$. Consider the extension Γ_n of $F(2, 2n)$ by ρ which is the semidirect product of $F(2, 2n)$ and the cyclic group $\langle \rho \rangle$ of order n. Then

$$\begin{aligned}\Gamma_n &\cong F(2, 2n) \leftthreetimes \langle \rho \rangle \\ &\cong \langle x_1, \dots, x_{2n}, \rho \mid \rho^n = 1,\ \rho^{-1} x_i \rho = x_{i+2},\ x_i x_{i+1} = x_{i+2},\ i = 1, \dots, 2n \rangle \\ &\cong \langle x_1, x_2, \rho \mid \rho^n = 1,\ \rho^{-1} x_1 \rho = x_1 x_2,\ \rho^{-1} x_2 \rho = x_2 x_1 x_2 \rangle.\end{aligned}$$

In particular, the relation $\rho^{-1} x_2 \rho = x_2 \rho^{-1} x_1 \rho$ holds. From this relation we can express x_1:

$$x_1 = \rho x_2^{-1} \rho^{-1} x_2.$$

Thus,

$$\Gamma_n \cong \langle x_2, \rho \mid \rho^n = 1,\ x_2^{-1} \rho^{-1} x_2 \rho = \rho x_2^{-1} \rho^{-1} x_2 x_2 \rangle.$$

Introducing b such that $x_2 = b\rho$, we rewrite the last relation as

$$\rho^{-1} b^{-1} \rho^{-1} b \rho = b^{-1} \rho^{-1} b \rho b.$$

Therefore, we come to the presentation

$$\Gamma_n \cong \langle b, \rho \mid \rho^n = b^n = 1,\ \rho^{-1} [b, \rho] = [b, \rho] b \rangle,$$

where $[b, \rho] = b^{-1}\rho^{-1}b\rho$. It is easy to check by a direct application of the Wirtinger algorithm [21] that

$$\Gamma \cong \langle b, \rho \mid \rho^{-1}[b, \rho] = [b, \rho]b \rangle$$

is the group of the figure-eight knot pictured in Fig. 1, wherein the generators b and ρ are marked. Thus [22], the group Γ_n is isomorphic to the fundamental group of the orbifold $\mathcal{O}(n)$ whose underlying space is the three-dimensional sphere and whose singular set is the figure-eight knot with the singularity index n.

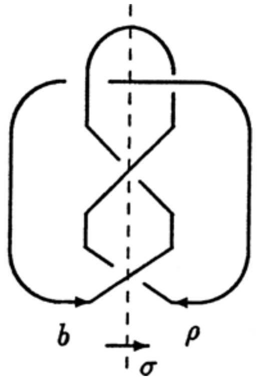


Fig. 1. The figure-eight knot.

Observe the following connection between the generators $x_1, \ldots, x_{2n}$ of the Fibonacci group $F(2, 2n) \cong \pi_1(M_n)$ and the generators b and ρ of the group $\Gamma_n \cong \pi_1(\mathcal{O}(n))$:

$$x_2 = b\rho, \quad x_1 = \rho x_2^{-1}\rho^{-1}x_2 = b^{-1}\rho^{-1}b\rho = [b, \rho].$$

Since $\rho^{-1}x_i\rho = x_{i+2}$, we obtain

$$x_{2i+2} = \rho^{-i}(b\rho)\rho^i, \quad x_{2i+1} = \rho^{-i}[b, \rho]\rho^i,$$

where $i = 0, \ldots, n-1$.

We recall that the symmetries of the figure-eight knot were first studied by M. Dehn in [19], where he demonstrated that the group Γ has eight outer automorphisms forming the dihedral group:

$$\mathbb{D}_8 = \langle \sigma, \tau \mid \sigma^2 = \tau^4 = (\sigma\tau)^2 = 1 \rangle.$$

In this case the action of the generators σ and τ on Γ is as follows:

$$\sigma(b) = \rho, \quad \sigma(\rho) = b, \quad \tau(b) = b^{-1}\rho b, \quad \tau(\rho) = \rho b \rho^{-1}.$$

These studies were continued by W. Magnus [20]. He proved that Γ has no other outer automorphisms. We note that the automorphism σ induces an involution of the figure-eight knot whose axis is pictured by the dotted line in Fig. 1. The automorphism τ induces an orientation-reversing symmetry of order four. The action of the automorphism group $\mathbb{D}_8$ on the generators b and ρ of Γ is given in Table 1. The action of $\mathbb{D}_8$ with respect to another system of generators was described in [23].

