All discrete $\mathcal{RP}$ groups whose generators have real traces

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Abstract

In this paper we give necessary and sufficient conditions for discreteness of a subgroup of $\mathrm{PSL}(2,\mathbb{C})$ generated by a hyperbolic element and an elliptic one of odd order with non-orthogonally intersecting axes. Thus we completely determine two-generator non-elementary Kleinian groups without invariant plane with real traces of the generators and their commutator. We also give a list of all parameters that correspond to such groups. An interesting corollary of the result is that the group of the minimal known volume hyperbolic orbifold $\mathbb{H}^3/\Gamma_{353}$ has real parameters.

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1 Introduction

The group of all Möbius transformations of the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is isomorphic to $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{\pm I\}$. The Poincaré extension gives the action of this group (as the group of all orientation preserving isometries) on hyperbolic 3-space

$$\mathbb{H}^3 = \{(z,t) \mid z \in \mathbb{C},\ t > 0\}$$

with the Poincaré metric

$$ds^2 = \frac{|dz|^2 + dt^2}{t^2}.$$

We study the class of $\mathcal{RP}$ *groups* (two-generator groups with real parameters):

$$\mathcal{RP} = \{\Gamma = \langle f, g\rangle \mid f, g \in \mathrm{PSL}(2,\mathbb{C});\ \beta, \beta', \gamma \in \mathbb{R}\},$$

where $\beta = \beta(f) = \mathrm{tr}^2 f - 4$, $\beta' = \beta(g) = \mathrm{tr}^2 g - 4$, $\gamma = \gamma(f,g) = \mathrm{tr}[f,g] - 2$.

Recall that an element $f \in \mathrm{PSL}(2,\mathbb{C})$ with real $\beta(f)$ is *elliptic*, *parabolic*, *hyperbolic*, or *π-loxodromic* according to whether $\beta(f) \in [-4,0)$, $\beta(f) = 0$, $\beta(f) \in (0,+\infty)$, or $\beta(f) \in (-\infty,-4)$. If $\beta(f) \notin [-4,\infty)$, then f is called *strictly loxodromic*. Among all strictly loxodromic elements only π-loxodromics have real $\beta(f)$.

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1.1. Elementary $\mathcal{RP}$ groups. Following [1], we call a subgroup Γ of $\mathrm{PSL}(2,\mathbb{C})$ *elementary* if there exists a finite Γ-orbit in $\mathbb{H}^3 \cup \overline{\mathbb{C}}$. Elementary groups are studied in [1, 22]. Among the $\mathcal{RP}$ groups, the following are elementary:

(1) Both generators are elliptic with intersecting axes.

(2) Both generators are elliptic of order 2 with mutually orthogonal skew axes or with disjoint axes lying in a hyperbolic plane.

(3) One generator is hyperbolic or π-loxodromic and the other is elliptic of order 2 whose axes intersect orthogonally.

(4) The generators (with real β, β', and γ) share a fixed point in $\overline{\mathbb{C}}$.

The list of all *discrete* elementary groups (not necessarily from the $\mathcal{RP}$ class) is given in [1], so there is no need to consider discrete elementary $\mathcal{RP}$ groups here.

1.2. $\mathcal{RP}$ groups with invariant hyperbolic plane.

(1) Any two-generator Fuchsian group, i.e., conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$, is an $\mathcal{RP}$ group. Discreteness conditions for such groups have been well-studied (see [13, 14, 24], for a complete list of references see [7]).

(2) Any non-Fuchsian two-generator subgroup of $\mathrm{PSL}(2,\mathbb{C})$ with invariant hyperbolic plane is also an $\mathcal{RP}$ group. Such a group contains elements that reverse orientation of the invariant plane and interchange the half-spaces bounded by it. Discreteness criteria and a classification theorem for all such discrete groups are given in [20].

In [8], the space of discrete $\mathcal{RP}$ groups with parameters $(\beta, -4, \gamma)$ is analyzed. Since one of the generators is elliptic of order 2 and parameters β and γ are real, it follows from [18, Theorems 1(i) and 2(i)] that all such groups are either elementary or have an invariant plane.

Our paper is devoted to the following class:

1.3. Truly spatial $\mathcal{RP}$ groups. We call a subgroup of $\mathrm{PSL}(2,\mathbb{C})$ *truly spatial*, if it is not elementary and has no invariant hyperbolic plane. All truly spatial $\mathcal{RP}$ groups are characterized by the following theorem (see also Table 1 in [18]):

Theorem 1. [18, Theorem 4] *Let* $\Gamma = \langle f, g\rangle$ *be an* $\mathcal{RP}$ *group.* Γ *is a truly spatial group if and only if*

$$(-1)^k\gamma < (-1)^{k+1}\beta\beta'/4 \text{ with } \gamma \neq 0,\ \beta \neq -4, \text{ and } \beta' \neq -4,$$

where $k \in \{0,1,2\}$ *is the number of π-loxodromic elements among* f *and* g.

The current paper is the last in a series of earlier papers [15]–[19] that gathered together give criteria (necessary and sufficient conditions) for discreteness of all truly spatial $\mathcal{RP}$ groups with real traces of both generators. Thus, we complete the description of all discrete $\mathcal{RP}$ groups with non-π-loxodromic generators. The result in terms of parameters can be found in Table 2 in Appendix, where all real triples (β, β', γ) for non-elementary discrete $\mathcal{RP}$ groups without invariant plane with non-π-loxodromic generators are listed.

Our main result is Theorem A below that gives criteria for discreteness of truly spatial $\mathcal{RP}$ groups generated by a hyperbolic element and an elliptic one of odd order with intersecting axes. The discrete groups from this theorem correspond to rows 21–41 of Table 2 in Appendix. In the theorem we assume without loss of generality that the elliptic generator is primitive; if not, we can replace it by its primitive power of the same order (cf. [18]).

Theorem A. *Let f be a primitive elliptic element of odd order n, $n \geq 3$, g be a hyperbolic element, and let their axes intersect non-orthogonally. Then*

(1) *There exist elements $h_1, h_2 \in \mathrm{PSL}(2,\mathbb{C})$ such that $h_1^2 = gfg^{-1}f$, $h_2^2 = f^{(n-1)/2}g^{-1}f^{-1}gf^{-(n+1)/2}gf^{-1}g^{-1}$, $(h_1f^{-1})^2 = 1$, and h_2gfg^{-1} is an elliptic element whose axis intersects the axis of f. There exists also an element $h_3 \in \mathrm{PSL}(2,\mathbb{C})$ such that $h_3^2 = f^{(n-1)/2}g^{-1}h_1^{-1}gf^{-(n-3)/2}h_1^{-1}$ and h_3h_1 is an elliptic element whose axis intersects the axis of f.*

(2) *$\Gamma = \langle f, g\rangle$ is discrete if and only if one of the following conditions holds:*

- (*i*) *h_1 is hyperbolic, parabolic, or a primitive elliptic element of even order $2m$ $(2/n+1/m < 1)$ and h_2 is hyperbolic, parabolic, or a primitive elliptic element of even order $2l$, $l \geq 2$;*
- (*ii*) *h_1 and h_2 are rotations through angles of π/m and π/l, respectively, and the ordered triple $\{n, m, l\}$ is one of the following: $\{5, 2, 5/2\}$; $\{3, m, m/3\}$, m is an integer, $m \geq 4$, $(m, 3) = 1$; $\{n, 3, n/3\}$, $n \geq 5$, $(n, 3) = 1$; $\{3, 5, 3/2\}$; $\{5, 2, 5/3\}$; $\{3, 5, 5/4\}$; $\{5, 3, 5/4\}$; $\{5, 2, 3/2\}$;*
- (*iii*) *$n = 3$, h_1 is hyperbolic, h_2 is a rotation through an angle of $4\pi/k$, h_2gfg^{-1} is a primitive elliptic element of order p and the ordered pair $\{k, p\}$ is one of the following: $\{6, 5\}$; $\{k, 3\}$, k is an integer, $k \geq 7$, $(k, 4) \leq 2$;*
- (*iv*) *h_1 is hyperbolic and h_2 is the cube of a primitive elliptic element $\tilde{h}_2$ of order $2n$ so that $\tilde{h}_2^2gfg^{-1}$ is a rotation through angle $4\pi/k$, where the ordered pair $\{n, k\}$ is one of the following: $\{5, 5\}$; $\{n, 4\}$, $n \geq 5$, $(n, 3) = 1$;*
- (*v*) *h_1 is a primitive elliptic element of odd order $\widetilde{m}$ $(1/n + 1/\widetilde{m} < 1/2)$ and h_3 is hyperbolic, parabolic, or a primitive elliptic element of even order $2k$, $k \geq 2$;*
- (*vi*) *$n = 3$, h_1 and h_3 are primitive elliptic elements of the same order $\widetilde{m}$, $\widetilde{m} \geq 7$, $(\widetilde{m}, 2) = 1$;*
- (*vii*) *h_1 is the square of a primitive elliptic element $\tilde{h}_1$ of order n, $n \geq 7$, and h_4 is hyperbolic, parabolic, or a primitive elliptic element of even order ≥ 4, where h_4 is defined as follows: $h_4^2 = f^{(n-3)/2}g^{-1}\tilde{h}_1gf^{-(n+1)/2}\tilde{h}_1f^{-1}$ and $h_4f\tilde{h}_1^{-1}$ is an elliptic element whose axis intersects the axis of f.*

The content of the paper is mainly the proof of this theorem that is given in Sections 3–6. In Section 8 we reformulate Theorem A in terms of parameters to get Theorem B.

Minimal volume hyperbolic 3-orbifold. Let Γ_{353} be a $\mathbb{Z}_2$-extension of the orientation preserving index 2 subgroup of the group generated by reflections in the faces of the hyperbolic tetrahedron with Coxeter diagram 3–5–3. This group is known to have minimal covolume (=0.03905...) among all arithmetic groups [2] and among all groups with a tetrahedral subgroup (the symmetry group of a regular tetrahedron) and groups containing elliptic elements of order $p \geq 4$ [11]. The latter fact is a great evidence of the Gehring-Martin conjecture that Γ_{353} has minimal covolume among all Kleinian groups. In Section 7 we prove that Γ_{353} is an $\mathcal{RP}$ group.

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2 Preliminaries

For the benefit of the reader, we will list some preliminary definitions and results that we will need in this paper.

For basic definitions in discrete groups we refer the reader to [22]. All formulas of hyperbolic geometry that we use in calculations were taken from [6].

We also use the following powerful instruments in our proof: discrete extensions of tetrahedral groups and minimal distances between axes of elliptic elements in discrete groups.

Denote by G_T the group generated by reflections in the faces of a compact Coxeter tetrahedron T. All discrete extensions of its orientation preserving index 2 subgroup Δ_T were obtained by Derevnin and Mednykh in [3].

The minimal distance $\rho_{min}(p, q)$ between the axes of two elliptic elements of orders p and q in a discrete group were obtained by Gehring and Martin in [10] and Gehring, Marshall, and Martin in [12]. Table 1 shows $\cosh \rho_{min}(p, q)$ for $p, q \leq 7$.

Table 1: $\cosh \rho_{min}(p,q)$, $p, q \le 7$

	2	3	4	5	6	7
2	1.000	1.019	1.088	1.106	1.225	1.152
3	1.019	1.079	1.155	1.376	1.155	1.198
4	1.088	1.155	1.366	1.203	1.414	1.630
5	1.106	1.376	1.203	1.447	1.701	1.961
6	1.225	1.155	1.414	1.701	2.000	2.305
7	1.152	1.198	1.630	1.961	2.305	1.656

In fact, for $p \ge 7$ and $q \ge 2$ the minimal distance is determined by the following formula [12]:

$$2\sin(\pi/p)\sin(\pi/q)\cosh\rho_{min} = \begin{cases} 1 & \text{if} \quad q \ne 3, q \ne p, \\ \cos(\pi/p) & \text{if} \quad q = 3, \\ \cos(2\pi/p) & \text{if} \quad q = p. \end{cases} \qquad (1)$$

We will also need the following geometric Lemmas.

Lemma 1. *Let l_1 and l_2 be two disjoint lines in $\mathbb{H}^2$ and let l be a transversal of them meeting them in points P and Q with corresponding angles of ψ and χ, respectively, where $\psi < \chi < \pi/2$. Let l_1' and l_2' be new lines passing through P and Q so that their corresponding angles are of $k\psi$ and $k\chi$, respectively, $0 < k < 1$ (see Figure 1a). Then l_1' and l_2' are also disjoint.*

Proof. By contradiction. Suppose l_1' and l_2' are parallel or intersecting. Then

$$\cosh PQ \le \frac{1 - \cos k\psi \cos k\chi}{\sin k\psi \sin k\chi} \qquad (2)$$

Note that (2) is an identity for parallel lines. Now prove that

$$f(k) = \frac{1 - \cos k\psi \cos k\chi}{\sin k\psi \sin k\chi}, \qquad 0 < \psi < \chi < \pi/2,$$

is increasing on $(0, 1]$. Indeed,

$$\begin{aligned} f'(k)(\sin k\psi \sin k\chi)^2 &= (\psi \sin k\psi \cos k\chi + \chi \sin k\chi \cos k\psi)\sin k\psi \sin k\chi + \\ &\quad (-1 + \cos k\psi \cos k\chi)(\psi \cos k\psi \sin k\chi + \chi \cos k\chi \sin k\psi) \\ &= (\chi \sin k\psi - \psi \sin k\chi)(\cos k\psi - \cos k\chi) > 0, \end{aligned}$$

since both expressions in parentheses are positive. (The first one is positive because $\sin(x)/x$ is decreasing on $(0, \pi/2)$.)

