The Tits alternative for generalised tetrahedron groups

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Abstract

A generalised tetrahedron group is the colimit of a triangle of groups whose vertex groups are generalised triangle groups and whose edge groups are finite cyclic. We prove an improved Spelling Theorem for generalised triangle groups which enables us to compute the precise Gersten-Stallings angles of this triangle of groups, and hence obtain a classification of generalised tetrahedron groups according to the curvature properties of the triangle. We also prove that the colimit of a negatively curved triangle of groups contains a nonabelian free subgroup. Finally, we apply these results to prove the Tits alternative for all generalised tetrahedron groups where the triangle is non-spherical: with three abelian-by-finite exceptions, every such group contains a nonabelian free subgroup.

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1 Introduction

A *generalised triangle group* is a group with a presentation of the form

$$\langle x, y \,|\, x^p = y^q = W(x,y)^r = 1\rangle,$$

where $W(x,y)$ is a cyclically reduced word in the free product $\langle x \,|\, x^p = 1\rangle * \langle y \,|\, y^q = 1\rangle$ and p, q, r are integers greater than 1.

A *generalised tetrahedron group* is defined to be a group admitting the following presentation:

$$\langle x, y, z \,|\, x^\ell = y^m = z^n = W_1(x,y)^p = W_2(y,z)^q = W_3(z,x)^r = 1\rangle,$$

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where each $W_i(a,b)$ is a cyclically reduced word involving both a and b and all powers are integers greater than 1.

These groups appear in many algebraic and geometric questions, for example, as subgroups of generalised triangle groups and as fundamental groups of certain orbifolds. Important special cases arise when the words W or W_i are each just the product of one of the two corresponding generators with the inverse of the other. The *triangle group*

$$\Delta(p,q,r) = \langle x, y \,|\, x^p = y^q = (xy)^r = 1\rangle$$

can be realised geometrically as a group generated by rotations through angles $2\pi/p$ and $2\pi/q$ about distinct points in the Euclidean, hyperbolic, or spherical plane. It is an index 2 subgroup of the group

$$\Delta^*(p,q,r) = \langle x, y, z \,|\, x^2 = y^2 = z^2 = (xy)^p = (yz)^q = (zx)^r = 1\rangle$$

generated by reflections in the sides of a (Euclidean, hyperbolic or spherical) triangle with angles π/p, π/q and π/r. (Note that $\Delta^*(p,q,r)$ is an example of a generalised tetrahedron group.)

Similarly, if $\mathcal{T}$ is a tetrahedron in 3-dimensional Euclidean, hyperbolic or spherical space whose dihedral angles are submultiples of π, then the reflections in the faces of $\mathcal{T}$ generate a discrete group of isometries. The index 2 subgroup of orientation-preserving isometries in this group is generated by rotations around the edges of any one of the faces of $\mathcal{T}$, and has a presentation of the form

$$\langle x, y, z \,|\, x^\ell = y^m = z^n = (xy^{-1})^p = (yz^{-1})^q = (zx^{-1})^r = 1\rangle.$$

We refer to this group as an *ordinary tetrahedron group.* (In the case of hyperbolic space, it is convenient for us to allow one or more of the vertices of $\mathcal{T}$ to lie on the boundary in this definition.)

A class of groups $\mathcal{C}$ is said to satisfy the *Tits alternative* if each group in $\mathcal{C}$ either contains a non-abelian free group of rank two or is virtually soluble (i.e., has a soluble subgroup of finite index). This property is named after J. Tits, who established [23] that it is satisfied by the class of linear groups. In particular, every ordinary tetrahedron or triangle group is linear, and so satisfies the Tits alternative.

The Tits alternative has been proved, for example, for the classes of one relator groups [13], mapping class groups of compact surfaces [12, 17], the outer automorphism groups of free groups of finite rank [1, 2], Coxeter groups [16, 18], subgroups of Gromov hyperbolic groups [9].

Conjecture (Rosenberger). *The class of generalised triangle groups satisfies the Tits alternative.*

This has been proved except in the case where $p \geq 2$, $q \geq 2$, $r = 2$, $1/p + 1/q > 1/2$ and $W(x,y)$ has length greater than eight in terms of the free product $\langle x\rangle * \langle y\rangle$ (see [8]). The same question can be asked about generalised tetrahedron groups.

Conjecture. *The class of generalised tetrahedron groups satisfies the Tits alternative.*

In [5], Edjvet, Howie, Rosenberger and Thomas proved that if $\mathcal{G}$ is a finite generalised tetrahedron group and at least one of p, q and r is greater than 3, then the presentation of $\mathcal{G}$ is equivalent to a presentation of an ordinary (spherical) tetrahedron group. In [21] Rosenberger, Scheer and Thomas classified finite generalised tetrahedron groups with a cubic relator, and in [20] Rosenberger and Scheer extended this to an almost complete classification of finite generalised tetrahedron groups.

There are also some sufficient conditions for generalised tetrahedron groups to contain a free subgroup (see [7]). These results cover a large class of generalised tetrahedron groups, but do not give the whole picture.

As was pointed out in [7], a generalised tetrahedron group $\mathcal{G}$ can be naturally realised as the colimit of a triangle of groups whose vertex groups are generalised triangle groups and whose edge groups are finite cyclic (see Sections 2 and 3.2). The class of generalised tetrahedron groups can thus be naturally subdivided into three subclasses, which we call negatively curved, Euclidean, and spherical, according to the curvature of the corresponding triangle of groups. In order to determine to which subclass a given group belongs, it is necessary to compute the Gersten-Stallings angles of the triangle [22].

In this paper, we present a spelling theorem (Theorem 3.2) for generalised triangle groups which improves on that given in [5] and enables us to give the precise values of the Gersten-Stallings angles as required. As a result, we are able to list all Euclidean and spherical generalised tetrahedron groups.