TABLE 1

	b	ρ
σ	ρ	b
τ	$b^{-1}\rho b$	$\rho b \rho^{-1}$
τ^2	ρ^{-1}	b^{-1}
τ^3	$\rho b^{-1}\rho^{-1}$	$b^{-1}\rho^{-1}b$
$\sigma\tau$	$\rho b \rho^{-1}$	$b^{-1}\rho b$
$\sigma\tau^2$	b^{-1}	ρ^{-1}
$\sigma\tau^3$	$b^{-1}\rho^{-1}b$	$\rho b^{-1}\rho^{-1}$

Comparing the presentations of Γ and Γ_n, we see that σ and τ induce automorphisms of Γ_n and consequently induce symmetries of the orbifold $\mathcal{O}(n)$. Connection between the symmetries of the figure-eight knot and the isometries of the Fibonacci manifolds is described in the following

Theorem 2.1. *Each symmetry of the figure-eight knot induces an isometry of every hyperbolic Fibonacci manifold.*

PROOF. Consider the generators σ and τ of the symmetry group $\mathbb{D}_8$ which act on b and ρ as above. Since $F(2,2n)$ is the normal subgroup of Γ_n, the images

$$\sigma(x_2) = \sigma(b\rho) = \rho b = x_2 x_1^{-1},$$
$$\tau(x_2) = \tau(b\rho) = b^{-1}\rho b \rho b \rho^{-1} = \rho(x_1^{-1}x_2x_1^{-1})\rho^{-1}$$

belong to $F(2,2n)$. Since for $i = 0, \dots, n-1$ we have the relations

$$x_{2i+2} = \rho^{-i}x_2\rho^i, \quad x_{2i+1} = \rho^{-i}(\rho x_2^{-1}\rho^{-1}x_2)\rho^i;$$

therefore, the images

$$\sigma(x_{2i+2}) = b^{-i}\sigma(x_2)b^i, \quad \sigma(x_{2i+1}) = b^{-i}(b\sigma^{-1}(x_2)b^{-1}\sigma(x_2))b^i,$$
$$\tau(x_{2i+2}) = (\rho b\rho^{-1})^{-1}\tau(x_2)(\rho b \rho^{-1})$$

and

$$\tau(x_{2i+1}) = (\rho b\rho^{-1})^{-i}((\rho b \rho^{-1})\tau^{-1}(x_2)(\rho b\rho^{-1})^{-1}\tau(x_2))(\rho b\rho^{-1})^i,$$

which can be obtained by conjugating the elements of $F(2,2n)$ by the elements of Γ_n, belong to the group $F(2,2n)$. Thus, σ and τ are automorphisms of $F(2,2n)$ and generate the group $\mathbb{D}_8$ of isometries of the manifold M_n. □

To simplify notation, henceforth we use the same symbols for automorphisms of groups and the symmetries of manifolds and orbifolds induced by these automorphisms.

Consider the natural extension of Γ_n by the above automorphisms which has the semidirect product structure:

$$\Omega(n) = \Gamma_n \leftthreetimes \mathbb{D}_8 \cong F(2,2n)\leftthreetimes\langle\rho,\sigma,\tau\rangle.$$

In the next section we demonstrate the importance of the group $\Omega(n)$ for our considerations.

§ 3. Maximality of the Group $\Omega(n)$

Let $\mathrm{Norm}(F(2,2n))$ be the normalizer of the Fibonacci group $F(2,2n)$ in the group $\mathrm{Isom}(\mathbb{H}^3)$ of all isometries of the Lobachevskiĭ space $\mathbb{H}^3$ and let $\mathrm{Isom}(M_n)$ be the isometry group of the Fibonacci manifold M_n.

We recall [9] that the isometry group of a three-dimensional hyperbolic manifold is isomorphic to the group of outer automorphisms of its fundamental group. Therefore,

$$\mathrm{Isom}(M_n) \cong \mathrm{Norm}(F(2,2n))/F(2,2n).$$

It was shown in Section 2 that the symmetries of the figure-eight knot induce isometries of the Fibonacci manifolds. Hence,

$$\mathrm{Norm}(F(2,2n)) \supset \Omega(n) \cong F(2,2n)\leftthreetimes\langle\rho,\sigma,\tau\rangle.$$

Theorem 3.1. *If $n \geq 6$ and Δ is a discrete group of isometries of $\mathbb{H}^3$ such that $\Gamma_n \subset \Delta$ then $\Delta \subset \Omega(n)$.*

PROOF. Recall that Γ_n is the fundamental group of the orbifold $\mathcal{O}(n)$. Consider a fundamental set for this group. The singular set of $\mathcal{O}(n)$ is the figure-eight knot which can be described as the two-bridge 5/2-knot. By [24], we can take a fundamental polyhedron for Γ_n to be the combinatorial polyhedron $\mathcal{P}_n$ pictured in Fig. 2. It is shown in [13] that for $n \geq 4$ the polyhedron $\mathcal{P}_n$ can be constructed in the Lobachevskiĭ space $\mathbb{H}^3$ as a nonconvex fundamental polyhedron for Γ_n. We describe its geometric parameters below.

Denote the vertices of $\mathcal{P}_n$ by $Q_0, Q_1, P_0, P_1, \ldots, P_9$, where Q_0 and Q_1 are the midpoints of the edges P_0P_5 and P_2P_7, respectively. The polyhedron $\mathcal{P}_n$ has four faces each of which is nonflat in general:

$$Q_0P_0P_1P_2P_3P_4P_5,\ Q_0P_0P_9P_8P_7P_6P_5,\ Q_1P_2P_3P_4P_5P_6P_7,\ Q_1P_2P_1P_0P_9P_8P_7,$$

and has the edges P_0P_5, P_2P_7, and P_iP_{i+1}, $i = 0, \ldots, 9$, with all subscripts reduced mod 10.