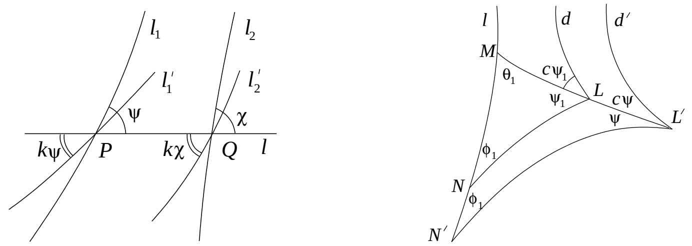
Therefore, $f(k)$ is increasing and we have

$$\frac{1 - \cos k\psi \cos k\chi}{\sin k\psi \sin k\chi} < \frac{1 - \cos \psi \cos \chi}{\sin \psi \sin \chi}, \quad 0 < k < 1. \qquad (3)$$

Since l_1 and l_2 are disjoint, PQ is greater than in case of parallel lines:

$$\cosh PQ > \frac{1 - \cos \psi \cos \chi}{\sin \psi \sin \chi}. \qquad (4)$$

Combining (2), (3), and (4), we have $\cosh PQ < \cosh PQ$. Contradiction. ☐



(a) (b)

Figure 1:

Lemma 2. *Let a triangle $\Delta = \Delta(\phi, \psi, \theta)$ in $\mathbb{H}^2$ have angles of $\phi \le \phi_0$, $\psi \le \psi_0$, $\theta \le \theta_0$ at vertices N, L, and M, respectively. Fix a real number c $(0 < c \le 1)$. Let $l = l(\phi, \psi, \theta)$ be the line through NM and let $d = d(\phi, \psi, \theta)$ be a half-line from L making an angle of $c\psi$ with LM so that d and Δ lie in different half-planes with respect to LM. Suppose d and l are disjoint for some $\phi = \phi_0$, $\psi = \psi_0$, $\theta = \theta_0$. Then d and l are also disjoint for any $\phi \le \phi_0$, $\psi \le \psi_0$, $\theta \le \theta_0$.*

Proof. Fix a point $(\phi_1, \psi_1, \theta_1)$ in 3-dimensional parameter space such that $\phi_1 \le \phi_0$, $\psi_1 \le \psi_0$, $\theta_1 \le \theta_0$. Suppose that the lemma holds at that point, i.e., $d = d(\phi_1, \psi_1, \theta_1)$ and $l = l(\phi_1, \psi_1, \theta_1)$ are disjoint. It is sufficient to show that d and l are also disjoint at (ϕ, ψ_1, θ_1), (ϕ_1, ψ, θ_1), and (ϕ_1, ψ_1, θ), where $\phi \le \phi_1$, $\psi \le \psi_1$, $\theta \le \theta_1$.

The first and the third cases are easy and true for any $c > 0$ that satisfies $c\psi_0 < \pi$.

In case (ϕ_1, ψ, θ_1), one can apply Lemma 1 with $l_2 = NL$, $l_1 = N'L'$, $l_2' = d$, $l_1' = d'$, $k = c < 1$ (see Figure 1b). For $c = 1$, the lines d and d' are disjoint, because they are images of NL and $N'L'$ under reflection in ML'. Hence, d' and l are also disjoint, since they lie on different sides of d. ☐

Polyhedra and links

A plane divides $\mathbb{H}^3$ into two components; we will call the closure of either of them a *half-space* in $\mathbb{H}^3$.

A connected subset P of $\mathbb{H}^3$ with non-empty interior is said to be a *(convex) polyhedron* if it is the intersection of a family $\mathcal{H}$ of half-spaces with the property that each point of P has a neighborhood meeting at most a finite number of boundaries of elements of $\mathcal{H}$.

Definition. Let P be a polyhedron in $\mathbb{H}^3$ and let ∂P be its boundary in $\mathbb{H}^3$. In (1)–(3) below we define the link for different "boundary" points of P (cf. [4]).

(1) Let $p \in \partial P$. Let S be a sphere in $\mathbb{H}^3$ with center p, whose radius is chosen small enough so that it only meets faces of P which contain p. Such a sphere exists by the local finiteness property we claim in the definition of a polyhedron. There is a natural way to endow S with a spherical geometry identifying S with $\mathbb{S}^2$ as follows. Map conformally $\mathbb{H}^3$ onto the unit ball $\mathbb{B}^3 = \{x \in \mathbb{R}^3 \mid |x| < 1\}$ so that p goes to 0 and after that change the scale of the sphere to be of radius 1. The *link* of p in P is defined to be the image of $S \cap P$ under the above identification (it is well-defined up to isometry).

(2) Let $\overline{\partial P}$ be the closure of ∂P in $\overline{\mathbb{H}}^3 = \mathbb{H}^3 \cup \overline{\mathbb{C}}$. Suppose $\overline{\partial P} \backslash \partial P \neq \emptyset$, and let $p \in \overline{\partial P} \backslash \partial P$. Then $p \in \overline{\mathbb{C}}$ (i.e., it is an ideal point). Let S be a horosphere centered at p that only meets those faces of P whose closures in $\overline{\mathbb{H}}^3$ contain p. We can identify S with Euclidean plane $\mathbb{E}^2$ using an isometry of $\mathbb{H}^3$ that sends p to ∞. The image of $S \cap P$ under such identification is called the *link* of the ideal boundary point p in P. Note that such a link is defined up to similarity.

(3) Suppose that there exists a hyperbolic plane S orthogonal to some faces $F_1, \ldots, F_t$ of P, and suppose that the other faces of P lie in the same open half-space which is bounded by S. If $t \geq 3$ then we say that S corresponds to an *imaginary vertex* p of P; and we define the *link* of p in P to be $S \cap P$.

Notice that the link of a proper (lying in $\mathbb{H}^3$), ideal, or imaginary vertex p in P is a spherical, Euclidean, or hyperbolic polygon (possibly of infinite area), respectively.

The surface S in the definition is called a *link surface* and is orthogonal to all faces of P that meet S; hence the group generated by reflections in these faces keeps S invariant and can be considered as the group of reflections in sides of a spherical, Euclidean, or hyperbolic polygon depending on the type of the vertex.

If such a polygon is a triangle with two primitive and one non-primitive angle, we use Figure 2 that gives a list of all such triangles in hyperbolic and spherical cases which 'generate' a discrete reflection group. The list of the hyperbolic triangles first appeared in Knapp's paper [21], then Matelski gave a nice geometric proof [23]. The list for the spherical case can be found in [5]. As for triangles with more than one non-primitive angle reflections in sides of which generate a discrete group, see [20, Lemma 2.1] for a hyperbolic list and [5] for a spherical one.

We denote a triangle with angles π/p, π/q, and π/r in any of spaces $\mathbb{S}^2$, $\mathbb{E}^2$ or $\mathbb{H}^2$ by (p, q, r). Note that the only Euclidean triangle with at least one fractional angle that gives a discrete group is $(6, 6, 3/2)$.

Definition. We define a *tetrahedron* T to be a polyhedron which in the projective ball model is the intersection of the hyperbolic space $\mathbb{H}^3$ with a usual Euclidean tetrahedron T_E (possibly with vertices on the sphere $\partial\mathbb{H}^3$ at infinity or beyond it) so that the intersection of each edge of T_E with $\mathbb{H}^3$ is non-empty.

Such a tetrahedron can be non-compact or have infinite volume, but the

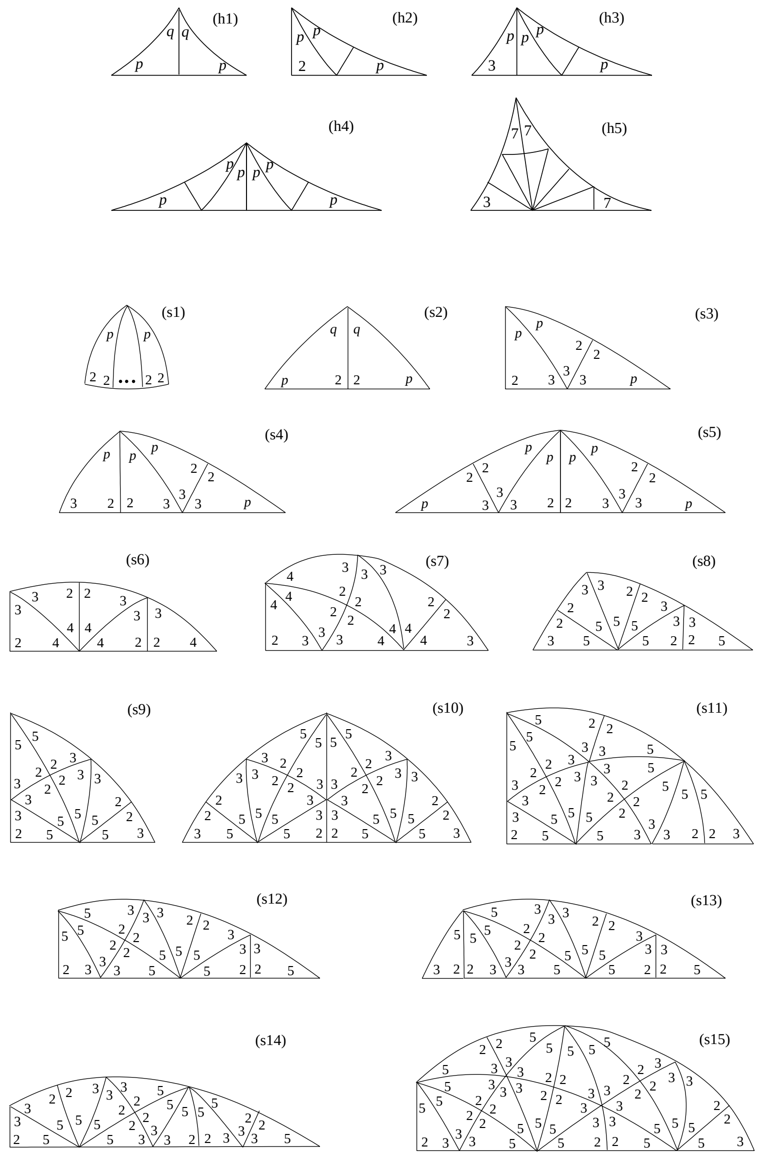


Figure 2: All hyperbolic and spherical triangles with two primitive and one non-primitive angles that generate a discrete group

links of its vertices are compact spherical, Euclidean, or hyperbolic triangles.

Let a tetrahedron T (possibly of infinite volume) have dihedral angles π/λ_1, π/λ_2, π/λ_3 at some face and let π/μ_1, π/μ_2, π/μ_3 be dihedral angles of T that are opposite to π/λ_1, π/λ_2, π/λ_3, respectively. We denote such a tetrahedron by $T = [\lambda_1, \lambda_2, \lambda_3; \mu_1, \mu_2, \mu_3]$ and the group generated by reflections in its faces by G_T.

3 Basic construction

In this section we describe the main steps of the proof of Theorem A and prepare for a detailed description of the list of all discrete groups.

The method of the proof is geometric and is based on using the Poincaré theorem [4]. However, the Poincaré theorem gives conditions on the dihedral angles of a fundamental polyhedron for a group. We would like to obtain conditions that depend only on the generators f and g of Γ. That is why we introduce auxiliary elements h_i, $i = 1, \dots, 4$, in Theorem A.

Each of h_i is defined by two relations. For example, for h_1 we have $h_1^2 = gfg^{-1}f$ and $(h_1 f^{-1})^2 = 1$. The first relation means that h_1 is a square root of $gfg^{-1}f$, but there are in general two square roots of $gfg^{-1}f$ in $\mathrm{PSL}(2,\mathbb{C})$. In order to rewrite conditions used in the Poincaré theorem, we need namely that one which is defined by the second relation $(h_1 f^{-1})^2 = 1$. The same is true for other elements h_i.

Existence of h_1 and h_2 is proved in Subsection 3 of this section and existence of h_3 is proved in Section 4.1. This proves the part (1) of Theorem A. Existence of h_4 that appears in item (2)(vii) of Theorem A is proved in Section 4.2.

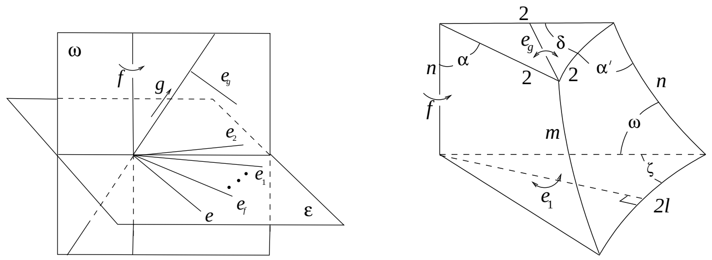
3.1. Construction of Γ^*. We start with construction of a group Γ^* containing Γ as a subgroup of finite index. Such a group is discrete if and only if Γ is.

Let f be a primitive elliptic element of odd order $n \geq 3$, g be a hyperbolic element, and let their axes intersect non-orthogonally. We denote elements and their axes by the same letters when it does not lead to any confusion. Let ω be a plane containing f and g, and let e be a half-turn with the axis which is orthogonal to ω and passes through the point of intersection of f and g.

For the group $\Gamma = \langle f, g \rangle$ we define two finite index extensions of it as follows: $\widetilde{\Gamma} = \langle f, g, e \rangle$ and $\Gamma^* = \langle f, g, e, R_\omega \rangle$ (we denote the reflection in a plane κ by R_κ). The groups Γ, $\widetilde{\Gamma}$, and Γ^* are either all discrete or all non-discrete.

3.2. New generators for Γ^*. Let e_f and e_g be half-turns such that $f = e_f e$ and $g = e_g e$. Axes e_f and e lie in some plane, denote it by ε, and intersect at an angle of π/n; ε and ω are mutually orthogonal; e_g is orthogonal to ω and intersects g, moreover, the distance between e_g and e is equal to half of the translation length of g.