The main result of our paper is the following theorem:

Theorem 1. *The class $\mathcal{C}$ of generalised tetrahedron groups realised by non-spherical triangle of groups satisfies the Tits alternative. More precisely, any $\mathcal{G} \in \mathcal{C}$ contains a non-abelian free subgroup unless $\mathcal{G}$ is isomorphic to the abelian-by-finite group $\Delta^*(p,q,r)$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.*

The paper is organised as follows. In Section 2 we discuss polygons of groups, in particular, we prove that the colimit of any negatively curved triangle of groups contains a non-abelian free subgroup. We also prove that the colimits of certain non-positively curved triangles and squares of groups contain non-abelian free subgroups.

Section 3 is devoted to results on generalised triangle groups. We prove Theorem 3.2 and present a complete list of presentations for Euclidean and spherical generalised tetrahedron groups.

Finally, in Section 4 we apply the results of Sections 2 and 3 to prove Theorem 1.

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2 Polygons of groups

Given two subgroups A and B of a group G, the inclusions $A \to G$ and $B \to G$ determine a homomorphism $\phi : A * B \to G$. If ϕ is injective, the (*Gersten-Stallings*) *angle* $(G; A, B)$ is defined to be 0, otherwise $(G; A, B)$ is defined to be π/n, where $2n$ is the minimal length of a non-trivial element in $\mathrm{Ker}(\phi)$.

Let Γ be a k-gon, to whose vertices are associated groups G_i and to whose edges are associated groups G_{ij} such that each G_{ij} is a proper subgroup of both G_i and G_j. The *colimit*, or *polygonal product* of Γ, is the group $\mathcal{G}$ given by generators and relations of the vertex groups together with relations which identify the subgroup G_{ij} of G_i with the subgroup G_{ij} of G_j.

In this paper, we are primarily concerned with the cases $k = 3$ and $k = 4$, known as *triangles of groups* and *squares of groups*, respectively.

These are special cases of the more general concept of a *complex of groups*, whose theory has been extensively developed in [4, 10, 11, 22] and in [3, Part III $\mathcal{C}$].

A k-gon of groups is said to be *non-spherical* if the sum over each vertex i of the Gersten-Stallings angles $(G_i; G_{i,i-1}, G_{i,i+1})$ (subscripts modulo k) is at most $(k-2)\pi$. If the angle sum is strictly less than $(k-2)\pi$ then we call the k-gon *negatively curved*.

Remark. The definition of a polygon of groups is often taken to include a *face group* F associated to the 2-dimensional cell of the polygon. This is regarded as a common subgroup of the edge groups. The Gersten-Stallings angles are then defined using length in the amalgamated free product $G_{i,i-1} *_F G_{i,i+1}$ rather than the free product $G_{i,i-1} * G_{i,i+1}$,etc. Our definition corresponds to the case $F = \{1\}$. While many of our arguments also hold in the more general setting, we shall consider in this paper only polygons of groups with trivial face groups for ease of exposition.

We will make frequent use of the well-known fact that non-spherical polygons (or, with an appropriate definition, more general complexes) of groups are *developable*, in the sense that each vertex group embeds into the colimit. Proofs may be found, for example, in [3, 4, 10, 11] or (in the triangle case) [22].

Theorem 2.1. *If Γ is a non-spherical k-gon of groups, then the vertex groups G_i embed in the colimit of Γ.*

Non-spherical triangles of groups

Consider a triangle of groups (see Figure 1).

Then such a triangle of groups is *spherical* if $(G_1; X, Y) + (G_2; Y, Z) + (G_3; Z, X) > \pi$, and *non-spherical* otherwise. Among non-spherical triangles we distinguish *Euclidean* and *negatively curved* triangles according to whether the sum of the angles is equal to or less than π.

Proposition 2.2. *Let Γ be a negatively curved triangle of groups. Then its colimit $\mathcal{G}$ contains a non-abelian free subgroup.*

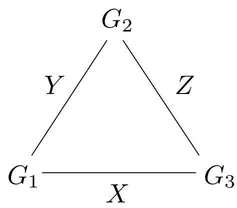

Figure 1:

Proof. We use the ideas of [6]. Let $\mathcal{G}$ be the colimit of the triangle of groups shown in Figure 1. If one of the edge groups, say X, is trivial, then Γ is in fact a tree, i.e., $\mathcal{G}$ is a free amalgamation product $G_1 *_Y G_2 *_Z G_3 = (G_1 *_Y G_2) *_{G_2} (G_2 *_Z G_3)$. Since Y and Z are proper subgroups of G_2, G_2 has infinite index in both $G_1 *_Y G_2$ and $G_2 *_Z G_3$. Therefore, $\mathcal{G}$ contains a free subgroup of rank 2.

Suppose that X, Y and Z are non-trivial groups. Then we may choose non-trivial elements $x \in X$, $y \in Y$ and $z \in Z$. Since the triangle of groups is negatively curved, there exists $\varepsilon > 0$ such that $\varepsilon < \pi - ((G_1; X, Y) + (G_2; Y, Z) + (G_3; Z, X))$. Consider $u = (xyz)^p$ and $v = (xzy)^p$, where $p = 4[1/\varepsilon] + 1$; then we show that u and v generate a free subgroup of $\mathcal{G}$.

Choose a presentation $\mathcal{P}$ for $\mathcal{G}$ which is a union of presentations $\mathcal{P}_i$ ($i = 1, 2, 3$) for the three vertex groups G_i, such that any two of these subpresentations intersect in a presentation for the appropriate edge group. Without loss of generality, we assume that x, y, z are all generators of $\mathcal{P}$, so that u, v are words in the generators of $\mathcal{P}$.

Assume that $w(u, v) = 1$ in $\mathcal{G}$. Consider a van Kampen diagram K over $\mathcal{P}$ whose boundary label is $w(u, v)$. Let D be an extremal disk of K. We divide D into G_1, G_2 and G_3 regions, where a G_i-region is a connected union of 2-cells labelled by the relations of $\mathcal{P}_i$. If two G_i-regions intersect at least at one edge, then we can amalgamate them into a single region. We continue in this way as often as possible, and so get a division of D into *maximal G_i-regions* for $i = 1, 2, 3$. (Note that the resulting division of D is not necessarily unique.)