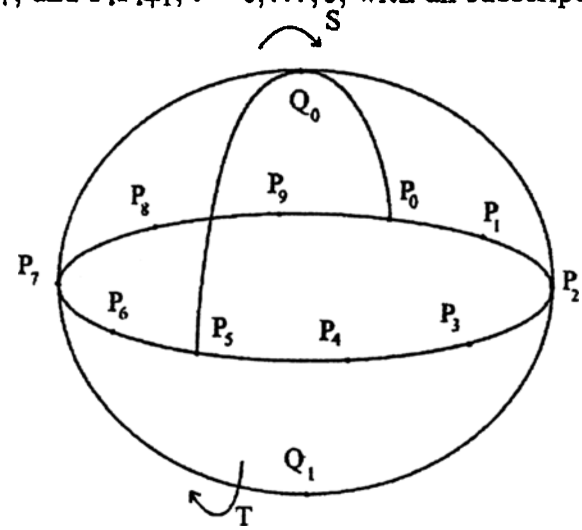


Fig. 2. The polyhedron $\mathcal{P}_n$.

The group Γ_n is generated by the isometries S and T. The isometry S is the rotation of order n about the edge P_0P_5. This isometry identifies the faces $Q_0P_0P_1P_2P_3P_4P_5$ and $Q_0P_0P_9P_8P_7P_6P_5$. The isometry T is the rotation of order n about the edge P_2P_7. This isometry identifies the faces $Q_1P_2P_3P_4P_5P_6P_7$ and $Q_1P_2P_1P_0P_9P_8P_7$. By the Poincaré theorem [25], we then have

$$\Gamma_n \cong \langle S, T \mid S^n = T^n,\ T^{-1}[S,T] = [S,T]S \rangle,$$

where $[S,T] = S^{-1}T^{-1}ST$. It is easy to check that the vertices $P_0, \ldots, P_9$ of $\mathcal{P}_n$ are divided into two equivalence classes with respect to the action of Γ_n:

$$\{P_0,\ P_6 = S^{-1}(P_0),\ P_4 = T^{-1}S^{-1}(P_0),\ P_2 = ST^{-1}S^{-1}(P_0),\ P_8 = TST^{-1}S^{-1}(P_0)\}$$

and

$$\{P_5,\ P_1 = S(P_5),\ P_9 = TS(P_5),\ P_7 = S^{-1}TS(P_5),\ P_3 = T^{-1}S^{-1}TS(P_5)\}.$$

Therefore, for each of the points $P_0, \ldots, P_9$ there is an elliptic element of order n in Γ_n whose axis passes through this point.

We now describe the geometric parameters of the polyhedron $\mathcal{P}_n$. Draw the additional edges Q_0P_i and Q_1P_i for $i = 0, \ldots, 9$. Then $\mathcal{P}_n$ can be represented as the union of the ten tetrahedra $\mathcal{T}_i = Q_0Q_1P_iP_{i+1}$, $i = 0, \ldots, 9$. It was shown in [13] that these tetrahedra can be divided into two classes that consist of pairwise isometric tetrahedra, $\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_4, \mathcal{T}_5, \mathcal{T}_6, \mathcal{T}_7, \mathcal{T}_9, \mathcal{T}_0\}$ and $\{\mathcal{T}_3, \mathcal{T}_8\}$, which are pictured in Fig. 3.

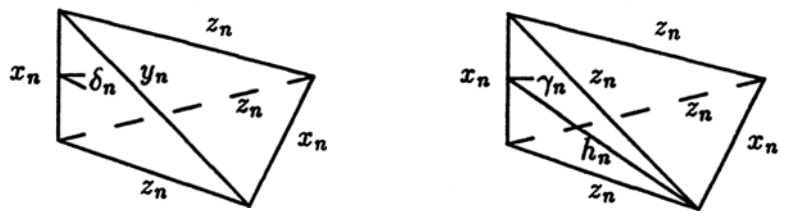


Fig. 3

By the formulae of non-Euclidean trigonometry, the lengths of edges and the dihedral angles of the tetrahedra in Fig. 3 can be determined from the following system of equations [13]:

$$\begin{cases} \sinh h_n \sin(\gamma_n/2) = \sinh(x_n/2), \\ \sinh h_n \sin\delta_n = \sinh x_n \sin(\pi/n), \\ \cosh h_n \cosh(x_n/2) = \cosh z_n, \\ \coth y_n \tanh(x_n/2) = \cot(\pi/n)\tan\delta_n, \\ \cosh z_n = \cosh x_n \cosh y_n, \\ \gamma_n + 4\delta_n = \pi. \end{cases}$$

Introducing the parameter

$$\alpha_n = \sqrt{4\sin^2(\pi/n) + 1},$$

we rewrite the solution of the system in the following form:

$$\cosh x_n = \frac{1}{\alpha_n - 1}, \quad \cosh h_n = \sqrt{\frac{2\alpha_n}{\alpha_n - 1}}\cos(\pi/n), \quad \cosh z_n = \frac{\alpha_n}{\alpha_n - 1}\cos(\pi/n),$$

$$\cosh y_n = \alpha_n \cos(\pi/n), \quad \cos 2\delta_n = \frac{1}{\alpha_n - 1}, \quad \gamma_n = \pi - 4\delta_n.$$