Consider ε and $\langle e, e_f \rangle$ (see Figure 3a). The group contains elements e, $e_f = fe$, $f^2 e, \dots$ Each element $f^k e$, $k = 0, 1, 2, \dots$ is a half-turn with an axis



(a) (b)

Figure 3:

lying in ε. Denote $e_1 = f^{(n-1)/2}e$ and $e_2 = f^{(n+1)/2}e$. Note that ω is a bisector of the two lines e_1 and e_2.

Let α be a hyperbolic plane such that $f = R_\omega R_\alpha$. Then $\Gamma^* = \langle e_1, e_g, R_\alpha, R_\omega \rangle$.

3.3. Construction of a fundamental polyhedron for Γ^* *corresponding to Theorem A(2)(i) and existence of* h_1 *and* h_2. First, note that there exists a plane δ which is orthogonal to the planes α, ω, and $\alpha' = e_g(\alpha)$. The plane δ passes through the common perpendicular to f and $e_g(f)$ orthogonally to ω. It is clear that $e_g \subset \delta$.

We need one more plane, denote it by ζ, for the construction of a fundamental polyhedron for Γ^*. To construct ζ we use an auxiliary plane κ that passes through e_1 orthogonally to α'. The plane ζ then passes through e_1 orthogonally to κ. (Note that in general $\zeta \neq \varepsilon$.) In fact, the plane α' and the line e_1 can either intersect, or be parallel, or disjoint. Note that if $\zeta \cap \alpha' \neq \emptyset$ then $e_1 \perp (\zeta \cap \alpha')$.

Let $\mathcal{P}$ be the convex polyhedron bounded by α, ω, α', δ, and ζ. Note that $\mathcal{P}$ can be compact or non-compact (see Figure 3b, where $\mathcal{P}$ is drawn as compact).

Consider the dihedral angles of $\mathcal{P}$. The angles between δ and ω, δ and α, δ and α' are of $\pi/2$; the angles formed by ω with α and α' are equal to π/n; the sum of the angles formed by ζ with ω and α is equal to π. The planes α and α' can either intersect or be parallel or disjoint. The same is true for ζ and α'. Denote the angle between α and α' by π/m and the angle between α' and ζ by $\pi/(2l)$; we use the symbols ∞ and $\overline{\infty}$ for l or m if the corresponding planes are parallel or disjoint, respectively. It is clear that if

$$m \ (2/n + 1/m < 1) \text{ and } l \ (l \geq 2) \text{ are integers, } \infty, \text{ or } \overline{\infty}, \qquad (5)$$

then $\mathcal{P}$ and elements e_1, e_g, R_α, R_ω, $R'_\alpha = e_g R_\alpha e_g$ satisfy the hypotheses of the Poincaré theorem, Γ^* is discrete, and $\mathcal{P}$ is its fundamental polyhedron.

Now we rewrite the condition (5) via conditions on elements of $\mathrm{PSL}(2,\mathbb{C})$. In fact, we wish to show that this sufficient condition for discreteness of Γ^* is equivalent to Item (i) of the theorem.

A natural choice of elements $R'_\alpha R_\alpha$ and $R''_\alpha R'_\alpha$, where $R''_\alpha = e_1 R_\alpha e_1$, does not provide enough information about $\mathcal{P}$. (Even if $R'_\alpha R_\alpha$ and $R''_\alpha R'_\alpha$ are primitive elliptic elements, the corresponding dihedral angles of $\mathcal{P}$ can be obtuse. We refer the reader to [18, pp. 257–258] for more detailed explanation.) Therefore, we choose the following square roots instead.

The first element we define is $h_1 = R_\xi R_\alpha = R'_\alpha R_\xi$, where ξ is the bisector of α and α' passing through e_g. Then

$$h_1^2 = R'_\alpha R_\alpha = R'_\alpha R_\omega R_\omega R_\alpha = f'f,$$

where

$$f' = R'_\alpha R_\omega = e_g R_\alpha e_g R_\omega = e_g R_\alpha R_\omega e_g = e_g f^{-1} e_g = e_g efee_g = gfg^{-1}. \tag{6}$$

Since ξ is orthogonal to ω, $(R_\xi R_\omega)^2 = 1$. On the other hand,

$$R_\xi R_\omega = h_1 R_\alpha R_\omega = h_1 f^{-1}.$$

The above equations imply two conditions

$$h_1^2 = gfg^{-1}f \qquad \text{and} \qquad (h_1 f^{-1})^2 = 1$$

that uniquely determine h_1 as an element of $\mathrm{PSL}(2,\mathbb{C})$. Moreover, h_1 is a primitive elliptic element of even order $2m$ $(2/n + 1/m < 1)$ if and only if the dihedral angle of $\mathcal{P}$ at the edge $\alpha \cap \alpha'$ is equal to π/m, $m \in \mathbb{Z}$ $(2/n + 1/m < 1)$; h_1 is parabolic (hyperbolic) if and only if α and α' are parallel (disjoint, respectively).

Similarly, we define the second element as $h_2 = R_\zeta R'_\alpha = R''_\alpha R_\zeta$. Then

$$h_2^2 = R''_\alpha R'_\alpha = R''_\alpha R_\omega R_\omega R'_\alpha = t(f')^{-1} = tgf^{-1}g^{-1}, \tag{7}$$

where

$$t = R''_\alpha R_\omega = e_1 R'_\alpha e_1 R_\omega = e_1 R'_\alpha R_\alpha e_1 = e_1 h_1^2 e_1.$$

Furthermore, since $e_1 = f^{(n-1)/2}e$ and $h_1^2 = gfg^{-1}f$,

$$t = f^{(n-1)/2} egfg^{-1} f^{(n+1)/2} e = f^{(n-1)/2} g^{-1} f^{-1} g f^{-(n+1)/2}. \tag{8}$$

From (7) and (8) we have $h_2^2 = f^{(n-1)/2} g^{-1} f^{-1} g f^{-(n+1)/2} g f^{-1} g^{-1}$.

We now need to choose the square root of $R''_\alpha R'_\alpha$ that corresponds to the dihedral angle of $\mathcal{P}$ made by α' and α''. The required h_2 is such that $h_2 f' = h_2 gfg^{-1}$ is an elliptic element whose axis intersects f.

With this choice, h_2 is a primitive elliptic element of even order $2l$, $l \geq 2$, if and only if the dihedral angle between α' and α'' is π/l, $l \in \mathbb{Z}$ $(l \geq 2)$; h_2 is parabolic (hyperbolic) if and only if α' and α'' are parallel (disjoint, respectively).

Therefore, the condition (5) is equivalent to

> h_1 is hyperbolic, parabolic or a primitive elliptic element of even order $2m$ $(2/n + 1/m < 1)$ and h_2 is hyperbolic, parabolic or a primitive elliptic element of even order $2l$ $(l \geq 2)$,

which is item (2)(*i*) of Theorem A.

3.4. Assume that the condition (5) does not hold. This means that $\mathcal{P}$ itself is not a fundamental polyhedron for Γ^*. Then discrete groups may appear in one of the following cases:

(1) m is fractional, $1/m + 2/n < 1$;

(2) m is an integer, $1/m + 2/n < 1$, and l is fractional, $l > 1$;

(3) $m \in \{\infty, \overline{\infty}\}$ and l is fractional, $l > 1$.

These cases require a further consideration and in Sections 4–6 we completely determine the list of discrete groups in each case.

Namely, in each of the cases (1)–(3) we assume first that Γ^* is discrete to exclude *some* groups that cannot be discrete, that is, we get necessary conditions for discreteness. It turns out that *all* remaining groups are discrete, that is the necessary conditions are also sufficient. Such discrete groups are listed in items (2)(*ii*)–(*vii*) of Theorem A. In order to prove sufficiency, we forget that Γ^* was assumed to be discrete and for each such group we give a polyhedron and generators of Γ^* satisfying the hypotheses of the Poincaré theorem. Hence, Γ^*, and therefore Γ, is discrete and the sufficient part of (2) in Theorem A is also proved.

4 m is fractional, $1/m + 2/n < 1$

We have assumed that $\Gamma^* = \langle e_1, e_g, R_\alpha, R_\omega \rangle$ is discrete, so each of its subgroups is also discrete. Hence $\langle R_\omega, R_\alpha, R'_\alpha \rangle$ is discrete. Notice that $\langle R_\omega, R_\alpha, R'_\alpha \rangle$ acts as a group of reflections in the sides of a hyperbolic triangle (n, n, m) which is the upper face of $\mathcal{P}$ (see Figure 3b). From the list of all triangles with two primitive angles that give a discrete group (see Figure 2), we have that for m fractional either

(a) $\widetilde{m} = 2m$ is odd; or

(b) $4m = n$.

In both cases Γ^* contains the reflection R_ξ, where ξ is the plane that bisects the dihedral angle of $\mathcal{P}$ at the edge $\alpha \cap \alpha'$. Moreover, R_δ also belongs to Γ^*, because ξ passes through e_g and $e_g = R_\xi R_\delta$.

4.1. $\widetilde{m} = 2m$ is odd

That is

$$h_1 \text{ is a primitive elliptic element of odd order } \widetilde{m},\ 1/n + 1/\widetilde{m} < 1/2. \tag{9}$$

Let $\mathcal{P}_1$ be a polyhedron bounded by ω, α, δ, $e_1(\delta)$, ξ, and $e_1(\xi)$. Let π/k be the dihedral angle of $\mathcal{P}_1$ at the edge $\xi \cap e_1(\xi)$, $k \in (1, \infty) \cup \{\infty, \overline{\infty}\}$. The other angles of $\mathcal{P}_1$ are submultiples of π (see Figure 4).

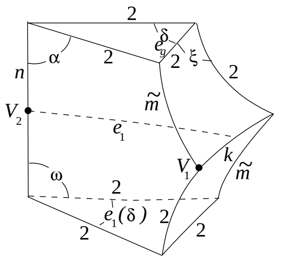


Figure 4: Polyhedron $\mathcal{P}_1$

If

$$k > 1 \text{ is an integer, } \infty, \text{ or } \overline{\infty} \tag{10}$$

then Γ^* is discrete. Its fundamental polyhedron is a half of $\mathcal{P}_1$.

Again, rewrite the condition (10) as a condition on elements of $\mathrm{PSL}(2,\mathbb{C})$. Let $h_3 = R_\beta R_\xi$, where β is the bisector of ξ and $e_1(\xi)$ passing through e_1. Then

$$\begin{aligned} h_3^2 &= (R_\beta R_\xi)^2 = (e_1 R_\xi e_1) R_\xi = e_1 R_\xi e_1 R_\omega R_\omega R_\xi = e_1 R_\xi R_\alpha e_1 h_1 f^{-1} \\ &= e_1 h_1 e_1 f h_1^{-1} = f^{(n-1)/2} e h_1 e f^{-(n-1)/2} f h_1^{-1}. \end{aligned}$$

Note that since $e_g h_1 e_g = h_1^{-1}$, we have $e h_1 e = g^{-1} h_1^{-1} g$ and, therefore,

$$h_3^2 = f^{(n-1)/2} g^{-1} h_1^{-1} g f^{-(n-3)/2} h_1^{-1}.$$

We have $h_3 = R_\beta R_\xi$ if and only if $h_3 h_1$ is an elliptic element whose axis intersects the axis of f.

The element h_3 is a primitive elliptic of even order $2k$, $k \geq 2$, if and only if the dihedral angle of $\mathcal{P}_1$ between ξ and $e_1(\xi)$ is π/k, $k \in \mathbb{Z}$, $k \geq 2$; h_3 is parabolic (hyperbolic) if and only if ξ and $e_1(\xi)$ are parallel (disjoint, respectively).

So we have that the conditions (9) and (10) are equivalent to Item (v) of Theorem A.

Assume that k is fractional. Then there are additional reflections in Γ^* in planes passing through the edge $\xi \cap e_1(\xi)$ which decompose $\mathcal{P}_1$. Consider the link of the vertex $V_1 = \alpha \cap \xi \cap e_1(\xi)$ (see Figure 4). It is either a spherical, Euclidean, or hyperbolic triangle $(2, \widetilde{m}, k)$, $\widetilde{m}$ is odd, depending on whether V_1 is a proper, ideal, or imaginary vertex, respectively. Since the corresponding link surface S_1 is invariant under $\langle R_\alpha, R_\xi, R'_\xi \rangle$, where $R'_\xi = e_1 R_\xi e_1$, the group naturally acts on S_1 as a group of reflections in the sides of the triangle $(2, \widetilde{m}, k)$. There are no discrete groups with $\widetilde{m}$ odd and k fractional if $(2, \widetilde{m}, k)$ is Euclidean. As for spherical and hyperbolic triangles, there are the following types of the link of V_1:

- spherical triangles (s3), (s7), (s9), (s11), (s12), (s14), (s15);
- hyperbolic triangle (h2) in Figure 2.

Lemma 3. *If $\widetilde{m}$ is odd and the bisector β of the dihedral angle of $\mathcal{P}_1$ as above formed by ξ and $e_1(\xi)$ is the plane of a reflection belonging to the discrete group Γ^*, then either $n = 3$ and $\widetilde{m} \geq 7$, or $n = 5$ and $\widetilde{m} \geq 5$.*

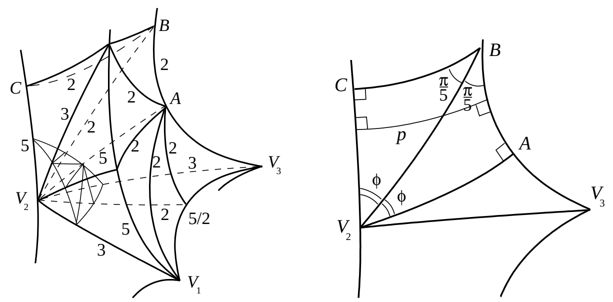
Proof. Let β intersect α at an angle of π/p, where $2 < p < \widetilde{m}$ (β lies between ξ and $e_1(\xi)$, p is not necessary an integer). Then $R_\beta R_\alpha$ is an elliptic element of order greater than 2. Moreover, the axis of this element meets f at $V_2 = f \cap e_1$. We have two elliptic elements f and $R_\beta R_\alpha$ with different intersecting axes whose orders are greater than 2. Since these elements belong to a discrete group, the orders are at most 5. So, for odd n we have $n = 3$ or $n = 5$. It follows from the upper face of $\mathcal{P}_1$ that $1/n + 1/\widetilde{m} < 1/2$. Thus, $n = 3$ implies $\widetilde{m} \geq 7$ and $n = 5$ implies $\widetilde{m} \geq 5$. □

In **Cases (s9)** and **(s15)** $\widetilde{m}$ should be 3, but this contradicts Lemma 3 and these cases disappear (the group is not discrete). We will use Lemma 3 also to specify n and $\widetilde{m}$ in Cases (s3) and (s14).