Since the vertex groups embed, it can be assumed that the maximal regions are simply connected. Let $\widehat{D}$ be the resulting diagram. We define an edge of $\widehat{D}$ to be a path whose edges are labelled by elements of either X, or Y, or Z.

Now place $\widehat{D}$ on the sphere and take its dual D^*. Let V_0 be the vertex corresponding to $\mathbb{S}^2 \backslash \widehat{D}$. We call a region of D^* *non-interior* if it involves V_0 and *interior* otherwise. Observe that each interior region is at least a 3-gon.

We give each corner at a vertex of D^* of degree d the angle $2\pi/d$. The curvature $c(\Delta)$ of a region Δ of degree k whose vertices have degrees d_1, d_2, ..., d_k is then defined by

$$c(\Delta) = (2 - k)\pi + \sum_{i=1}^{k} \frac{2\pi}{d_i}.$$

Then

$$\sum_{\Delta\subset D^*} c(\Delta) = 2\pi\chi(\mathbb{S}^2) = 4\pi.$$

Let Δ be an interior region of D^* of degree 3. Then

$$c(\Delta) = -\pi + \left(\frac{2\pi}{d_1} + \frac{2\pi}{d_2} + \frac{2\pi}{d_3}\right) \le -\pi + (G_1; X, Y) + (G_2; Y, Z) + (G_3; Z, X) < 0.$$

It is not difficult to see that the curvature of an interior region of degree $k > 3$ is also negative. Thus, the sum of the curvatures of interior regions is negative.

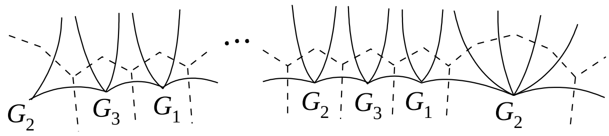

Figure 2:

Consider non-interior regions. Observe that a non-interior region can be a 2-gon. The maximal sum of the curvatures of non-interior regions is achieved when the number of 2-gons is maximal.

Note that the cancellations in w can happen only in cases $[u^{-1}v]^{\pm 1} = [(z^{-1}y^{-1}x^{-1})^{p-1}z^{-1}y^{-1}x^{-1}xzy(xzy)^{p-1}]^{\pm 1}$. Consider a path labelled by a part of $w(u,v)$ where a cancellation takes place, for example, $(z^{-1}y^{-1}x^{-1})^{p-1}z^{-1}y^{-1}zy$. The curvature of such a chain S of non-interior regions of D^* along this path is maximal when $z^{-1}y^{-1}zy$ is a part of a maximal G_2-region and regions of D^* of degree 2 and 3 are arranged consecutively.

Since p is odd, the chain S starts with G_2 and ends in G_2 (see Figure 2). Moreover, the number of 3-gons in S is $N_3 = 3(p-1)/2$ and the number of 2-gons in S is $N_2 = 3(p-1)/2 + 2$. The sum of angles at each vertex of S different from V_0, say labelled by G_1, is not greater than $3(G_1; X, Y)$ and the sum of angles at V_0 is $2\pi N_S/d_0$, where d_0 is the degree of V_0 and $N_S = N_2 + N_3$. Note that we have $(p-1)/2$ maximal G_1- and G_3-regions and $(p-1)/2 + 1$ maximal G_2-region for S. Then

$$\begin{aligned}
c(S) &\le -N_3\pi + \frac{3(p-1)}{2}\left((G_1; X, Y) + (G_2; Y, Z) + (G_3; Z, X)\right) \\
&\quad +2(G_2; Y, Z) + \frac{2\pi}{d_0}N_S \\
&= \frac{3(p-1)}{2}\left(-\pi + (G_1; X, Y) + (G_2; Y, Z) + (G_3; Z, X)\right) \\
&\quad +2(G_2; Y, Z) + \frac{2\pi}{d_0}N_S \\
&< -\frac{3(p-1)}{2}\varepsilon + 2\frac{\pi}{2} + \frac{2\pi}{d_0}N_S < -6 + \pi + \frac{2\pi}{d_0}N_S < \frac{2\pi}{d_0}N_S.
\end{aligned}$$

For a chain S of regions of D^* along a path which does not contain cancellations $c(S) < \frac{2\pi}{d_0} N_S$. Furthermore, the number of non-interior regions of D^* is d_0. Thus,

$$\sum_{\Delta \subset D^*} c(\Delta) < 2\pi.$$

We arrive at a contradiction. □

The previous result is not true in general for non-spherical triangles of groups. However, in many cases it is still possible to obtain free subgroups.

The following is an example of a result in that direction.

Proposition 2.3. *Let Γ be a non-spherical triangle of groups, as in Figure 1. Suppose that $y \neq 1$ and $z \neq 1$ are elements of the edge groups Y, Z respectively, such that yz is not a subword of a relation of the vertex group G_2 of minimal length (as a word in the free product $Y * Z$). Then the colimit $\mathcal{G}$ of Γ contains a non-abelian free subgroup.*

Sketch proof. Follow the proof of Proposition 2.2, but putting $u = (xyz)^p$ and $v = (xz^{-1}y^{-1})^p$ for some $p \gg 0$.

The same analysis as in the proof of Proposition 2.2 works, with two differences. Firstly, we cannot assume that Γ is negatively curved, so possibly $\varepsilon = 0$. But, to compensate, we can assume that the angles at the G_2-vertices of S are less than $(G_2; Y, Z)$, since neither yz nor $z^{-1}y^{-1}$ can be a subword of a minimal length relation. The difference is at least $\frac{\pi}{6} - \frac{\pi}{7} = \frac{\pi}{42}$, since $(G_2; Y, Z) \geq \frac{\pi}{6}$. Hence the inequality calculation for $c(S)$ becomes

$$c(S) \leq -\frac{3(p-1)\pi}{84} + \pi + \frac{2\pi}{d_0} N_S < \frac{2\pi}{d_0} N_S,$$

provided we choose $p \geq 30$. □

Lemma 2.4. *Let Γ be a triangle of groups shown in Figure 1 such that*

(i) $(G_1; X, Y) = (G_2; Y, Z) = (G_3; X, Z) = \pi/3$;

(ii) there exist non-trivial elements $x \in X$, $y \in Y$ such that $x^2 \neq 1$, $y^2 \neq 1$;

(iii) for all α, β, γ, $xy^2xy^\alpha x^\beta y^\gamma \neq 1$ *and* $yx^2yx^\alpha y^\beta x^\gamma \neq 1$ *in G_1.*

If $\mathcal{G}$ is the colimit of Γ then $\mathcal{G}$ contains a non-abelian free subgroup.