As we see from Fig. 3, the faces of the tetrahedra are divided into two classes which consist of pairwise isometric triangles. With respect to this division, we define two sets of triangles which form the surface of the polyhedron $\mathcal{P}_n$:

$$\mathcal{W}_n^1 = \{\, Q_0P_1P_2,\ Q_0P_2P_3,\ Q_0P_3P_4,\ Q_0P_6P_7,\ Q_0P_7P_8,\ Q_0P_8P_9,\\ Q_1P_3P_4,\ Q_1P_4P_5,\ Q_1P_5P_6,\ Q_1P_8P_9,\ Q_1P_9P_0,\ Q_1P_0P_1 \},$$

with edge lengths x_n, z_n, and z_n, and

$$\mathcal{W}_n^2 = \{Q_0P_0P_1,\ Q_0P_9P_0,\ Q_0P_4P_5,\ Q_0P_5P_6,\ Q_1P_1P_2,\ Q_1P_2P_3,\ Q_1P_6P_7,\ Q_1P_7P_8\},$$

with edge lengths x_n, y_n, and z_n.

Consider the extension of Γ_n by an isometry $h \in \mathrm{Isom}(\mathbb{H}^3)$ such that the group $\langle \Gamma_n, h\rangle$ is discrete. Recall that $S \in \Gamma_n$ is an elliptic element of order n and denote the axis of S by l. Since $\mathcal{P}_n$ is a fundamental polyhedron for Γ_n, without loss of generality we can assume that $h(l) \cap \mathcal{P}_n \neq \varnothing$. In this case we have two possibilities: either $h(l)$ passes through a vertex of some triangle of the set $\mathcal{W}_n^1 \cup \mathcal{W}_n^2$ or $h(l)$ intersects some triangle of $\mathcal{W}_n^1 \cup \mathcal{W}_n^2$ in an interior point. We show that the second possibility cannot occur.

Indeed, $h(l)$ is the axis of an element of order n which is conjugate with S. On the other hand, as was pointed out above, each of the vertices of the polyhedra $\mathcal{T}_0, \dots, \mathcal{T}_9$ and so each of the vertices of the triangles in $\mathcal{W}_n^1 \cup \mathcal{W}_n^2$ belong to the axis of an element of order n. F. Gehring and G. Martin in [14] estimated the distances between the axes of elliptic elements in a discrete group. Namely, the following estimate holds for the distance ρ_6 between two axes of elements of order 6:

$$\sinh \rho_6 \geq \sqrt{3}.$$

The estimate for the distance ρ_n between the two axes of elements of order $n \geq 7$ is given by

$$\sinh \rho_n \geq \frac{\sqrt{2\cos(2\pi/n) - 1}}{2\sin^2 \pi/n}.$$

Proposition 3.1. *If $n \geq 6$ and T is a triangle in $\mathcal{W}_n^1$ then for every point of T there exists a vertex of T such that the distance between the point and this vertex is less then ρ_n.*

PROOF. Rewrite the Gehring–Martin estimates as follows:

$$\cosh \rho_6 \geq c_6 = 2, \quad \cosh \rho_n \geq c_n = \frac{1}{2\sin^2(\pi/n)} - 1$$

for $n \geq 7$. Whence we infer the asymptotic estimate

$$c_n \sim \frac{n^2}{2\pi^2}, \quad n \to \infty.$$

Consider a triangle $T = ABC$ in $\mathcal{W}_n^1$ with edges $BC = x_n$, $AB = z_n$, and $AC = z_n$. This triangle is pictured in Fig. 4. Denote by R_n^1 the radius of the circumscribed circle with center O. Then $OA = OB = R_n^1$. Drop the perpendiculars OD and OE from O to the edges AB and BC. Denote by φ_n the value of the angle DAO.

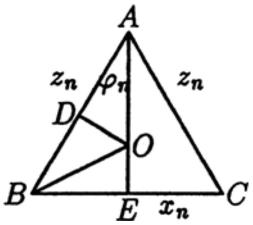


Fig. 4

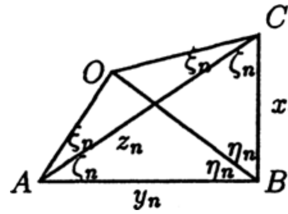


Fig. 5

With the above notations, from the right-angled triangle ABE we find

$$\sin \varphi_n \sinh z_n = \sinh(x_n/2),$$

and from the right-angled triangle AOD we see that

$$\tanh R_n^1 \cos \varphi_n = \tanh\,(z_n/2).$$

Hence,

$$\tanh R_n^1 = \frac{\cosh z_n - 1}{\sqrt{\cosh^2 z_n - \frac{1}{2}\cosh x_n + \frac{3}{2}}}.$$

Using the above expressions for $\cosh z_n$ and $\cosh x_n$ in terms of α_n, after obvious simplifications we obtain

$$\cosh R_n^1 = \frac{\sqrt{\alpha_n} \cdot \sqrt{2\alpha_n \cos^2(\pi/n) - \alpha_n + 1}}{\sqrt{\alpha_n - 1} \cdot \sqrt{-3\alpha_n + 4\alpha_n \cos(\pi/n) + 2}}.$$

This formula easily implies the following asymptotic estimate:

$$\cosh R_n^1 \sim \frac{n}{\sqrt{3}\pi}, \quad n \to \infty.$$

Thus, c_n has quadratic growth in n and $\cosh R_n^1$ has linear growth in n. Computing the values of the functions c_n and $\cosh R_n^1$ for small n and comparing their graphs, we deduce the following inequality for all $n \geq 6$: $\cosh R_n^1 < c_n$. The values of c_n and $\cosh R_n^1$ for $n = 6, \ldots, 16$ are presented in Table 2.