Let us describe the construction for **Case (s3)** in detail. All the other decompositions are made in a similar manner and we will give only a short description for each of them.

By Lemma 3, we have $n = 5$ and $\widetilde{m} = 5$. Let us first consider a non-compact polyhedron $\widetilde{\mathcal{P}}_1$ bounded by ω, α, ξ and $e_1(\xi)$. The reflection planes δ and $e_1(\delta)$ will be added later.

Since the bisector β is a reflection plane, there exists a reflection plane η passing through e_1 orthogonally to β. From the link of V_1, there exists a reflection plane η_1 through $\alpha \cap \beta$ that makes an angle of $\pi/3$ with α (see Figure 5a). The link of the vertex V_2 formed by α, ω and β is a spherical triangle $(5, 3, 3/2)$, which can be decomposed into smaller triangles only as (s8). Then $\widetilde{\mathcal{P}}_1$, and therefore $\mathbb{H}^3$, is decomposed into tetrahedra $T = [2, 2, 5; 2, 3, 5]$ (see the last paragraph in Section 2, which explains this notation). Each of T is a fundamental polyhedron for the group $\langle R_\omega, R_\alpha, R_\xi, e_1 \rangle = G_T < \Gamma^*$.



(a) (b)

Figure 5:

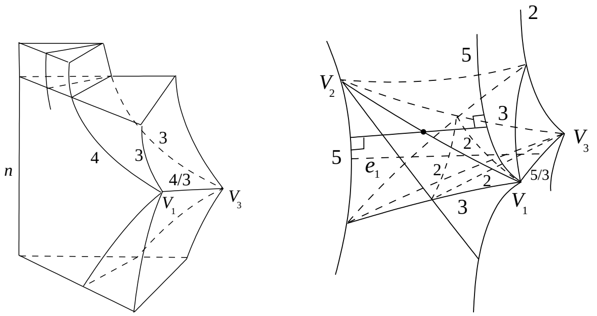
Now we determine the position of δ. Consider the face of $\widetilde{\mathcal{P}}_1$ lying in ω. In Figure 5b we have drawn traces of the reflection planes in ω. (Note that not all of these traces are reflection lines in ω.) The angle at B of the quadrilateral $ABCV_2$ is $2\pi/5$. From the link of V_2,

$$\cos\phi = \frac{\cos(\pi/3) + \cos(\pi/2)\cos(\pi/5)}{\sin(\pi/2)\sin(\pi/5)} = \frac{1}{2\sin(\pi/5)} > \frac{1}{2\sin(\pi/4)} = \frac{1}{\sqrt{2}}.$$

That is $2\phi < \pi/2$. Thus, the common perpendicular p to $CV_2 = f$ and $AB = \omega \cap \xi$ lies inside $ABCV_2$. Since δ passes through p, δ intersects the interior of one of the tetrahedra T. By [3], $\Gamma^* = \langle G_T, R_\delta \rangle$ is not discrete.

In **Case (s14)**, $n = 5$ and $\widetilde{m} = 5$, the space is decomposed into tetrahedra $T = [2, 2, 3; 2, 5, 3]$. By the same reason as in Case (s3), Γ^* is not discrete. So, it remains to consider only Cases (s7), (s11), (s12) without the central plane and the hyperbolic case (h2).

Cases (s7) and **(s11)**. Here $k = p/3$, $p = 4$ or 5, $\widetilde{m} = 3$, $n \geq 7$. There are two planes through $\xi \cap e_1(\xi)$. Consider one of them (denote it by η). From the links of V_1 and V_3 (see Figure 6a for Case (s7)) one can easily find the dihedral angles which η forms with α and ω. Since $n \geq 7$ and Γ^* is supposed to be discrete, the vertex formed by α, ω, and η is imaginary and its link is a hyperbolic triangle $\Delta = (n, 4, 3/2)$ for Case (s7) and $\Delta = (n, 3, 5/3)$ for (s11); $\langle R_\alpha, R_\eta, R_\omega \rangle$ acts as a group of reflections in the sides of Δ. But there are no such discrete groups. Thus, Γ^* is not discrete.



(a) (b)

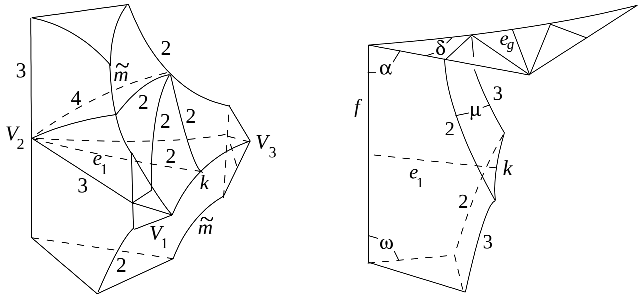
Figure 6:

Case (s12). Here $k = 5/3$, $\widetilde{m} = 5$, $n \geq 5$. Consider the polyhedron of infinite volume bounded by α, ω, ξ, and $e_1(\xi)$. The link of the vertex V_2 is a spherical, Euclidean, or hyperbolic triangle $(n, 5, 3/2)$. Since the group generated by the reflections in its sides is discrete, $n = 5$. It is easy to find the decomposition of the space into tetrahedra $T = [2, 3, 5; 2, 3, 2]$ (see Figure 6b). One can see that the common perpendicular p to the lines f and $\alpha \cap \xi$ intersects

V_1V_2 in the middle. Thus, the plane δ (which is orthogonal to α, ω, and ξ) passes through p and cuts T into two parts. But $\langle G_T, R_\delta \rangle < \Gamma^*$ is not discrete by [3].

Case (h2). Here $k = \widetilde{m}/2$, $\widetilde{m} \geq 7$, $(\widetilde{m}, 2) = 1$, $n = 3, 5$.

Assume first that $n = 3$. Consider the link of V_1. The bisector β of the dihedral angle formed by ξ and $e_1(\xi)$ passes through e_1. Since $R_\beta \in \Gamma^*$, there is a plane η passing through e_1 orthogonally to β and $R_\eta \in \Gamma^*$. Note that η is orthogonal to ξ. The link of the vertex V_2 formed by α, ω, and β is a spherical triangle $(3, 3, 3/2)$ and is divided uniquely into 4 triangles $(2, 3, 4)$ by reflection planes.



(a) (b)

Figure 7: Discrete groups for Items (*vi*) and (*vii*) of Theorem A.

Let $\widetilde{T}$ be the prism bounded by ω, β, η, ξ, and the link surface S of V_3 (see Figure 7a). Since $R_\delta \in \Gamma^*$, it is clear that the hyperbolic plane S becomes a reflection plane, $\Gamma^* = \langle R_\omega, R_\beta, R_\eta, R_\xi, R_S \rangle$, and $\widetilde{T}$ is a fundamental polyhedron for Γ^*. This is item (2)(*vi*) of the theorem.

Now assume that $n = 5$. In this case the polyhedron $\widetilde{\mathcal{P}}_1$ is decomposed into infinite volume tetrahedra $T = [2, 2, 5; 2, 3, \widetilde{m}]$. Similarly to Case (s3), the plane δ intersects the interior of T so that Γ^* is not discrete.

4.2. $m = n/4$, $n \geq 7$

Consider the polyhedron $\mathcal{P}_2$ bounded by α, ω, δ, $e_1(\delta)$, μ, and $e_1(\mu)$ (see Figure 7b for determining the plane μ which is orthogonal to δ; $\mathcal{P}_2$ is drawn as a compact). Denote the dihedral angle of $\mathcal{P}_2$ formed by μ and $e_1(\mu)$ by π/k. Note that $\mathcal{P}_2$ is congruent to the polyhedron $\mathcal{P}_1$ with $\widetilde{m} = 3$, see Subsection (a) above, but we see that there are no discrete groups in that case with fractional k. However, if

$$k > 1 \text{ is an integer}, \infty \text{ or } \overline{\infty}, \tag{11}$$

then Γ^* is discrete.

We show how to get Item (*vii*) of Theorem A. The procedure is analogous to the usual one. First of all, note that now the element h_1 defined in Section 3 is a rotation through $\pi/m = 4\pi/n$ and we take a primitive elliptic $\tilde{h}_1$ of order n so that $h_1 = \tilde{h}_1^2$. Further, define h_4 as follows:

$$\begin{aligned} h_4^2 &= (e_1 R_\mu e_1) R_\mu = (e_1 R_\mu e_1 R_\omega)(R_\omega R_\mu) \\ &= (e_1 f \tilde{h}_1^{-1} f e_1)(\tilde{h}_1 f^{-1}) = f^{(n-3)/2} g^{-1} \tilde{h}_1 g f^{-(n+1)/2} \tilde{h}_1 f^{-1}. \end{aligned}$$

Note that to obtain the last equality we have used the obvious fact that $e_g \tilde{h}_1^{-1} e_g = \tilde{h}_1$.

In order to define h_4 uniquely we need the last condition: $h_4 f \tilde{h}_1^{-1}$ must be an elliptic element whose axis intersects f. Now we see that the conditions "$m = n/4$, $n \geq 7$" and (11) are equivalent to Item (*vii*) of the theorem.

5 m is an integer $(2/n + 1/m < 1)$ and l is fractional $(l > 1)$

Let $\widetilde{\mathcal{P}}$ be a non-compact polyhedron bounded by α, ω, α', and $\alpha'' = e_1(\alpha')$. The intersection of α, α', and α'' forms a vertex V_1. Its link is either a spherical, Euclidean, or hyperbolic triangle (n, m, l). The subgroup $\langle R_\alpha, R'_\alpha, R'_\alpha \rangle$ of the discrete group Γ^* is also discrete and acts as the group generated by reflections in the sides of the triangle (n, m, l), where n is odd, m is an integer, and l is fractional.

There are no discrete groups if V_1 is ideal. As for proper or imaginary V_1, using Figure 2 and taking into account the fact that $2/n + 1/m < 1$, we have the following list to consider (we also indicate triangles from Figure 2 that correspond to V_1):

(1) $n \geq 5$, $m = n$, $l = k/2$, k is odd, $k \geq 3$ (h1, s2);

(2) $n \geq 5$, $m = 2$, $l = n/2$ (h2, s3);

(3) $n = 3$, $m \geq 4$, $(m, 3) = 1$, $l = m/3$ (h3, s4);

(4) $n \geq 5$, $(n, 3) = 1$, $m = 3$, $l = n/3$ (h3, s4);

(5) $n \geq 5$, $m = n$, $l = n/4$ (h4, s5);

(6) $n = 3$, $m = 5$, $l = 3/2$ (s8);

(7) $n = 5$, $m = 3$, $l = 3/2$ (s8);

(8) $n = 5$, $m = 2$, $l = 5/3$ (s12);

(9) $n = 3$, $m = 5$, $l = 5/4$ (s13);

(10) $n = 5$, $m = 3$, $l = 5/4$ (s13);

(11) $n = 5$, $m = 2$, $l = 3/2$ (s14);

(12) $n = 3$, $m = 7$, $l = 7/2$ (h5);

(13) $n = 7$, $m = 3$, $l = 7/2$ (h5).

For each case we proceed as follows. We start with reflections in α, ω, α', and α''. Using the link of V_1 we find additional reflection planes and dihedral angles that they make with α and α'. Then using new links we find new reflection planes and construct a decomposition of $\widetilde{\mathcal{P}}$ into subpolyhedra (usually tetrahedra) T with primitive angles, or show that the group is not discrete. If the bisector ζ of the dihedral angle of $\widetilde{\mathcal{P}}$ formed by α' and α'' is a reflection plane in Γ^*, then since it passes through e_1, there is another reflection plane η through e_1 that is orthogonal to ζ. We include η in the decomposition. If the decomposition gives a discrete group G_T generated by reflections in the faces of T, we find the positions of e_1 and e_g and determine whether the group $\langle G_T, e_1, e_g \rangle = \Gamma^*$ is discrete. When construction of such a decomposition is not difficult we give only a description of T.

Remark 1. If two axes of elliptic elements intersect non-orthogonally in a discrete group, then the orders of both elements are at most 5. In particular, if Γ^* is discrete and the bisector ζ is a reflection plane in Γ^* so that ζ meets α non-orthogonally, then $n = 3$ or 5 and the order of $R_\alpha R_\zeta$ is $3, 4$, or 5.

In **Case (1)**, $n \geq 5$, $m = n$, $l = k/2$, k is odd, $k \geq 3$. The bisector ζ is the only reflection plane passing through $\alpha' \cap \alpha''$. It is orthogonal to both α and ω and so to $f = \alpha \cap \omega$. Since ζ passes through e_1, there is a reflection plane η passing through e_1 orthogonally to ζ. Since ζ is orthogonal to f, the plane η contains f and is the bisector of the dihedral angle formed by α and ω. Since also $m = n$, the polyhedron is symmetric with respect to η. Besides R_η, Γ^* contains reflections $e_g R_\eta e_g$ and $R_\eta(e_g R_\eta e_g) R_\eta$ (see Figure 8 for proper V_1). Note that the plane η' of the last reflection passes through $\alpha \cap \alpha'$ and makes an angle of $\pi/(2n)$ with α.