Proof. We follow closely the proof of Proposition 2.2. If Z is trivial, then Γ is a tree and $\mathcal{G}$ contains a free subgroup. Suppose Z is not trivial. Take $x \in X$, $y \in Y$ as in the statement of the lemma and $z \in Z \setminus \{1\}$.

Let $u = zxyzx^{-1}y^{-1}$ and $v = zx^{-1}y^{-1}zxy$. We shall show that u and v generate a free group. Suppose there is a nontrivial relation $w(u, v) = 1$ in $\mathcal{G}$.

Consider an extremal disk D of a van Kampen diagram for $w(u, v)$. Place the diagram $\widehat{D}$ of maximal G_i-regions on the sphere and take its dual D^*. Clearly, the curvature of the interior regions of D^* is non-positive. Let us show that the curvature of non-interior regions is also non-positive. Note that a chain of non-interior regions of D^* corresponding to a path on the boundary of $\widehat{D}$ labelled by uv and vu has non-positive curvature. Consider a chain S of non-interior regions corresponding to a path labelled by $uv^{-1} = zxyzx^{-1}y^{-2}x^{-1}z^{-1}yxz^{-1}$. It is clear that $c(S)$ is maximal when the number of 2-gons is maximal, i.e., the regions are arranged as in Figure 3.

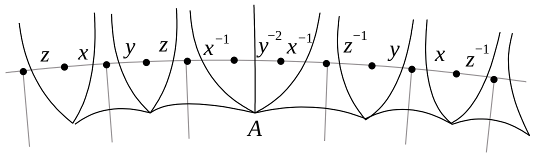

Figure 3:

Since xy^2x is not a part of a length 6 relation in G_1, each angle at the vertex A is not greater than $\pi/4$. Angles at other vertices of D^* are not greater than $\pi/3$. So, $c(S) \leq -5\pi + 12\pi/3 + 4\pi/4 + 22\pi/d_0 = 22\pi/d_0$, where d_0 is the degree of the vertex corresponding to $\mathbb{S}^2 \backslash \widehat{D}$. For a chain labelled by $v^{-1}u$ the argument is similar. Then $\sum_{\Delta \subset D^*} \leq 2\pi$ and we arrive at a contradiction. $\square$

Corollary 2.5. *A group* $\mathcal{G} = \langle x, y, z \,|\, x^\ell = y^2 = z^2 = (xy)^2 = (yz)^3 = (xzx^\eta zx^{-\eta}z)^2 = 1\rangle$ *with* $3 \leq \ell \leq 5$ *contains a free subgroup of rank* 2.

Proof. The normal closure K of x has the presentation

$$K = \langle a, b, c \mid a^\ell = b^\ell = c^\ell = ab^\eta a^\eta b^{-1} a^{-\eta} b^{-\eta} = bc^\eta b^\eta c^{-1} b^{-\eta} c^{-\eta} = ac^\eta a^\eta c^{-1} a^{-\eta} c^{-\eta} = 1\rangle.$$

The group K can be realised as the colimit of a triangle of groups with the vertex groups

$$\begin{aligned} K_1 &= \langle a, b \,|\, a^\ell = b^\ell = ab^\eta a^\eta b^{-1} a^{-\eta} b^{-\eta} = 1\rangle, \\ K_2 &= \langle b, c \,|\, b^\ell = c^\ell = bc^\eta b^\eta c^{-1} b^{-\eta} c^{-\eta} = 1\rangle, \\ K_3 &= \langle a, c \,|\, a^\ell = c^\ell = ac^\eta a^\eta c^{-1} a^{-\eta} c^{-\eta} = 1\rangle. \end{aligned}$$

Since $(G_3; X, Z) = \pi/6$, all angles of the triangle are $\pi/3$.

For $3 \leq \ell \leq 5$ and $1 \leq \eta < \ell$, we can easily check that no word of the form ab^2a or ba^2b is a subword of a length 6 identity in K_1. In order to do this, we map K_1 onto one of the finite groups S_ℓ or $\mathbb{Z}_\ell$ and check if $ab^2ab^ia^jb^k$ presents the identity in the image. It turns out that $\phi(ab^2ab^ia^jb^k) \neq 1$ for all $i, j, k = 1, \ldots, \ell$ for at least one $\phi : K_1 \to F$, where F is S_ℓ or $\mathbb{Z}_\ell$. Then ab^2a is also not a part of a length 6 identity in K_1.

The same is true for K_2 and K_3. Then K, and therefore, $\mathcal{G}$ contains a free subgroup by Lemma 2.4. $\square$

Non-spherical squares of groups

Proposition 2.6. *Let Γ be a square of groups such that*

1. *the edge groups are non-trivial;*
2. *at least one of the edge groups contains 3 or more elements;*
3. *all the Gersten-Stallings angles are at most $\pi/2$.*

If $\mathcal{G}$ is the colimit of Γ, then $\mathcal{G}$ contains a non-abelian free subgroup.