TABLE 2

n	c_n	$\cosh R_n^1$	$\cosh R_n^2$
6	2	1.4812...	1.4362...
7	1.6559...	1.6323...	1.5782...
8	2.4142...	1.7867...	1.7245...
9	3.2743...	1.9447...	1.8748...
10	4.2360...	2.1060...	2.0287...
11	5.2993...	2.2700...	2.1855...
12	6.4641...	2.4365...	2,3449...
13	7.7302...	2.6050...	2.5063...
14	9.0978...	2.7753...	2.6696...
15	10.5667...	2.9471...	2.8343...
16	12.1370...	3.1201...	3.0003...

This completes the proof of Proposition 3.1. □

Proposition 3.2. *If $n \geq 6$ and T is a triangle in $\mathcal{W}_n^2$ then for every point of T there exists a vertex of T such that the distance between the point and this vertex is less then ρ_n.*

PROOF. To find the radius of the circumscribed circle of T, we use the following property.

Lemma 3.1. *Let ABC be a triangle in $\mathbb{H}^2$ with respective angles α, β, and γ at the vertices A, B, and C. Assume that the edge subtending the angle β has the length b. Then the following formula holds for the radius R of the circumscribed circle*

$$\tanh R = \frac{\cosh b - 1}{\sinh b \cos(\frac{\alpha-\beta+\gamma}{2})}.$$

PROOF. Denote the center of the circumscribed circle by O and consider the triangles ABO, BCO, and CAO. Introduce the following notations: $\xi = \angle OAC = \angle OCA$, $\eta = \angle OBC = \angle OCB$, and $\zeta = \angle OAB = \angle OBA$. Suppose that O lies outside the triangle ABC (see Fig. 5). In this case the following relations hold:

$$-\xi + \zeta = \alpha, \quad \eta + \zeta = \beta, \quad -\xi + \eta = \gamma.$$

From them we can express ξ, η, and ζ through α, β, and γ. By the cosine theorem for hyperbolic triangles [25] we have

$$\cosh R = \cosh R \cosh b - \sinh R \sinh b \cos \xi;$$

hence,

$$\tanh R = \frac{\cosh b - 1}{\sinh b \cos \xi},$$

where $\xi = (-\alpha + \beta - \gamma)/2$. A similar result holds in the case when the center O of the circumscribed circle is inside ABC. Thus, the lemma is proven. □

We now return to the proof of Proposition 3.2. Consider $T = ABC$ in $\mathcal{W}_n^2$ with edges $AB = y_n$, $AC = z_n$, and $BC = x_n$. It is pictured in Fig. 5. Denote the radius of the circumscribed circle by R_n^2. By Lemma 3.1,

$$\tanh R_n^2 = \frac{\cosh z_n - 1}{\sinh z_n \cos(\frac{\angle A + \angle C - \pi/2}{2})}.$$

Using the formulae of hyperbolic trigonometry [25] and the above expressions for x_n, y_n, and z_n through the parameter α_n, we obtain

$$\cosh R_n^2 = \frac{\sqrt{\alpha_n}\sqrt{\alpha_n \cos(\pi/n) + 1}}{\sqrt{\alpha_n^2 \cos(\pi/n) + 3\alpha_n - 2 - 2\alpha_n \cos(\pi/n)}}.$$

It is easy to see from this that the following asymptotic formula holds:

$$\cosh R_n^2 \sim \frac{n}{\sqrt{3}\pi}, \quad n \to \infty.$$

Thus, the function c_n has quadratic growth in n and the function $\cosh R_n^2$ has linear growth in n. Computing the values of c_n and $\cosh R_n^1$ for small n and comparing their graphs, we deduce that the following inequality holds for all $n \geq 6$: $\cosh R_n^2 < c_n$. The values of c_n and $\cosh R_n^1$ for $n = 6, \dots, 16$ are presented in Table 2. Proposition 3.2 is thus proven. □

Since for each vertex of an arbitrary triangle in $\mathcal{W}_n^1 \cup \mathcal{W}_n^2$ there is an elliptic element of order n whose axis passes through this vertex; in virtue of Propositions 3.1 and 3.2, the axis $h(l)$ cannot pass through interior points of these triangles.