Now we show that V_1 cannot be proper, that is, $n \neq 5$. Indeed, if $n = 5$ then there are two rotations $R_\alpha R'_\eta$ of order $2n = 10$ and $R'_\alpha R_\zeta$ of order k (k is odd) with axes intersecting at V_1, which is impossible in discrete groups.

V_1 cannot be imaginary as well. Indeed, from [20] it follows that the hyperbolic link of V_1 is a triangle (n, n, n), but this means that $k = 2n$ is even.

So, Case (1) does not give discrete groups.

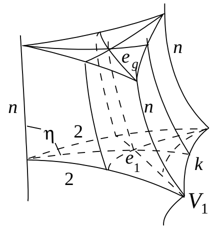


Figure 8:

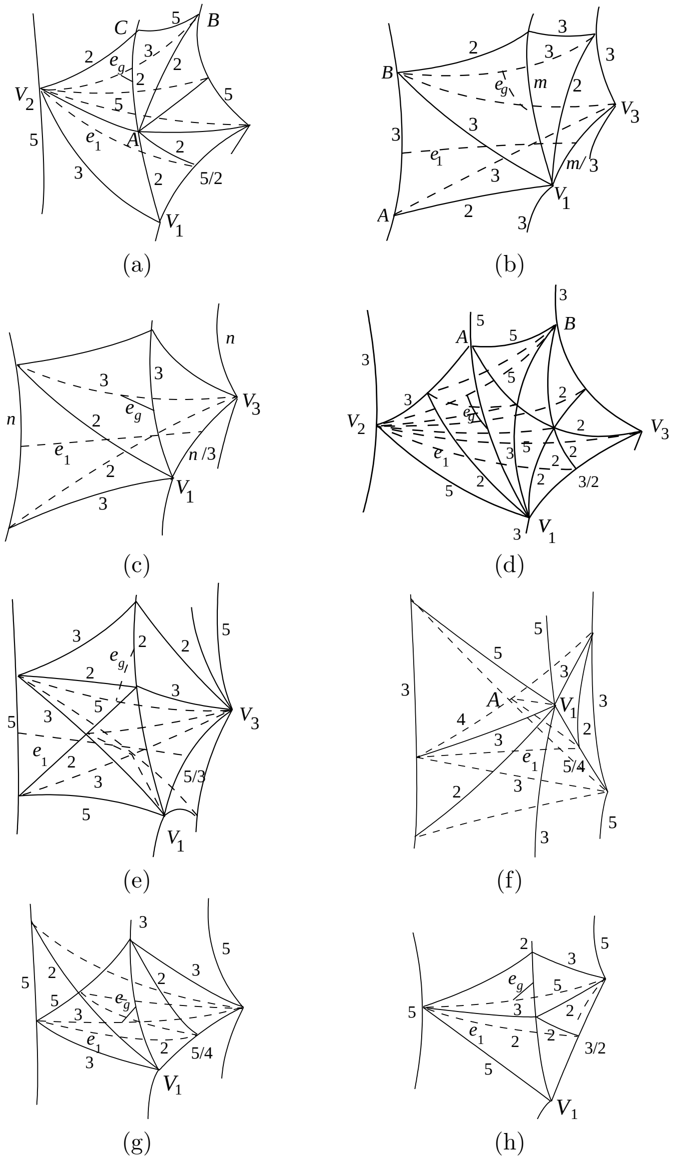


Figure 9: Discrete groups for Item (*ii*) of Theorem A.

In **Case (2)**, $n \geq 5$, $m = 2$, $l = n/2$. The link of the vertex V_2 formed by α, ω, and ζ is a spherical triangle $(n, 3, 3/2)$. By Remark 1, $n = 5$. In this case V_1 is proper and $\widetilde{\mathcal{P}}$ is decomposed into tetrahedra $T = [2, 3, 5; 2, 2, 5]$ so that e_1 is one of its edges (see Figure 9a).

One can see that the line passing through the midpoints of the edges AC and BV_2 in the tetrahedron $ABCV_2$ is orthogonal to ω and to V_1C and thus coincides with e_g. By [3], the group $\langle G_T, e_g\rangle = \Gamma^*$ is discrete and a half of T is its fundamental polyhedron. This is a group from the list in item (2)(*ii*) of the theorem.

In **Case (3)**, $n = 3$, $m \geq 4$, $(m, 3) = 1$, $l = m/3$, $\widetilde{\mathcal{P}}$ is decomposed into tetrahedra (possibly of infinite volume) $T = [2, 3, m; 2, 3, 3]$ (see Figure 9b for $m = 4, 5$). By construction, e_1 is orthogonal to f and to V_1V_3 and passes through the midpoint of AB. It is clear that $\langle G_T, e_1\rangle$ is discrete. Note that e_g is conjugate to e_1 and $\langle G_T, e_1\rangle = \Gamma^*$. This is one of the groups from item (2)(*ii*) of the theorem.

In **Case (4)**, $n \geq 5$, $(n, 3) = 1$, $m = 3$, $l = n/3$, $\widetilde{\mathcal{P}}$ is decomposed into tetrahedra (possibly of infinite volume) $T = [2, 3, n; 2, 3, n]$ (see Figure 9c for $n = 5$). One can see that e_1 is orthogonal to the opposite edges of T with the dihedral angles of π/n, and e_g is orthogonal to the opposite edges with dihedral angles of $\pi/3$. Therefore, $\langle G_T, e_1, e_g\rangle = \Gamma^*$ is discrete [3] and a quarter of T is a fundamental polyhedron for Γ^*. This is one of the groups from item (2)(*ii*) of the theorem.

In **Case (5)**, $n \geq 5$, $m = n$, $l = n/4$. Analogously to Case (1), there exists a reflection plane η passing through e_1 and f. Conjugating η by $R_\eta e_g$, we have a new reflection plane through V_1. However, the link of V_1 cannot be divided by additional reflection planes so that the corresponding reflection group to be discrete (see [20]). Thus, Γ^* is not discrete.

In **Case (6)**, $n = 3$, $m = 5$, $l = 3/2$. Construct the decomposition of $\widetilde{\mathcal{P}}$ into tetrahedra (see Figure 9d). Note that e_1 coincides with one of the edges of a tetrahedron in the decomposition. Consider the tetrahedron $ABV_1V_2 = [5, 3/2, 5; 5, 3/2, 5]$. It is easy to see that the line passing through the midpoints of AV_1 and BV_2 is orthogonal to ω and to AV_1. Thus, it coincides with e_g. Moreover, ABV_1V_2 is divided by reflection planes into four tetrahedra $T = [2, 3, 5; 2, 3, 2]$ so that e_g passes through the midpoints of the opposite edges of T with dihedral angles of $\pi/5$ and $\pi/2$. By [3], $\langle G_T, e_g\rangle = \Gamma^*$ is discrete and a half of T is its fundamental polyhedron. This is one of the groups from item (2)(*ii*) of the theorem.

In **Case (7)**, $n = 5$, $m = 3$, $l = 3/2$. The bisector ζ is a reflection plane and makes dihedral angles of $2\pi/5$ and $3\pi/5$ with α, and ω, respectively. Hence, the link of the vertex V_2 formed by α, ω and ζ is a spherical triangle $(5, 5/2, 5/3)$. But the reflections in its sides do not generate a discrete group [5]. Thus, Γ^* is not discrete.

In **Case (8)**, $n = 5$, $m = 2$, $l = 5/3$. In this case $\widetilde{\mathcal{P}}$ is decomposed into tetrahedra $T = [2, 3, 5; 2, 3, 2]$ and e_1 passes through the midpoints of the opposite edges of T with dihedral angles of $\pi/5$ and $\pi/2$. One can see that e_g is conjugate to e_1 (see Figure 9e, where we give the most important elements of the decomposition). From [3], $\langle G_T, e_g\rangle = \langle G_T, e_1\rangle = \Gamma^*$ is discrete and a half of T is its fundamental polyhedron. This is one of the groups from item (2)(*ii*) of the theorem.

In **Case (9)**, $n = 3$, $m = 5$, $l = 5/4$, $\widetilde{\mathcal{P}}$ is decomposed into tetrahedra $T = [2, 3, 5; 2, 2, 4]$ (see Figure 9f). Note that e_1 coincides with an edge of T and therefore, $e_1 \in G_T$. Moreover, the edge AV_1 is orthogonal to the edge $\alpha\cap\alpha'$ and lies in a plane orthogonal to ω. Hence, e_g passes through AV_1 and therefore, $e_g \in G_T$. Thus, $\Gamma^* = G_T$ is discrete; this is one of the groups from item (2)(*ii*) of the theorem.

In **Case (10)**, $n = 5$, $m = 3$, $l = 5/4$. In this case $\widetilde{\mathcal{P}}$ is decomposed into tetrahedra $T = [2, 3, 5; 2, 2, 5]$ (see Figure 9g). Since the bisector ζ is a reflection plane, e_1 coincides with one of the edges of T and therefore, $e_1 \in G_T$. Furthermore, e_g passes through the midpoints of the opposite edges of T with dihedral angles of $\pi/3$ and $\pi/2$. From [3], $\Gamma^* = \langle G_T, e_g\rangle$ is discrete and a half of T is a fundamental polyhedron for Γ^*. This is one of the groups from item (2)(*ii*) of the theorem.

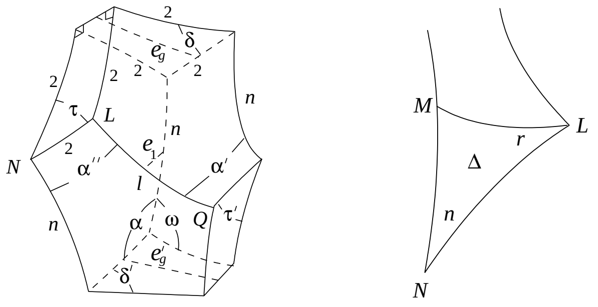
In **Case (11)**, $n = 5$, $m = 2$, $l = 3/2$. In this case $\widetilde{\mathcal{P}}$ is decomposed into tetrahedra $T = [2, 3, 5; 2, 3, 2]$ (see Figure 9h). Again $e_1 \in G_T$. It is not difficult to see that e_g passes through the midpoints of the edges of T with dihedral angles of $\pi/5$ and $\pi/2$. $\Gamma^* = \langle G_T, e_g\rangle$ is discrete (see [3]) and a half of T is a fundamental polyhedron for Γ^*. This is one of the groups from item (2)(*ii*) of the theorem.

In **Cases (12)** and **(13)**, the link of a vertex formed by α, ω, and ζ is a spherical triangle $(n, 7/3, 7/4)$. Obviously, the group generated by reflections in sides of such a triangle is not discrete. Thus, Γ^* is not discrete.

6 $m \in \{\infty, \overline{\infty}\}$ and l is fractional $(l > 1)$

Suppose $m = \infty$. Then the link of a vertex formed by α, α', and α'' is a hyperbolic triangle (n, l, ∞), where l is fractional. Clearly, there are no such discrete groups (cf. [23]).

From here on, $m = \overline{\infty}$, l is fractional, and Γ^* is discrete. Start with $\widetilde{\mathcal{P}}$ (see Section 5). The planes α, ω, $\alpha' = e_g(\alpha)$, $\alpha'' = e_1(\alpha')$ are reflection planes and make the following angles: α and ω, α and α'', α' and ω intersect at π/n, α' and α'' at π/l; but now the planes α and α' as well as α'' and ω are disjoint. To draw this non-compact polyhedron we use four additional planes to compactify it: δ (which is orthogonal to α, ω, and α'), τ (orthogonal to α, α', α''), $\delta' = e_1(\delta)$, and $\tau' = e_1(\tau)$ (see Figure 10a).



(a) (b)

Figure 10:

The compact polyhedron is also symmetric with respect to e_1. Note that τ intersects δ and $p = \tau\cap\delta$ is the common perpendicular for α and α'. Remember that a priori τ and δ are not reflection planes, so they can make any angle, which does not contradict discreteness; moreover, p can intersect NL (as in Case (16) below).

Consider τ and $\langle R_\alpha, R'_\alpha, R''_\alpha\rangle$. The group acts in τ as a group generated by reflections in the sides of a polygon D of infinite area with three sides and two angles of π/n and π/l at vertices N and L, respectively. Since l is fractional, there are additional reflection lines in τ that correspond to reflection planes through $LQ = \alpha'\cap\alpha''$ in $\widetilde{\mathcal{P}}$.

Remark 2. Note that if a plane through LQ intersects α, it also intersects $\alpha\cap\tau$. Hence, if ζ is the bisector of the dihedral angle of $\widetilde{\mathcal{P}}$ formed by α' and α'', it passes through e_1 and therefore, intersects f, α, and $\alpha\cap\tau$.

Remark 3. The bisector ζ from Remark 2 cannot be orthogonal to α (otherwise α and α' meet at an angle of π/n, but they must be disjoint). If ζ is a reflection plane in Γ^*, then $n = 3$ or 5 and the order of $h = R_\alpha R_\zeta$ is 3, 4 or 5 by Remark 1 (see Section 5).

Now we will produce a complete list of possible decompositions of τ by reflection planes. Suppose $\pi/l = s\pi/r$, where s and r are integers, $(s, r) = 1$, $r > s \geq 2$. Then the line through L cutting D and making an angle of π/r with LN intersects the opposite side $\alpha\cap\tau$ of D at some point M (because ζ also intersects $\alpha\cap\tau$ by Remark 2 and $s \geq 2$). Denote by Δ the triangle with vertices L, M, N. It has two primitive angles π/n and π/r at N and L, respectively, so either Δ is one of the hyperbolic triangles in Figure 2 or all its angles are primitive. It is clear that the decomposition of τ is specified by n, l, and Δ.