Proof. Without loss of generality, assume that G_{12} has order at least 3. The condition on Gersten-Stallings angles means, for example, that the intersection of G_{12} and G_{23} in G_2 is trivial. Similarly, the intersection of G_{23} and G_{34} in G_3 is trivial. It follows that the subgroups G_{12} of G_2 and G_{34} of G_3 generate their free product in the group

$$A = G_2 \underset{G_{23}}{*} G_3.$$

By a similar argument, G_{12} and G_{34} generate their free product in the group

$$B = G_1 \underset{G_{41}}{*} G_4.$$

Hence we may write the colimit $\mathcal{G}$ as a free product with amalgamation

$$\mathcal{G} = A \underset{F}{*} B,$$

where $F \cong G_{12} * G_{34}$. Since $|G_{12}| \geq 3$ and $|G_{34}| \geq 2$, F and hence $\mathcal{G}$ contains a nonabelian free subgroup. $\square$

3 Generalised tetrahedron groups as triangles of groups

3.1 Spelling theorem

In general, it is not easy to calculate angles between subgroups in a group. The following theorem is very useful for this in the case of a generalised triangle group.

Theorem 3.1 (Spelling theorem, [5]). *Let H be the generalised triangle group defined by the presentation*

$$\langle x, y \mid x^p = y^q = W(x,y)^r = 1 \rangle,$$

where

$$W(x,y) = x^{\alpha_1} y^{\beta_1} \dots x^{\alpha_k} y^{\beta_k} \ (0 < \alpha_i < p, 0 < \beta_i < q),$$

and let

$$V(x,y) = x^{\gamma_1} y^{\delta_1} \dots x^{\gamma_\ell} y^{\delta_\ell} \ (0 < \gamma_i < p, 0 < \delta_i < q)$$

be a nonempty word that is equal to 1 in H. Then $\ell \geq k(r-1)+1$.

The proof of this is straightforward, but the idea behind it can be extended to yield a strengthened version. To motivate this extended proof, we first recall the proof of Theorem 3.1 from [5] (slightly modified).

Proof. Let

$$X = \begin{pmatrix} e^{i\pi/p} & \lambda \\ 0 & e^{-i\pi/p} \end{pmatrix}, \qquad Y = \begin{pmatrix} e^{i\pi/q} & 0 \\ 1 & e^{-i\pi/q} \end{pmatrix}$$

be two matrices in $SL(2, \mathbb{C}[\lambda])$. Then $\mathrm{tr}(X) = 2\cos(\pi/p)$, $\mathrm{tr}(Y) = 2\cos(\pi/q)$, and $\mathrm{tr}(W(X, Y))$ is a polynomial $\tau(\lambda)$ of degree k in λ. If Λ is any quotient ring of $\mathbb{C}[\lambda]$ in which $\tau(\lambda) = 2\cos(m\pi/r)$ for some $m = 1, \ldots, r-1$, then $x \mapsto X$, $Y \mapsto Y$ defines a representation $\rho : H \to PSL(2, \Lambda)$. Moreover, the lower left entry of $V(X, Y)$ is given by a polynomial $\sigma(\lambda)$ of degree $\ell - 1$, and necessarily $\sigma(\lambda) = 0$ in Λ, since ρ is a representation.

We now take $\Lambda = \mathbb{C}[\lambda]/I$, where I is the ideal generated by

$$f(\lambda) = \prod_{m=1}^{r-1} (\tau(\lambda) - 2\cos(m\pi/r)).$$

Since σ is a nonzero polynomial in $\mathbb{C}[\lambda]$ that belongs to I, it is divisible by f, and hence has degree greater than or equal to that of f. In other words

$$\ell - 1 \geq (r-1)k,$$

as claimed. $\square$

Now we push the idea behind Theorem 3.1 a little further to improve the lower bound on the length of V to $\mathrm{length}(V) \geq rk = \mathrm{length}(W^r)$ (which is clearly sharp).

Theorem 3.2. *Let H be a group with the following presentation:*

$$\langle x, y \mid x^p = y^q = W(x, y)^r = 1\rangle,$$

where

$$W(x, y) = x^{\alpha_1} y^{\beta_1} \ldots x^{\alpha_k} y^{\beta_k} \ (0 < \alpha_i < p, 0 < \beta_i < q),$$

and let

$$V(x, y) = x^{\gamma_1} y^{\delta_1} \ldots x^{\gamma_\ell} y^{\delta_\ell} \ (0 < \gamma_i < p, 0 < \delta_i < q)$$

be a nonempty word that is equal to 1 *in H. Then $\ell \geq rk$.*

Proof. Let

$$X = \begin{pmatrix} e^{i\pi/p} & \lambda \\ 0 & e^{-i\pi/p} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} e^{i\pi/q} & 0 \\ 1 & e^{-i\pi/q} \end{pmatrix}$$

be two matrices in $SL(2, \mathbb{C}[\lambda])$. Then $\mathrm{tr}(X) = 2\cos(\pi/p)$, $\mathrm{tr}(Y) = 2\cos(\pi/q)$, and $\mathrm{tr}(W(X, Y))$ is a polynomial $\tau(\lambda)$ of degree k in λ. Let

$$F_1(\lambda) = \prod_{\substack{m \text{ odd} \\ m \in \{1, \ldots, r-1\}}} (\tau(\lambda) - 2\cos(m\pi/r)),$$

$$F_2(\lambda) = \prod_{\substack{m \text{ even} \\ m\in\{1,\ldots,r-1\}}} (\tau(\lambda) - 2\cos(m\pi/r)).$$

We take Λ_i to be $\mathbb{C}[\lambda]/I_i$, where I_i is the ideal generated by F_i, $i = 1,2$. Then $\rho_1 : H \to PSL(2,\Lambda_1)$, $\rho_2 : H \to PSL(2,\Lambda_2)$ are two representations of H defined by $x \mapsto X$ and $y \mapsto Y$.

Split $V(X,Y)$ into two parts: $V(X,Y) = V_1(X,Y)V_2(X,Y)$, where $V_1(X,Y) = X^{\gamma_1}Y^{\delta_1}\ldots X^{\gamma_{[r/2]k}}Y^{\delta_{[r/2]k}}$ and $V_2(X,Y) = X^{\gamma_{[r/2]k+1}}Y^{\delta_{[r/2]k+1}}\ldots X^{\gamma_\ell}Y^{\delta_\ell}$.