We now consider the case when $h(l)$ passes through a vertex of the polyhedron $\mathcal{P}_n$. As was mentioned above, the vertices $P_0, \dots, P_9$ are divided into two classes with respect to the action of the group Γ_n: those equivalent to P_0 and those equivalent to P_1. Considering $\tilde{h} = hh'$, where $h' \in \Gamma_n$, we therefore derive that $\tilde{h}(Q_0)$ coincides with one of the following four vertices: Q_0, Q_1, P_0, and P_1. Observe that there exists an isometry of $\mathbb{H}^3$ which is induced by an automorphism of the group $\Gamma_n \lambda \langle \tau \rangle$ such that the image of Q_0 is P_1 and the image of Q_1 is P_0. Indeed, according to the action of the automorphism τ on Γ_n we have

$$\tau(S) = S^{-1}TS, \quad \tau(T) = TST^{-1}.$$

Denote by g the automorphism that results from conjugating τ by the element TS^{-1}. Then

$$g(S) = ST^{-1}(S^{-1}TS)TS^{-1} = T^{-1}S^{-1}T = (T^{-1}ST)^{-1},$$

$$g(T) = ST^{-1}(TST^{-1})TS^{-1} = S.$$

Therefore, g induces an isometry of $\mathbb{H}^3$ such that the image of the axis of S is the axis of the element $T^{-1}ST$, and the image of the axis of T is the axis of S. Observe [13] that the segment $[Q_0, Q_1]$ is a common perpendicular to the axis of S (at Q_0) and the axis of T (at Q_1). The segment $[P_0, P_1]$ is a common perpendicular to the axis of $T^{-1}ST$ (at P_1) and the axis of S (at P_0). Thus, $g(Q_0) = P_1$ and $g(Q_1) = P_0$. Considering $\tilde{h} = hh''$, $h'' \in \Gamma_n \lambda \langle \tau \rangle$, we therefore conclude that $\tilde{h}(Q_0)$ coincides with Q_0 or Q_1. Recall that the group Γ_n has an automorphism σ such that

$$\sigma(S) = T, \quad \sigma(T) = S.$$

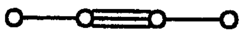
This automorphism induces an isometry of $\mathbb{H}^3$ which carries the axis of S to the axis of T and vice versa. Since the segment $[Q_0, Q_1]$ is a common perpendicular to these axes, we obtain $\sigma(Q_0) = Q_1$ and $\sigma(Q_1) = Q_0$. Considering $\tilde{h} = hh'''$, $h''' \in \Gamma_n \lambda \langle \sigma, \tau \rangle = \Omega(n)$, we then obtain $\tilde{h}(Q_0) = Q_0$. Thus, the axes l and $\tilde{h}(l)$ of order $n \geq 6$ have the common point Q_0 and so coincide by [15]. Under the action of $\tilde{h}$ the axis of S goes to itself. Multiplying $\tilde{h}$ by an appropriate power of S, we may assume that $\tilde{h}(Q_1) \in \mathcal{P}_n$. Therefore, $\tilde{h}(Q_1) = Q_1$ and under the action of the isometry $\tilde{h}$ the axis of T goes to itself. Thus, the isometry $\tilde{h}$ has two invariant skew lines and their common perpendicular $[Q_0, Q_1]$ is fixed. Multiplying $\tilde{h}$, if necessary, by a symmetry from $\mathbb{D}_8$, we infer that $\tilde{h} = id$; hence, $h \in \Omega(n)$. The theorem is proven. □

We recall that a discrete group is said to be a *maximally discrete group* if it is not a proper subgroup of another discrete group.

Corollary 3.1. *For $n \geq 6$ the group $\Omega(n)$ is a maximally discrete group.*

PROOF. Indeed, assume that Δ is a discrete group such that $\Omega(n) \subset \Delta$. Then $\Gamma_n \subset \Omega(n) \subset \Delta$. On the other hand, by Theorem 3.1 we have $\Delta \subset \Omega(n)$. Thus, $\Omega(n)$ is a maximally discrete group. □

REMARK. For $n = 4, 5$ the groups $\Omega(n)$ are maximally discrete groups as well [8]. This fact follows from their arithmeticity and the comparison of hyperbolic volumes. Observe that the Fibonacci group $F(2, 10)$ is commensurable with the Coxeter group with the following Coxeter scheme:



This Coxeter group is generated by reflections in the faces of a compact hyperbolic tetrahedron.

Theorem 3.2. *The isometry group* $\mathrm{Isom}(M_n)$ *of the hyperbolic Fibonacci manifold* M_n, $n \geq 4$, *consists of* $8n$ *elements and has the following presentation*

$$\mathrm{Isom}(M_n) = \langle x, y \mid x^{2n} = y^4 = (yx)^2 = (y^{-1}x)^2 = 1 \rangle.$$

PROOF. In the cases $n = 4, 5, 6, 8, 12$ when the Fibonacci group is arithmetic, the claim was proven by C. Maclachlan and A. Reid [8]. We prove the claim for all $n \geq 6$. We recall that

$$\mathrm{Isom}(M_n) \cong \mathrm{Norm}(F(2, 2n))/F(2, 2n),$$

where $\mathrm{Norm}(F(2, 2n))$ is the normalizer of $F(2, 2n)$ in the isometry group of $\mathbb{H}^3$. On the one hand, by Theorem 2.1 we have

$$\mathrm{Norm}(F(2, 2n)) \supset \Omega(n).$$

On the other hand, the group $\mathrm{Norm}(F(2, 2n))$ is discrete as a finite index extension of Γ_n. Therefore, by Corollary 3.1 we have

$$\mathrm{Norm}(F(2, 2n)) \subset \Omega(n).$$

Thus,

$$\mathrm{Norm}(F(2, 2n)) = \Omega(n) \cong F(2, 2n)\lambda(\mathbb{Z}_n \lambda \mathbb{D}_8),$$

and therefore

$$\mathrm{Norm}(F(2, 2n))/F(2, 2n) \cong \mathbb{Z}_n \lambda \mathbb{D}_8.$$

The claimed 2-generator presentation of the group follows from [11]. □

§ 4. Quotient Orbifolds

In this section we consider some quotient orbifolds resulting from the action of isometries on the hyperbolic Fibonacci manifolds. As was indicated in the previous section, the quotient space of the manifold M_n by the action of the isometry ρ is the orbifold $\mathcal{O}(n)$ whose underlying space is the three-dimensional sphere and whose singular set is the figure-eight knot with the singularity index n. Thus, $\mathcal{O}(n) = M_n/\langle \rho \rangle$.