Now suppose that Δ is chosen and placed into τ so that it has primitive angles $\phi = \pi/n$ (n is odd) at N and $\psi = \pi/r$ at L (see Figure 10b). We find the smallest s such that α' and α are disjoint and $(s, r) = 1$. Then we check if α and ζ intersect and if Remark 3 holds. If both conditions hold, we include such a decomposition in our list and replace s with $s+1$. If just $\alpha\cap\zeta \neq \emptyset$, we do not include the decomposition of τ in the list but still replace s with $s+1$. Then check again if $(s,r)=1$, $\alpha\cap\zeta\neq\emptyset$, and Remark 3 holds. Using Remark 2, we stop our process for the fixed position of Δ as soon as the planes α and ζ are parallel or disjoint for some $s=\widetilde{s}$. We do not include decompositions of τ with such a triangle for all $s\geq\widetilde{s}$. Note that we can detect the behavior of α and ζ (as well as α' and α) by considering their projections onto τ, since α, α', and ζ are orthogonal to τ.

Note that one picture in Figure 2 can give us two different sequences of triangles, say for (h3), we can take first $n=3$ and then $n=p\geq 7$. If the same triangle can be placed in τ in a different way, we proceed with a new position (in fact, we consider this new position as a new triangle Δ in our algorithm).

It remains to explain why not all triangles with two primitive angles give at least one decomposition of τ and appear as Δ in our list.

We start with triangles (h2)–(h5) in Figure 2; triangles of type (h1) will be considered later. Obviously, triangles of type (h2) do not give any suitable decomposition. If a picture, say (h3), gives us a sequence of triangles that depends on a parameter p, then we start with the smallest $p=p_0$ (in fact, $p_0=7$) and consider this triangle as $\Delta=\Delta(p_0)$. If for the chosen triangle the process of producing the list of decompositions stops with some $s=\widetilde{s}$, there is no need to consider all other triangles $\Delta(p)$ (of the same type and placed in a similar way into τ) for all $p\geq p_0$ and $s\geq\widetilde{s}$ by Lemma 2.

Using the above algorithm, one can see that if Δ is chosen from (h2)–(h5) in Figure 2, then the only possible decompositions that satisfy both Remark 2 and the condition $\alpha\cap\zeta\neq\emptyset$ are those listed as (1)–(4) in Proposition 1 below. To produce the remaining cases in the list we need also the following

Lemma 4. *If Δ has angles $\pi/5$ at N, $\pi/4$ at L, and $\pi/3$ at M, then $\alpha\cap\zeta=\emptyset$ for $s=3$.*

Proof. The proof is a straightforward calculation. □

Now consider the two-parameter family of triangles (h1) in Figure 2. Here q and $p=n$ are odd $(1/q+1/p<1/2)$. Fix $q=3$ first. Then we have a one-parameter family and the process above gives decompositions (5) and (6) in Proposition 1. For $q=5$, we obtain (7) and (8). For $q\geq 7$, using Remark 3 (when $s=2$) and Lemmas 2 and 4 (when $s=3$), one can show that the triangles (h1) do not give suitable decompositions.

It remains only to consider triangles with all primitive angles. We start with a right-angled triangle Δ, then $\angle M=\pi/2$. If $n=3$ we use the above algorithm to get (9) and (10) in the proposition below. Analogously, using Remark 3 and Lemma 2 for $n\geq 5$, we get (11) and (12) and show that there are no other decompositions with right-angled triangles.

Suppose that the greatest angle of Δ is $\pi/3$. It is easier to start now with $s=3$ to see what happens. Since the angle at L should be less than π, i.e., $s\pi/r<\pi$, we have $r\geq 4$. Then Lemma 4 combined with Lemma 2 gives that we do not have other decompositions except (13) for $s=3$, and therefore, for all $s\geq 3$. The last case to consider is $s=2$, that is the case where the bisector

ζ is the only reflection plane through LQ. Using Remark 3 (which is now a criterion to stop), we get the rest of the list. We have proved the following

Proposition 1. *Let f and g be as in Theorem A. If $\Gamma=\langle f,g\rangle$ is discrete and $gfg^{-1}f$ is hyperbolic, then either l is an integer, or n, l, and Δ as above belong to the following list:*

(1) $n=3$, $l=8/3$, $\Delta=(3,8,8/3)$;

(2) $n=3$, $l=7/3$, $\Delta=(3,7,7/3)$;

(3) $n=7$, $l=7/3$, $\Delta=(7,7,7/4)$;

(4) $n=7$, $l=7/5$, $\Delta=(7,7,7/4)$;

(5) $n\geq 7$, $l=n/5$, $(n,5)=1$, $\Delta=(n,n,3/2)$;

(6) $n\geq 9$, $l=n/7$, $(n,7)=1$, $\Delta=(n,n,3/2)$;

(7) $n=5$, $l=5/2$, $\Delta=(5,5,5/2)$;

(8) $n\geq 5$, $l=n/3$, $(n,3)=1$, $\Delta=(n,n,5/2)$;

(9) $n=3$, $l=r/4$, $r\geq 7$, $(r,2)=1$, $\Delta=(3,r,2)$;

(10) $n=3$, $l=r/5$, $r\geq 7$, $(r,5)=1$, $\Delta=(3,r,2)$;

(11) $n\geq 5$, $l=r/3$, $r\geq 4$, $(r,3)=1$, $\Delta=(n,r,2)$;

(12) $n=5$, $l=r/4$, $r\geq 5$, $(r,2)=1$, $\Delta=(5,r,2)$;

(13) $n=3$, $l=r/3$, $r\geq 4$, $(r,3)=1$, $\Delta=(3,r,3)$;

(14) $n=3$, $l=r/2$, $r\geq 5$, $(r,2)=1$, $\Delta=(3,r,3)$;

(15) $n=3$, $l=r/2$, $r\geq 3$, $(r,2)=1$, $\Delta=(3,r,4)$;

(16) $n=3$, $l=r/2$, $r\geq 3$, $(r,2)=1$, $\Delta=(3,r,5)$;

(17) $n=5$, $l=r/2$, $r\geq 3$, $(r,2)=1$, $\Delta=(5,r,3)$;

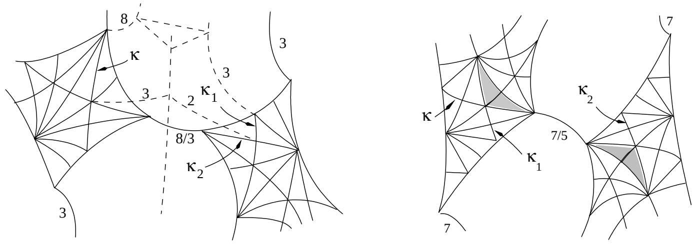
(18) $n=5$, $l=r/2$, $r\geq 3$, $(r,2)=1$, $\Delta=(5,r,4)$;

(19) $n=5$, $l=r/2$, $r\geq 3$, $(r,2)=1$, $\Delta=(5,r,5)$.

Later we will see that Cases (16), (9), and (14) (maybe with some further restrictions) lead to Item (*iii*) and Cases (8), (11) lead to Item (*iv*) of the theorem and that there are no other discrete groups among (1)–(19).

In **Case (1)**, $n=3$, $l=8/3$, $\Delta=(3,8,8/3)$. Let κ, κ_1, and κ_2 be three planes in the decomposition of $\widetilde{\mathcal{P}}$ shown in Figure 11a (if a plane, say κ, is orthogonal to τ or $\tau'=e_1(\tau)$, we show often just $\kappa\cap\tau$ and $\kappa\cap\tau'$ in figures).

Suppose κ and κ_1 are parallel or disjoint. Then κ_1 is disjoint from α (because κ and α are also disjoint and α and κ_1 lie in different half-spaces bounded by κ).



(a) (b)

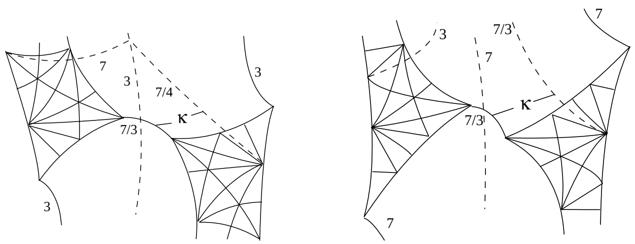
Figure 11:

The same argument shows that α and ω are disjoint (both α and ω are disjoint from κ_1 and lie in different half-spaces with respect to κ_1). This contradicts the fact that $f = R_\omega R_\alpha$ is elliptic.

Suppose κ and κ_1 intersect and form a dihedral angle of π/k. We see that the link formed by α', κ, and κ_1 is $(8, 3, k)$. So, $k = 8/3$ (Figure 2(h3)) or k is an integer, $k \geq 2$. The case $k = 8/3$ is impossible, because the link formed by κ, κ_1, and κ_2 is a spherical triangle $(3, 2, 8/3)$ that cannot appear in a discrete group.

The case $k \geq 3$ is also impossible, because two planes α and ω are disjoint from κ and κ_1, respectively, so even if α and ω intersect, the angle between them is smaller than between κ and κ_1. If $k = 2$ (κ is orthogonal to κ_1), then it is easy to see that $R_\kappa(\alpha)$ is disjoint from κ_1, so α itself is also disjoint from κ_1. We see again that α and ω are disjoint, because they lie in different half-spaces with respect to κ_1 and κ_1 and ω are disjoint.

In **Case (4)**, $n = 7$, $l = 7/5$, $\Delta = (7, 7, 7/4)$. Consider κ_1 and $\kappa_2 = e_1(\kappa_1)$ (see Figure 11b). These two planes are either disjoint or parallel (and then α and ω are disjoint), or intersect at an angle of $\pi/7$ to make the only possible compact link $(7, 7, 3/2)$ together with κ. But in the latter case α and ω cannot make the same angle $\pi/7$ contradicting the fact that $n = 7$. Therefore, Γ^* is not discrete.



(a) (b)

Figure 12:

In **Cases (2)** and **(3)**, $n = 3$, $\Delta = (3, 7, 7/3)$ or $n = 7$, $\Delta = (7, 7, 7/4)$ and $l = 7/3$. Let κ be a plane in the decomposition of $\widetilde{\mathcal{P}}$ making a dihedral angle of $\pi/7$ with α' (see Figure 12). The link formed by κ, α, and ω is a spherical triangle $(3, 7, 7/4)$ for Case (2) or a hyperbolic triangle $(3, 7, 7/3)$ for Case (3). However, the group generated by reflections in its sides is not discrete (there are no such triangles in Figure 2). Thus, in both cases Γ^* is not discrete.

In **Case (5)**, $n \geq 7$, $l = n/5$, $(n, 5) = 1$, $\Delta = (n, n, 3/2)$; $\widetilde{\mathcal{P}}$ is decomposed into infinite volume tetrahedra $T = [2, 3, n; 2, 3, n]$ (see Figure 13).

Now we determine the position of the axis e_g. It passes through A and is orthogonal to ω. Since e_g is not orthogonal to τ (the perpendicular to τ through A is the line $\kappa_1 \cap \kappa_3$), the distance between e_g and the axis $LQ = \alpha' \cap \alpha''$ of an elliptic element of order n is less than AL, where

$$\cosh AL = 1/(2\sin(\pi/n)) = \cosh \rho_{min}(2, n), \quad n \geq 7.$$

We arrive at a contradiction; Γ^* is not discrete.

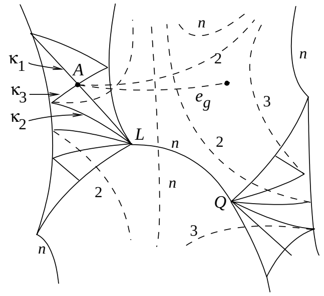


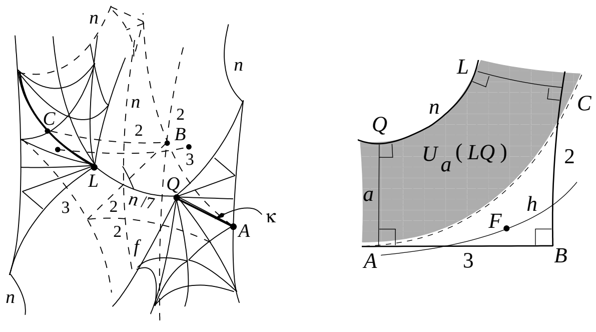
Figure 13:

In **Case (6)**, $n \geq 9$, $l = n/7$, $(n, 7) = 1$, $\Delta = (n, n, 3/2)$. Construct a decomposition of $\widetilde{\mathcal{P}}$ into infinite volume tetrahedra $T = [2, 2, 3; 2, n, 3]$ (see Figure 14a).

The points of intersection of e_g with τ and ω lie in different half-spaces bounded by the plane κ. So, e_g intersects κ. Moreover, the point $F = e_g \cap \kappa$ lies within the pentagon $ABCLQ$ in Figure 14b. Since e_g makes an angle of less than $\pi/6$ with κ, there exists an elliptic element $h \in \langle G_T, e_g \rangle$ of order $q > 3$ with the axis lying in κ.

On one hand, h cannot intersect the a-neighborhood $U_a(LQ)$ of the axis of order n passing through LQ, where $a = AQ = \rho_{min}(3, n) < \rho_{min}(q, n)$, for all $q > 3$ (see Table 1 and the formulas (1) or [12]). On the other hand, h cannot meet AB since the projection of e_g on κ is orthogonal to h and AB. Contradiction. Γ^* is not discrete.