Let $f(\lambda)$ and $g(\lambda)$ be the lower left entries of $V_1(X,Y)$ and $V_2(X,Y)$, respectively. Then

$$\deg(f(\lambda)) = [r/2]k - 1 \quad \text{and} \quad \deg(g(\lambda)) = \ell - [r/2]k - 1.$$

Suppose that, for $i = 1,2$ we have $V(X,Y) = \varepsilon_i I$ in $SL(2,\Lambda_i)$ (where $\varepsilon_i = \pm 1$). Rewriting this as $V_1(X,Y) = \varepsilon_i V_2(X,Y)^{-1}$, we see that $f(\lambda) = -\varepsilon_i g(\lambda)$ in Λ_i, and so, for some $A_i(\lambda) \in \mathbb{C}[\lambda]$,

$$f(\lambda) + \varepsilon_i g(\lambda) = A_i(\lambda)F_i(\lambda) \text{ in } \mathbb{C}[\lambda].$$

If $r = 2t$ is even, then since $\deg(F_1(\lambda)) = tk$, $\deg(F_1(\lambda)) > \deg(f(\lambda))$, so either $A_1(\lambda) = 0$ or $\deg(f(\lambda)) < \deg(g(\lambda))$. In either case $\ell \geq 2tk = rk$.

Hence we may assume that $r = 2t+1$ is odd. In this case $\deg(F_1(\lambda)) = \deg(F_2(\lambda)) = tk > \deg(f(\lambda))$.
If $\deg(f(\lambda)) \geq \deg(g(\lambda))$ then $\ell \leq 2tk = (r-1)k$ which contradicts Theorem 3.1. Hence, $\deg(f(\lambda)) < \deg(g(\lambda))$. In particular, $f(\lambda) \neq \pm g(\lambda)$, so $A_1(\lambda) \neq 0 \neq A_2(\lambda)$.

If $\varepsilon_1 = \varepsilon_2$ then $A_1(\lambda)F_1(\lambda) = A_2(\lambda)F_2(\lambda)$ in $\mathbb{C}[\lambda]$. Since F_1 and F_2 have no common root in $\mathbb{C}$, they are coprime in the unique factorization domain $\mathbb{C}[\lambda]$. It follows from the equation $A_1F_1 = A_2F_2$ that A_1 is a multiple of F_2, so $\deg(A_1(\lambda)) \geq \deg(F_2(\lambda)) = tk$. Hence

$$\deg(g(\lambda)) = \deg(f(\lambda) + \varepsilon_1 g(\lambda)) = \deg(A_1(\lambda)F_1(\lambda)) \geq 2tk.$$

Thus $\ell = \deg(g(\lambda)) + tk + 1 \geq 3tk + 1 > rk$.

Hence we are reduced to the case where $r = 2t+1$ is odd, $A_1(\lambda) \neq 0 \neq A_2(\lambda)$, and $\varepsilon_1 \neq \varepsilon_2$. Now we have

$$\begin{aligned} 2f(\lambda) &= A_1(\lambda)F_1(\lambda) + A_2(\lambda)F_2(\lambda) \\ &= (A_1(\lambda) + A_2(\lambda))F_2(\lambda) + A_1(\lambda)(F_1(\lambda) - F_2(\lambda)). \end{aligned} \tag{1}$$

Note that $F_1(\lambda) - F_2(\lambda)$ has degree $(t-1)k$. Therefore, from (1),

$$\begin{aligned} \deg((A_1(\lambda) + A_2(\lambda))F_2(\lambda)) &\leq \max\{\deg(f(\lambda)), \deg(A_1(\lambda)(F_1(\lambda) - F_2(\lambda)))\} \\ &= \max\{tk - 1, \ell - tk - k - 1\}. \end{aligned}$$

If $A_1(\lambda) + A_2(\lambda) \neq 0$ then

$$\begin{aligned} & tk = \deg(F_2(\lambda)) \leq \deg((A_1(\lambda) + A_2(\lambda))F_2(\lambda)) \leq \ell - tk - k - 1, \\ & \ell \geq 2tk + k + 1 = rk + 1. \end{aligned}$$

If, however, $A_1(\lambda) + A_2(\lambda) = 0$ then $2f(\lambda) = A_1(\lambda)(F_1(\lambda) - F_2(\lambda))$ and

$$\begin{aligned}
&\deg(f(\lambda)) = \deg(A_1(\lambda)) + \deg(F_1(\lambda) - F_2(\lambda));\\
&tk - 1 = \ell - tk - k - 1;\\
&\ell = 2tk + k = rk.
\end{aligned}$$

Thus, $\ell \geq rk$. $\square$

Corollary 3.3. *Let $\mathcal{G} = \langle a, b, c \,|\, a^\ell = b^m = c^n = (a^\alpha b^\beta)^p = W_2(b, c)^q = W_3(c, a)^r = 1\rangle$ be a generalised tetrahedron group realised as a non-spherical triangle of groups, where $\ell \leq m$. If any of the following conditions hold, then $\mathcal{G}$ contains a non-abelian free subgroup.*

1. *β is not coprime to m;*
2. *α is not coprime to ℓ;*
3. *$\ell \geq 3$;*
4. *$m \geq 4$ and $n \geq 3$.*

Proof. We apply Proposition 2.3, with $G_2 = \langle a, b \,|\, a^\ell = b^m = (a^\alpha b^\beta)^p = 1\rangle$.

1. Put $y = a$, $z = b$. In this case G_2 is a free product with amalgamation

$$G_2 = \langle a, d \,|\, a^p = d^s = (a^\alpha d)^t = 1\rangle \underset{d=b^\gamma}{*} \langle b \,|\, b^q = 1\rangle$$

   for some s, t, γ. Combining the Spelling Theorem 3.2 with the Normal Form Theorem for free products with amalgamation (see for example [15, Section I.11]), it is not difficult to see that any minimal length relation in G_2 must be a word in a and d, so cannot contain a subword ab. Hence Proposition 2.3 applies.