Consider the involution $\tau^2 \in \mathbb{D}_8$. The quotient space of the orbifold $\mathcal{O}(n)$ by the action of this involution was discussed, for example, in [7, 26].

Proposition 4.1. *The quotient space $M_n/\langle \rho, \tau^2 \rangle$ of the hyperbolic Fibonacci manifold M_n by the action of the group of isometries generated by ρ and τ^2 is the orbifold whose underlying space is the three-dimensional sphere and whose singular set is the two-component link 6_2^2 pictured in Fig. 6 with singularity indices 2 and n on its components.*

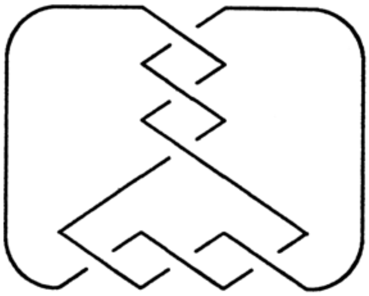


Fig. 6. The singular set of the orbifold $M_n/\langle\rho,\tau^2\rangle$.

PROOF. The claim follows from the fact that $M_n/\langle\rho,\tau^2\rangle \cong \mathcal{O}(n)/\langle\tau^2\rangle$. □

Consider the involution $\sigma \in \mathbb{D}_8$ inducing the symmetry of the figure-eight knot with the axis pictured by the dotted line in Fig. 1.

Proposition 4.2. *The quotient space $M_n/\langle\rho,\sigma\rangle$ of the hyperbolic Fibonacci manifold M_n by the action of the group of isometries generated by ρ and σ is the orbifold whose underlying space is the three-dimensional sphere and whose singular set is the spatial graph pictured in Fig. 7.*

PROOF. The claim follows from the fact that $M_n/\langle\rho,\sigma\rangle \cong \mathcal{O}(n)/\langle\sigma\rangle$. □

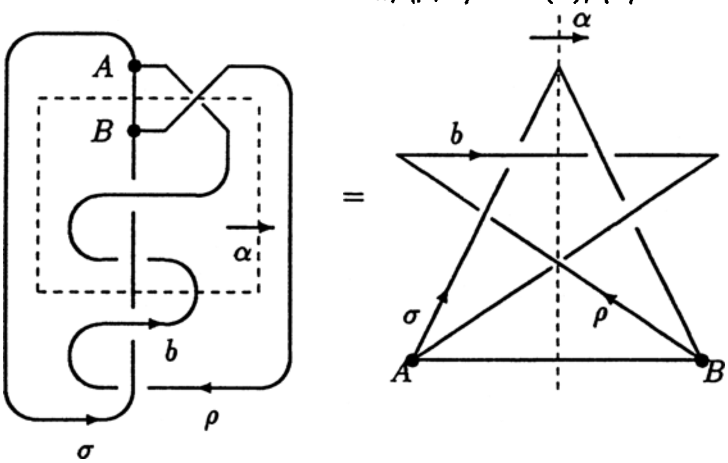


Fig. 7. The singular set of the orbifold $M_n/\langle\rho,\sigma\rangle$.

Considering the symmetry σ of Fig. 1, we obtain the graph that is pictured on the left of Fig. 7 and has two vertices and three edges connecting these vertices. Therefore, this graph is the Θ-graph according to the terminology of spatial graph theory [27]. This graph can also be described as the torus knot 5_1 with a bridge AB (see the right side of Fig. 7). The points A and B are the images of the intersection points of the singular set of the orbifold $\mathcal{O}(n)$ with the axis of the involution σ. Observe that the arc AB is the image of the tunnel of the figure-eight knot in the sense of [28]. In this case the two arcs of the singular set of the orbifold $M_n/\langle\rho,\sigma\rangle$, which are images of the axis of the involution σ, have singularity indices 2. The third arc, which is the image of the singular set of the orbifold $\mathcal{O}(n)$, has the singularity index n. The group of the orbifold $M_n/\langle\rho,\sigma\rangle$ has the presentation

$$\Delta(n) = F(2,2n)\lambda\langle\rho,\sigma\rangle \cong \langle\rho,b,\sigma \mid \rho^n = b^n = \sigma^2 = 1,\ \rho^{-1}[b,\rho] = [b,\rho]b,\ \sigma\rho\sigma = b\rangle,$$

where the generators ρ, b, and σ correspond to Fig. 7. Indeed, the relation $\rho^{-1}[b,\rho] = [b,\rho]b$ follows from the fact that the loop around the bridge AB is an element of order two. The relation $\sigma\rho\sigma = b$ follows by the Wirtinger algorithm [21].