In **Case (7)**, $n = 5$, $l = 5/2$, $\Delta = (5, 5, 5/2)$. The link of the vertex formed by α, ω, and ζ, where ζ is the bisector of the dihedral angle formed by α' with

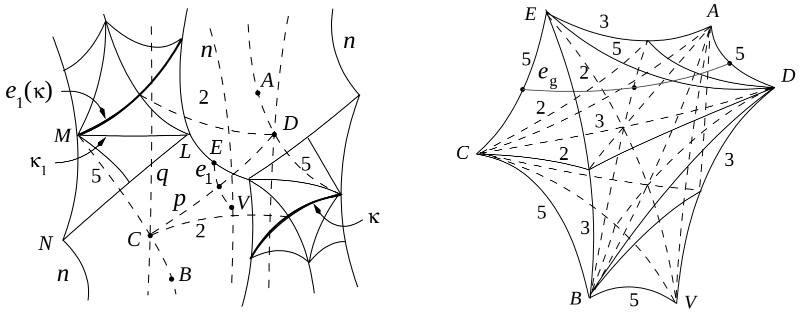


(a) (b)

Figure 14:

α'', is a spherical triangle $(5, 5/2, 5/3)$. But the group generated by reflections in its sides is not discrete [5]. Thus, Γ^* is not discrete.

In **Case (8)**, $n \geq 5$, $l = n/3$, $(n, 3) = 1$, $\Delta = (n, n, 5/2)$. Consider two reflection planes κ and $e_1(\kappa)$ from the decomposition (see Figure 15a). Denote by π/p and π/q the angles that κ makes with $e_1(\kappa)$ and α, respectively. It is not difficult to show that if π/p is less than or equal to $\pi/2$ (or the planes are parallel or disjoint) then α and ω are disjoint.



(a) (b)

Figure 15:

So we can assume that $\pi/p > \pi/2$. The link of the vertex formed by α, ω, and κ is a triangle $\Delta = (5/2, n, q)$. Suppose Δ is spherical, then immediately $n = 5$ and from Figure 2, $q = 2$ or $q = 3/2$. Therefore, from the link of C, either $p = 2$ for $q = 2$ or $p = 5/2$ for $q = 3/2$. In both cases $\pi/p \leq \pi/2$ and thus, α and ω are disjoint, which is impossible. Suppose now that Δ is hyperbolic. By [20, Lemma 2.1], $q = n \geq 5$ and the link of C made by α, κ, and κ_1 is a spherical triangle $(2, 5/3, q/(q-1))$. Then $q = n = 5$ and therefore, $p = 3/2$

(see [5]).

Consider $n = 5$, $p = 3/2$. Then $\widetilde{\mathcal{P}}$ can be decomposed into compact tetrahedra $T = [2, 2, 3; 2, 5, 3]$ (see Figure 15b). Here, e_1 coincides with an edge of T (actually, with EV). Determine the position of e_g. For $n = 5$, e_g intersects LM in the midpoint. Further, CE makes equal alternate interior angles with $\kappa_1 \cap \alpha$ and $\kappa_1 \cap \alpha'$ and hence, LM bisects CE. Therefore, e_g also bisects CE. The tetrahedron $ACDE = [5, 5, 3/2; 5, 5, 3/2]$ consists of four copies of T. Consider the line l passing through the middles of CE and AD in $ACDE$. Then l intersects an edge with the dihedral angle $\pi/2$ that is opposite to both CE and AD. Thus, l is orthogonal to CE and AD. Moreover, l bisects the dihedral angle formed by $\kappa = ACD$ and $e_1(\kappa_1) = AED$ and therefore, l is orthogonal to ω. So, e_g coincides with l. By [3], $\Gamma^* = \langle G_T, e_g \rangle$ is discrete. This is one of the groups from item (2)(*iv*) of the theorem.

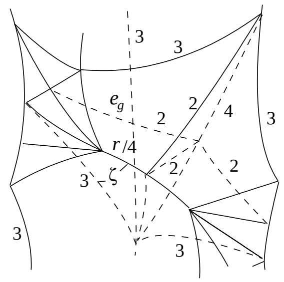


Figure 16:

In **Case (9)**, $n = 3$, $l = r/4$, $r \geq 7$, $(r, 2) = 1$, $\Delta = (3, r, 2)$. Taking into account the fact that the bisector ζ is a reflection plane, it is not difficult to construct the decomposition of $\widetilde{\mathcal{P}}$ into tetrahedra $T = [2, 3, r; 2, 2, 4]$ with one imaginary vertex. By construction, e_1 coincides with an edge of T with the dihedral angle of $\pi/2$. Moreover, one can see that e_g coincides with another edge of T with the dihedral angle of $\pi/2$ (see Figure 16). Hence, $\Gamma^* = G_T$ is discrete and T is its fundamental polyhedron. It is one of the groups from item (2)(*iii*) of the theorem.

In **Case (10)**, $n = 3$, $l = r/5$, $r \geq 7$, $(r, 5) = 1$, $\Delta = (3, r, 2)$. There is a plane κ from the decomposition of $\widetilde{\mathcal{P}}$ which passes through $\alpha' \cap \alpha''$ and makes an angle of $2\pi/r$ with α'. The link of the vertex formed by α, ω, and κ is a spherical triangle $(3, r, 3/2)$, $r \geq 7$. But the group generated by reflections in the sides of this triangle is not discrete. Thus, Γ^* is not discrete.

In **Case (11)**, $n \geq 5$, $l = r/3$, $r \geq 4$, $(r, 3) = 1$, $\Delta = (n, r, 2)$. One can construct the decomposition of $\widetilde{\mathcal{P}}$ into tetrahedra $T = [2, n, n, ; 2, n, r]$ of infinite volume (see Figure 17). Let κ_1 and κ_2 be reflection planes that pass through $\alpha' \cap \alpha''$ and make angles of π/r and $2\pi/r$ with α', respectively. Then T is bounded by α, κ_1, κ_2, and ω.

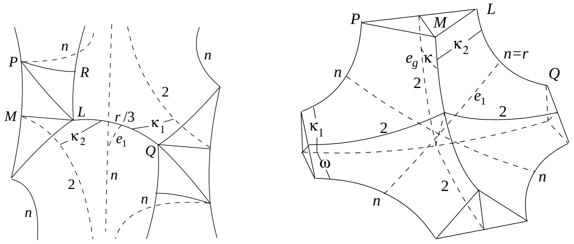


Figure 17:

(1) Assume $n = r \geq 5$. Then e_g lies in κ_1. Moreover, e_g maps the face of T lying in κ_1 onto itself. There are two more reflection planes that meet the interior of T. One of them, denote it by κ, passes through e_g and is orthogonal to κ_1 (κ passes also through $\alpha \cap \kappa_2$), its image with respect to e_1 is the other reflection plane that passes through $\kappa_1 \cap \omega$. Therefore, T is decomposed into four tetrahedra $T_1 = [2, 2, 4; 2, n, 4]$ (see Figure 17b). Obviously, $\Gamma^* = \langle G_{T_1}, e_1 \rangle$ is discrete and a half of T_1 is a fundamental polyhedron for Γ^*. This is item (2)(iv) of Theorem A.

(2) Assume $n > r \geq 4$. In this case e_g does not lie in a reflection plane. Then e_g maps κ_2 to a plane κ_3; moreover, κ_3 makes an angle of π/n with ω. Let $h = e_g R_\omega R_{\kappa_2} e_g$ and let $e'_g = R_{\kappa_1} e_g R_{\kappa_1}$. Let $A = e_g \cap \omega$ and $A' = e'_g \cap \omega$. The distance from A to h (denote it by t) is less than the distance from A' to the axis of $h' = e'_g R_\omega R_{\kappa_2} e'_g$ (denote it by x, see Figure 18).

If h and h' intersect, then for $n \geq 7$, Γ^* is not discrete. Suppose $n = 5$ and h and h' intersect at an angle ϕ.

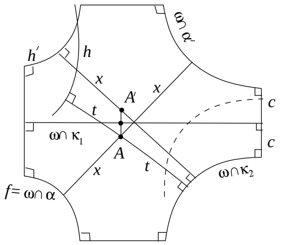


Figure 18:

Elementary calculations show that for $n = 5$, $r = 4$, we have $1 > \cos\phi > \cos(\pi/6)$, that is Γ^* is not discrete.

So we may assume that $n > 5$ and h and h' are disjoint. Let $h_1 = e'_g h e'_g$. The elliptic element h_1 has order $n > 5$. The axis h_1 then cannot intersect (or be parallel to) the axes of order n through $\omega \cap \kappa_2$ and $\omega \cap \alpha'$ (h_1 is drawn in

Figure 18 by the dashed line). Thus,

$$\rho(h_1, \omega \cap \kappa_2) + \rho(h_1, \omega \cap \alpha') < 2c,$$

where c can be defined from the link of the imaginary vertex made by κ_1, κ_2, and ω. However,

$$c = \operatorname{arccosh}\left(\frac{\cos(\pi/r)}{\sin(\pi/n)}\right) < \operatorname{arccosh}\left(\frac{\cos(2\pi/n)}{2\sin^2(\pi/n)}\right) = \rho_{min}(n, n)$$

for all $n > r \geq 8$. We arrive at a contradiction, Γ^* is not discrete.

(c) If $r > n \geq 5$ the arguments are analogous and the group is also not discrete.

In **Case (12)**, $n = 5$, $l = r/4$, $r \geq 5$, $(r, 2) = 1$, $\Delta = (5, r, 2)$. In this case $\widetilde{\mathcal{P}}$ is decomposed into tetrahedra $T = [2, 2, 3; 2, 5, r]$ of infinite volume (see Figure 19). Note that e_1 coincides with the edge DV of T.

Elementary calculations show that e_g intersects ω orthogonally inside the quadrilateral $ABCV$. Besides, e_g intersects either the triangular face CDV or the quadrilateral $AVDQ$. Show that in either case Γ^* is not discrete.

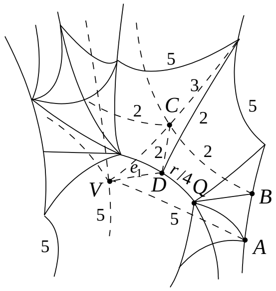


Figure 19:

(a) The axis e_g intersects CDV. Then the angle of intersection is less than $\pi/6$ and hence, there exists an elliptic element $h \in \Gamma^*$ of order $q > 3$ with the axis lying in CDV. Since h and CV are disjoint, h intersects CD and DV.

Let $H = h \cap CD$. Then $CD = CH + HD$. From $\triangle CDV$, $\cosh CD = 2\cos(\pi/5)/(\sqrt{3}\sin(\pi/r))$. On the other hand, the distance between h and an elliptic element of order 3 or $r \geq 5$ cannot be greater than the corresponding minimal distance between axes of elliptic elements in a discrete group and therefore,

$$CH \geq \rho_{min}(q, 3) \qquad \text{and} \qquad HD \geq \rho_{min}(q, r). \tag{12}$$

However,

$$\frac{2\cos(\pi/5)}{\sqrt{3}\sin(\pi/r)} < \cosh(\rho_{min}(q, r) + \rho_{min}(q, 3)), \qquad \text{for } q > 3, r \geq 5. \tag{13}$$

Therefore, from (12)–(13) it follows that $CD < CH + HD$. Contradiction. So, Γ^* is not discrete.

(b) The axis e_g intersects $AVDQ$. Then there exists an elliptic element $h \in \Gamma^*$ of order q with the axis lying in $AVDQ$ such that h does not intersect AV, and there exists a reflection plane κ that passes through h and cuts the interior of $ABCVDQ$. Consider the following situations:

(i) if h and DQ are disjoint and $q \geq 3$, then we estimate the distances between h and the axes of orders 5 and r as above and conclude that the group is not discrete.

(ii) if h and DQ are disjoint and $q = 2$, then the line of intersection of κ and $BCDQ$ is the axis of an elliptic element of order $\widetilde{q} \geq 3$. Similarly to part (a) above,

$$\cosh AQ = \cot(\pi/5)\cot(\pi/r) < \cosh(\rho_{min}(\widetilde{q}, 5) + \rho_{min}(\widetilde{q}, r))$$

for all $\widetilde{q} \geq 3$, $r \geq 5$. Therefore, the group is not discrete.

(iii) if h intersects DQ at some point (possibly at infinity), then $r = 5$ and κ cuts off a finite volume tetrahedron from the trihedral angle with the vertex D. Since Γ^* is discrete, $\langle G_T, R_\kappa\rangle$ is also discrete and $\mathbb{H}^3$ is tesselated by finite volume Coxeter tetrahedra. There are nine compact Coxeter tetrahedra, but the group generated by reflections in faces of any of them does not contain subgroups generated by reflections in the sides of triangles $(2, 2, 3)$ and $(2, 2, 5)$ simultaneously. As for non-compact tetrahedra, only $T[2, 2, 5; 2, 3, 6]$ contains such subgroups, but in this case an axis of order 6 passes through CV and thus, the stabilizer of V contains elements of orders 5 and 6, which is impossible in a discrete group. So, Γ^* is not discrete.

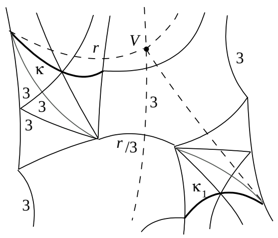


Figure 20:

In **Case (13)**, $n = 3$, $l = r/3$, $r \geq 4$, $(r, 3) = 1$, $\Delta = (3, r, 3)$. In this case the bisector ζ is not a reflection plane. Let $V = \zeta \cap f$ and let κ and κ_1 be reflection planes such that κ passes through $\alpha \cap \zeta$ and makes an angle of π/r with α and $\kappa_1 = e_1(\kappa)$ (Figure 20). Since ζ intersects f, κ also intersects f at the same point V. Since the elliptic element $R_\alpha R_\kappa$ of order r belongs to $St_{\Gamma^*}(V) \subset O(3)$, we have $r = 4, 5$.

Suppose $r = 5$. Consider the link Δ of V formed by α, ω, and κ. Then $\Delta = (3, 5/4, p)$, $p > 0$. Since the group is supposed to be discrete, $p = 2, 5/2, 3$, or 5. But this means that the area of Δ is at least a half of the area of the digonal link of V formed by α and ω, which is impossible. Arguments for $r = 4$ are analogous and Γ^* is not discrete.