2. Similar to Part 1.

3. By Parts 1 and 2, we may assume that $\alpha = \beta = 1$, so that G_2 is a triangle group. Since $3 \leq \ell \leq m$, the only relations of minimal length in G_2 are cyclic conjugates of $(ab)^{\pm p}$, so no such relation contains a^2b as a subword. Hence Proposition 2.3 applies.

4. Again, we may assume that $\alpha = \beta = 1$ and G_2 is a triangle group. By Part 3 we may assume that $\ell = 2$. Since $n \geq 3$, the only minimal length relations in G_2 are cyclic conjugates of $(ab)^{\pm p}$, and since $m \geq 4$ no such word has ab^2 as a subword. Hence Proposition 2.3 applies.

$\square$

3.2 Generalised tetrahedron groups realised by Euclidean and spherical triangles of groups

A generalised tetrahedron group

$$\mathcal{G} = \langle x, y, z \mid x^\ell = y^m = z^n = W_1(x,y)^p = W_2(y,z)^q = W_3(x,z)^r = 1 \rangle$$

can be realised as the colimit of a triangle of groups with generalised triangle vertex groups

$$\begin{aligned} G_1 &= \langle x, y \mid x^\ell = y^m = W_1(x,y)^p = 1 \rangle, \\ G_2 &= \langle y, z \mid y^m = z^n = W_2(y,z)^q = 1 \rangle, \\ G_3 &= \langle x, z \mid z^n = x^\ell = W_3(x,z)^r = 1 \rangle, \end{aligned}$$

and with edge groups $X = \left\langle x \,|\, x^\ell = 1 \right\rangle$, $Y = \langle y \,|\, y^m = 1 \rangle$ and $Z = \langle z \,|\, z^n = 1 \rangle$. Let

$$\begin{aligned} W_1(x,y) &= x^{\alpha_1} y^{\beta_1} \dots x^{\alpha_{k_1}} y^{\beta_{k_1}}, \\ W_2(y,z) &= y^{\gamma_1} z^{\delta_1} \dots y^{\gamma_{k_2}} z^{\delta_{k_2}}, \\ W_3(x,z) &= x^{\eta_1} z^{\theta_1} \dots x^{\eta_{k_3}} z^{\theta_{k_3}}. \end{aligned}$$

By Theorem 3.2, if $V(x,y) = 1$ in G_1, then $V(x,y)$ has length at least $p\,k_1$. Then the angle between the two edge groups in G_1 is

$$(G_1; X, Y) = \frac{\pi}{p\,k_1}.$$

Similarly, we have

$$(G_2; Y, Z) = \frac{\pi}{q\,k_2} \qquad \text{and} \qquad (G_3; X, Z) = \frac{\pi}{r\,k_3}.$$

Hence,

$$(G_1; X, Y) + (G_2; Y, Z) + (G_3; X, Z) = \frac{\pi}{p\,k_1} + \frac{\pi}{q\,k_2} + \frac{\pi}{r\,k_3}.$$

Therefore, if the triangle of groups is spherical or Euclidean, then

$$(G_1; X, Y) + (G_2; Y, Z) + (G_3; X, Z) \geq \pi$$

gives

$$\frac{\pi}{p\,k_1} + \frac{\pi}{q\,k_2} + \frac{\pi}{r\,k_3} \geq \pi.$$

So we can determine p, q, r and k_i that give spherical or Euclidean triangles of groups. We have the following lists of presentations:

Euclidean.

E1. $\langle x, y, z \,|\, x^\ell = y^m = z^n = (x^\alpha y^\beta)^2 = (y^\gamma z^\delta)^3 = (x^\eta z^\theta)^6 = 1\rangle$

E2. $\langle x, y, z \,|\, x^\ell = y^m = z^n = (x^\alpha y^\beta)^2 = (y^\gamma z^\delta)^4 = (x^\eta z^\theta)^4 = 1\rangle$

E3. $\langle x, y, z \,|\, x^\ell = y^m = z^n = (x^\alpha y^\beta)^3 = (y^\gamma z^\delta)^3 = (x^\eta z^\theta)^3 = 1\rangle$

E4. $\langle x, y, z \,|\, x^\ell = y^m = z^n = (x^\alpha y^\beta)^2 = (y^\gamma z^\delta)^3 = (x^{\eta_1} z^{\theta_1} x^{\eta_2} z^{\theta_2})^3 = 1\rangle$

E5. $\langle x, y, z \,|\, x^\ell = y^m = z^n = (x^\alpha y^\beta)^2 = (y^\gamma z^\delta)^4 = (x^{\eta_1} z^{\theta_1} x^{\eta_2} z^{\theta_2})^2 = 1\rangle$

E6. $\langle x, y, z \,|\, x^\ell = y^m = z^n = (x^\alpha y^\beta)^2 = (y^{\gamma_1} z^{\delta_1} y^{\gamma_2} z^{\delta_2})^2 = (x^{\eta_1} z^{\theta_1} x^{\eta_2} z^{\theta_2})^2 = 1\rangle$

E7. $\langle x, y, z \,|\, x^\ell = y^m = z^n = (x^\alpha y^\beta)^2 = (y^\gamma z^\delta)^3 = (x^{\eta_1} z^{\theta_1} x^{\eta_2} z^{\theta_2} x^{\eta_3} z^{\theta_3})^2 = 1\rangle$

Spherical.