It is easy to see from Fig. 7 that the spatial Θ-graph has an involution α whose axis is pictured by the dotted line. The involution α induces a symmetry of the orbifold $M_n/\langle\rho,\sigma\rangle$. From Fig. 7 we

see that

$$\alpha(b) = b^{-1}, \quad \alpha(\rho) = b\rho^{-1}b^{-1}.$$

Thus, the involution α is conjugate to the involution $\sigma\tau^2$ (see Table 1). Thus, the quotient orbifolds $M_n/\langle\rho,\sigma,\alpha\rangle$ and $M_n/\langle\rho,\sigma,\tau^2\rangle$ are isometric. □

Proposition 4.3. *The quotient space $M_n/\langle\rho,\sigma,\tau^2\rangle$ of the hyperbolic Fibonacci manifold M_n by the action of the group of isometries generated by ρ, σ, and τ^2 is the orbifold whose underlying space is the three-dimensional sphere and whose singular set is pictured in Fig. 8.*

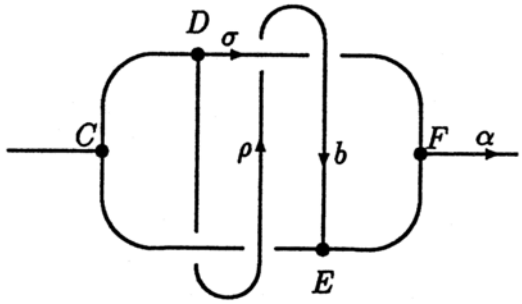


Fig. 8. The singular set of the orbifold $M_n/\langle\rho,\sigma,\tau^2\rangle$.

PROOF. The claim follows from the fact that $M_n/\langle\rho,\sigma,\tau^2\rangle \cong (\mathcal{O}(n)/\langle\sigma\rangle)/\langle\tau^2\rangle$. □

The graph pictured in Fig. 8 has four vertices and six edges. As an abstract graph it is isomorphic to the complete graph on four vertices. On the other hand, this graph can be described as the figure-eight knot with two arcs CD and EF which are his tunnels. The vertices C, E, and F are the images of the intersection points of the singular set of the orbifold $\mathcal{O}(n)/\langle\sigma\rangle$ with the axis of the involution α. The vertex D is the image of the point A in Fig. 7. One edge of the graph has singularity index n and the other edges have singularity index 2. The group of the orbifold $M_n/\langle\rho,\sigma,\tau^2\rangle$ has the following presentation:

$$\begin{aligned}\Pi(n) &= F(2,2n)\leftthreetimes\langle\rho,\sigma,\tau^2\rangle \\ &\cong \langle\rho,b,\sigma,\alpha \mid \rho^n = b^n = \sigma^2 = \alpha^2 = 1,\ \rho^{-1}[b,\rho] = [b,\rho]b, \\ &\qquad \sigma\rho\sigma = b, \alpha^{-1}b\alpha = b^{-1},\ \alpha^{-1}\rho\alpha = b\rho^{-1}b^{-1}\rangle,\end{aligned}$$

where the elements ρ, b, σ, and α correspond to Fig. 8. Another presentation of the group $\Pi(n)$ can be obtained by the Wirtinger algorithm:

$$\Pi(n) \cong \langle\rho,\sigma,\alpha \mid \rho^n = \sigma^2 = \alpha^2 = 1, \qquad (\sigma\rho\sigma\rho^{-1}\sigma\alpha)^2 = (\rho^{-1}\sigma\alpha)^2 = (\sigma\alpha\rho^{-1}\alpha^{-1})^2 = 1\rangle.$$

As we can see from Fig. 8, the singular set of the orbifold $M_n/\langle\rho,\sigma,\tau^2\rangle$ has an antipodal involution χ such that $\chi(C) = F$, $\chi(D) = E$, $\chi(E) = D$, and $\chi(F) = C$. Theorem 3.2 readily yields $\langle\rho,\sigma,\tau^2,\chi\rangle \cong \operatorname{Isom}(M_n)$.

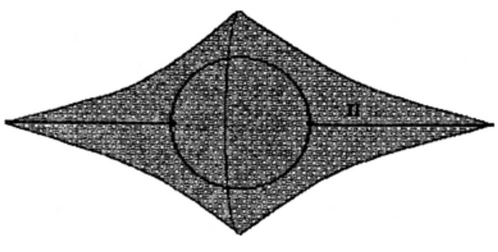


Fig. 9. The quotient orbifold $M_n/\operatorname{Isom}(M_n)$.

Since the underlying space of the orbifold $M_n/\langle \rho, \sigma, \tau^2 \rangle$ is the three-dimensional sphere, the underlying space of the quotient orbifold $M_n/\operatorname{Isom}(M_n)$ is the pseudo-manifold P^3 which can be represented as the union of two cones whose common base is the projective plane. This pseudo-manifold and the singular set of the orbifold are schematically pictured in Fig. 9, where one singularity index equals n and the others equal 2.

Proposition 4.4. *The quotient space $M_n/\operatorname{Isom}(M_n)$ of the hyperbolic Fibonacci manifold M_n by the action of its isometry group $\operatorname{Isom}(M_n)$ is the orbifold whose underlying space is the pseudo-manifold P^3 and whose singular set in pictured in Fig. 9.*

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