S1. $\langle x, y, z \,|\, x^\ell = y^m = z^n = (x^\alpha y^\beta)^2 = (y^\gamma z^\delta)^2 = W_3(x, z)^r = 1\rangle$

S2. $\langle x, y, z \,|\, x^\ell = y^m = z^n = (x^\alpha y^\beta)^2 = (y^\gamma z^\delta)^3 = (x^\eta z^\theta)^r = 1\rangle$, $r = 3, 4, 5$

S3. $\langle x, y, z \,|\, x^\ell = y^m = z^n = (x^\alpha y^\beta)^2 = (y^\gamma z^\delta)^3 = (x^{\eta_1} z^{\theta_1} x^{\eta_2} z^{\theta_2})^2 = 1\rangle$

4 Proof of Theorem 1

Theorem 1. *The class $\mathcal{C}$ of generalised tetrahedron groups realised by non-spherical triangle of groups satisfies the Tits alternative. More precisely, any $\mathcal{G} \in \mathcal{C}$ contains a non-abelian free subgroup unless $\mathcal{G}$ is isomorphic to the abelian-by-finite group $\Delta^*(p, q, r)$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.*

Proof. If $\mathcal{G}$ is realised by a negatively curved triangle of groups, then the theorem follows from Proposition 2.2. Hence we may assume that the triangle is Euclidean, that is we are in one of the cases E1-E7.

Clearly $\ell = m = n = 2$ implies that $\mathcal{G} = \Delta^*(p, q, r)$ is abelian-by-finite. Hence we can assume that at least one of ℓ, m, n is greater than 2. It is convenient to group the seven cases E1 to E7 according to the distribution of Gersten-Stallings angles.

Case E3

Corollary 3.3 implies that $\mathcal{G}$ has a free subgroup, except possibly if ℓ, m, n are (in some order) $2, 2, 3$. But then $\mathcal{G}$ is an ordinary tetrahedron group acting on hyperbolic 3-space and hence contains a non-abelian free subgroup.

Cases E2, E5, E6

$$\mathcal{G} = \langle x, y, z | x^\ell = y^m = z^n = R_1(x,y)^2 = R_2(y,z)^2 = R_3(z,x)^2 = 1\rangle,$$

for some words R_1, R_2 and R_3, where R_1, R_2 and R_3 have free-product lengths 2, 4 and 4 in $\mathbb{Z}_\ell * \mathbb{Z}_m$, $\mathbb{Z}_m * \mathbb{Z}_n$ and $\mathbb{Z}_n * \mathbb{Z}_\ell$ respectively.

If $n = 2$ then the normal closure of x and y can be expressed as the colimit of a non-spherical square of two-generator groups, where the edge groups are cyclic, generated by x, y, zxz and zyz respectively. Since at least one of ℓ, m is greater than 2 by hypothesis, we see by Proposition 2.6 that $\mathcal{G}$ contains a nonabelian free subgroup.

Hence we may assume that $n \geq 3$. If $\ell = m = 2$, then the normal closure of z can be expressed as the colimit of a non-spherical square of groups, with edge-groups generated by conjugates of z. Hence $\mathcal{G}$ contains a free subgroup by Proposition 2.6.

On the other hand, if $\ell \geq 3$ and $m \geq 3$, then $\mathcal{G}$ contains a free subgroup by Corollary 3.3. Hence we may assume that $\ell = 2$ and $m \geq 3$. By results of Rosenberger [19], the vertex group G_2, and hence also $\mathcal{G}$, contains a free subgroup unless $n = 3$. But then the normal closure of y and z in $\mathcal{G}$ is again a generalised tetrahedron group, of type E2 or E6, with all three generators of order 3. By Corollary 3.3 again, there is a free subgroup.

Cases E1, E4, E7

Corollary 3.3 implies that $\mathcal{G}$ has a free subgroup, except possibly if $m = 3$ and $\ell = n = 2$ (possible only in case E1) or if $m = 2$ and $n \leq 3$.

If $m = 3$ and $\ell = n = 2$ then $\mathcal{G}$ is an ordinary tetrahedron group acting on hyperbolic 3-space and hence the result follows.

Suppose then that $m = 2$ and $n \leq 3$. We treat the cases E1, E4 and E7 separately.

In E1, we must have $\{\ell, n\} = \{2, 3\}$. Again, $\mathcal{G}$ is an ordinary tetrahedron group acting on hyperbolic space, and so contains a nonabelian free subgroup. In case E4, we again have $\{\ell, n\} = \{2, 3\}$. There are two possibilities:

$$\text{(i)} \;\; \mathcal{G} = \langle x, y, z \,|\, x^2 = y^2 = z^3 = (xy)^2 = (yz)^3 = (xzxz^2)^3 = 1\rangle.$$

In this case the normal closure of y and z is again a generalised tetrahedron group, of type E3, with generators of orders $2, 3, 3$, so $\mathcal{G}$ contains a free subgroup.

$$\text{(ii)} \; \mathcal{G} = \langle x, y, z \,|\, x^3 = y^2 = z^2 = (xy)^2 = (yz)^3 = (xzx^2z)^3 = 1\rangle.$$

In this case the normal closure of x and $w = yz$ is a generalised triangle group $\langle x, w \,|\, x^3 = w^3 = (xwx^2w^2)^3 = 1\rangle$, which contains a free subgroup by [19].

In case E7, using the results of [14], G_3 and hence $\mathcal{G}$ contains a free subgroup except in a small number of cases. Combining this with Corollary 2.5, and with our assumption that $m = 2$ and $n \leq 3$, we are reduced to two cases:

(i) $\mathcal{G} = \langle x, y, z \,|\, x^2 = y^2 = z^3 = (xy)^2 = (yz)^3 = (xzxzxz^2)^2 = 1 \rangle$.

(ii) $\mathcal{G} = \langle x, y, z \,|\, x^4 = y^2 = z^2 = (xy)^2 = (yz)^3 = (xzxzx^2z)^2 = 1 \rangle$.

In (i) we note that the subgroup generated by y, z and $w = xz^2x$ is the colimit of a triangle of groups with all three Gersten-Stallings angles equal to $\pi/3$, where the edge groups are generated by y, z and w respectively, and one of the vertex groups is the binary octahedral group

$$H = \langle z, w \,|\, z^3 = w^3 = zwz^2wzw^2 = 1 \rangle.$$

Noting that neither zw^2z nor wz^2w is a subword of a length six relation in H, we apply Lemma 2.4 to see that $\mathcal{G}$ contains a nonabelian free subgroup.

In (ii) we can add the relation $x^2 = 1$ to get an epimorphism onto a free product with amalgamation that contains a free subgroup. □

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