In **Case (14)**, $n = 3$, $l = r/2$, $r \geq 5$, $(r, 2) = 1$, $\Delta = (3, r, 3)$. It is easy to construct the decomposition of $\widetilde{\mathcal{P}}$ into tetrahedra $T = [2, 2, 4; 3, 3, r]$ of infinite volume (see Figure 21).

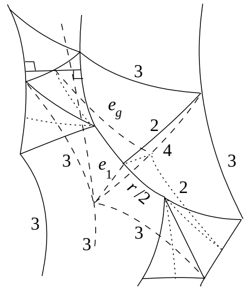


Figure 21:

In this decomposition, e_1 coincides with one of the edges of T and e_g lies in the face F of T opposite to the vertex $(2, 2, r)$. Moreover, e_g maps F onto itself. Hence, there is a reflection plane κ passing through e_g orthogonally to F so that κ decomposes T into two tetrahedra $T_1 = [2, 2, 4; 2, 3, 2r]$ of infinite volume (see Figure 21, where new lines in the decomposition are dotted, compare with Case (9)). Thus, $\Gamma^* = G_{T_1}$ is discrete and is one of the groups from item (2)(*iii*) of the theorem.

In **Case (15)**, $n = 3$, $l = r/2$, $r \geq 3$, $(r, 2) = 1$, $\Delta = (3, r, 4)$. One can construct the decomposition of $\widetilde{\mathcal{P}}$ into infinite volume tetrahedra $T = [2, 2, 3; 3, 4, r]$ (see Figure 22). In this decomposition, e_1 coincides with one of the edges of T. We determine the position of e_g. Let η be the plane in the decomposition that is orthogonal to the bisector ζ. Let $A = \eta \cap \alpha' \cap \omega$, $B = \eta \cap \alpha' \cap \zeta$, and $V = \eta \cap \zeta \cap \omega$. Consider the face F of $\widetilde{\mathcal{P}}$ lying in ω. F is split by the axis AV of order 3 into two parts so that AV makes equal (alternate interior) angles with $f = \alpha \cap \omega$ and $\alpha' \cap \omega$. Then e_g intersects AV in the midpoint.

Since e_g is orthogonal to AV, it maps A to V and therefore, the stabilizer $St_{\Gamma^*}(A)$ of the point A in Γ^* contains an element of order 4. Hence, AB is the axis of an elliptic element of order 4 and from the link of B formed by α', ζ, and η we see that $r = 4$, which contradicts $(r, 2) = 1$. Thus, Γ^* is not discrete.

In **Case (16)**, $n = 3$, $l = r/2$, $r \geq 3$, $(r, 2) = 1$, $\Delta = (3, r, 5)$. Since the bisector ζ is a reflection plane, there is a reflection plane η passing through e_1 orthogonally to ζ (see Figure 23a). The link of the vertex A formed by η, α',

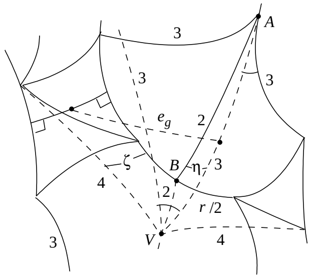
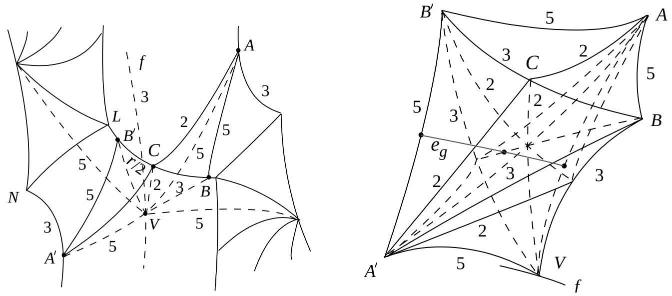


Figure 22:

and ω is a spherical triangle $(2,3,5/2)$. Hence, there is a reflection plane κ through $\eta\cap\omega$ that makes an angle of $\pi/5$ with both ω and η. One can see that the link of the vertex B formed by α', α'', and κ is a triangle $(3,r,5/2)$, $r\geq 3$. However, if $r\geq 5$, then $(3,r,5/2)$ is a hyperbolic triangle that does not lead to a discrete group.



(a) (b)

Figure 23:

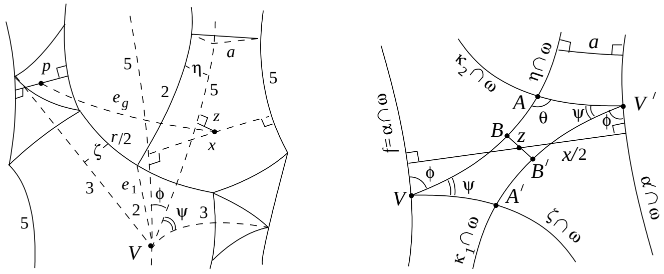
Consider $r=3$. Then the tetrahedron $ABCV$ is compact and splits up into three tetrahedra $T=[2,2,3;2,5,3]$ (Figure 23b). In the decomposition of $\widetilde{\mathcal{P}}$ into T, e_1 coincides with an edge of T.

We determine the position of e_g. First, note that for $r=3$, e_g intersects LN in the midpoint. Further, $A'B'$ makes equal alternate interior angles with $\alpha\cap\alpha'$ and $\alpha'\cap\alpha''$ and hence, LN bisects $A'B'$. Therefore, e_g also bisects $A'B'$.

Consider the line l passing through the midpoints of $A'B'$ and AV. It bisects the edge with the dihedral angle $\pi/2$ that is opposite to both AV and $A'B'$ in the small tetrahedra. Thus, l is orthogonal to AV and to $A'B'$. Moreover, l makes an angle of $\pi/10$ with $\eta=AA'V$ and, therefore, is orthogonal to ω. So, e_g coincides with l.

By [3], $\langle G_T, e_g\rangle=\Gamma^*$ is discrete. This is one of the groups from item (2)(*iii*) of the theorem.

In **Case (17)**, $n=5$, $l=r/2$, $r\geq 3$, $(r,2)=1$, $\Delta=(5,r,3)$. In this case $\widetilde{\mathcal{P}}$ is decomposed into tetrahedra $T=[2,2,5;3,5,r]$ of infinite volume (see Figure 24a).



(a) (b)

Figure 24:

Let z denote the distance between e_g and $\eta\cap\omega$, an axis of order 5. From the decomposition of τ, we can calculate the length of the common perpendicular p to α and α':

$$\cosh p=\cos(\pi/r)+\cos(\pi/5).$$

Then from the plane δ, which is orthogonal to α, α', and ω, we calculate the length of the common perpendicular x to $f=\alpha\cap\omega$ and $\alpha'\cap\omega$:

$$\cosh x=\frac{\cosh p+\cos^2(\pi/5)}{\sin^2(\pi/5)}.$$

From the link of the vertex V formed by ζ, α, and ω,

$$\cos\phi=\frac{\cos(2\pi/3)+\cos^2(\pi/5)}{\sin^2(\pi/5)}=\frac{2\cos^2(\pi/5)-1}{2\sin^2(\pi/5)}.$$

Analogously, from the link of a vertex formed by α', η, ω,

$$\cosh a=\cot^2(\pi/5).$$

Then

$$\sinh z=\frac{\cosh a-\cos\phi}{2\sinh(x/2)}=\frac{1}{4\sin^2(\pi/5)\sinh(x/2)}.$$

If $r>3$, then $z<\rho_{min}(2,5)$ and the group is not discrete. Assume that $r=3$. Let κ_1 and κ_2 be reflection planes such that $R_{\kappa_1}=e_gR_\eta e_g$ and $R_{\kappa_2}=e_gR_\zeta e_g$. Then η, κ_1, κ_2, and ζ cut off a quadrilateral $V_1AV_1'A'$ from the plane ω (Figure 24b). Calculate $\theta=\angle A$ in $VAV'A'$:

$$\cos\theta=-\cos\psi\cosh(2z)+\sin\psi\sinh(2z)\sinh B'V',$$

where

$$\cos\psi = \cot(\pi/5)/\sqrt{3} \quad \text{and} \quad \sinh B'V' = \frac{\sinh(x/2) - \cos\phi \sinh z}{\sin\phi \cosh z}.$$

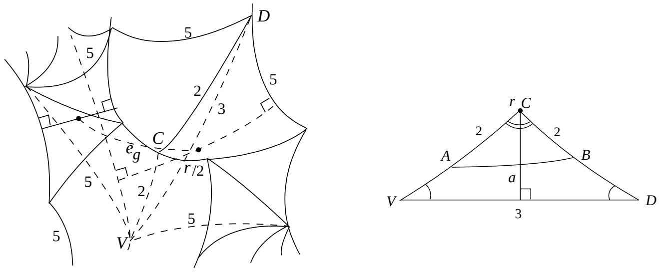
On the other hand, the link of the vertex A formed by η, κ_2, and ω is a spherical triangle $(3, 5, q)$. In a discrete group, $q = 2, 5/3, 3/2$, or $5/4$. Then from such a link

$$\cos\theta(q) = \frac{\cos(\pi/q) + \cos(\pi/3)\cos(\pi/5)}{\sin(\pi/3)\sin(\pi/5)} = \frac{2\cos(\pi/q) + \cos(\pi/5)}{\sqrt{3}\sin(\pi/5)} \neq \cos\theta$$

for all possible q. We arrive at a contradiction and thus, Γ^* is not discrete.

In **Case (18)**, $n = 5$, $l = r/2$, $r \geq 3$, $(r, 2) = 1$, $\Delta = (5, r, 4)$. The link of a vertex formed by α, ω, and the bisector ζ is a spherical triangle $(5, 4, 4/3)$. But the group generated by reflections in its sides is not discrete. Thus, Γ^* is not discrete.

In **Case (19)**, $n = 5$, $l = r/2$, $r \geq 3$, $(r, 2) = 1$, $\Delta = (5, r, 5)$. It is not difficult to construct the decomposition of $\widetilde{\mathcal{P}}$ into tetrahedra $T = [2, 2, 3; 5, 5, r]$ of infinite volume. The axis e_1 coincides with an edge of T. We determine the position of e_g (Figure 25a). Consider the face F of $\widetilde{\mathcal{P}}$ lying in ω. F is split into two parts by the line $l = \eta \cap \omega$, where η is a reflection plane passing through e_1 orthogonally to the bisector ζ. Since the (acute) angle that l makes with f is greater than the (acute) angle that l makes with $\omega \cap \alpha'$, e_g intersects η in an interior point of VCD. Moreover, the angle between e_g and η is less than $\pi/6$.



(a) (b)

Figure 25:

Then there is an elliptic element $h \in \Gamma^*$ of order $q > 3$ with an axis lying in η and disjoint from VD. Let h intersect the sides VC and CD of $\triangle VCD$ in points A and B, respectively (Figure 25b). Clearly, h cannot intersect both the sides at right angles. So, $St_{\Gamma^*}(A)$ and $St_{\Gamma^*}(B)$ cannot be both $(2, 2, q)$. Therefore, $q = 4$ or 5. Note that VCD is an isosceles triangle with

$$\angle V = \angle D = \arccos((1 + \sqrt{5})/2) \quad \text{and} \quad \angle C = \pi/r.$$

Suppose $r = 3$. For $q = 5$ calculate the altitude a from C in VCD: $\cosh a = 4\cos(\pi/5)/\sqrt{3}$. But then $a/2 < \rho_{min}(3, 5)$ and thus, the group is not discrete. Suppose $q = 4$. The axis h of order 4 intersects the axes of order 2 at A and B. Since in a discrete group axes of orders 2 and 4 intersect at $\pi/4$ or $\pi/2$ and ABC is a hyperbolic triangle, we have $\angle A = \angle B = \pi/4$ in $\triangle ABC$. Then the distance between AB and VD is less than $\rho_{min}(3, 4)$. Thus, Γ^* is not discrete.

As for $r \geq 5$, it turns out that $a - \rho_{min}(q, r) < \rho_{min}(q, 3)$ for $q = 4, 5$. Thus, the group is not discrete.

We have considered all the cases and so Theorem A is proved. □

7 On the minimal volume hyperbolic 3-orbifold

Let G_T be the group generated by reflections in the faces of the hyperbolic tetrahedron $T = [2, 2, 3; 2, 5, 3]$ and let Δ_T be its orientation preserving index 2 subgroup. It is well-known that Δ_T can be generated by two elements a and b of order 3 whose axes are the mutually orthogonal edges of T with dihedral angles $\pi/3$ (see eg. [15]). The $\mathbb{Z}_2$-extension $\Gamma_{353} = \langle \Delta_T, e \rangle$, where e is a half-turn whose axis is orthogonal to the opposite edges of T with dihedral angles $\pi/2$ and $\pi/5$, is discrete by [3]. Γ_{353} has minimal co-volume among all Kleinian groups containing elements of order $p \geq 4$ and among all groups with a tetrahedral subgroup [11]. It is also known that Γ_{353} has minimal co-volume in the class of arithmetic Kleinian groups [2].

The group $\Gamma_{353} = \langle a, b, e \rangle$ is generated by a and e because $b = eae$. For this choice of generators, the parameter $\gamma(a, e)$ is not real, since the axes of a and e are not mutually orthogonal. (For geometric meaning of the parameters see [18, Theorems 1–3].) However, the following statement is true.

Corollary 1 *The group Γ_{353} is an $\mathcal{RP}$ group.*

Proof. The idea is to find another generating pair for Γ_{353}. We will show that Γ_{353} is isomorphic to one of the discrete $\mathcal{RP}$ groups that appear in the proof of Theorem A. Consider, for example, the group from Item (*ii*) of Theorem A with $n = 5$, $m = 2$, and $l = 3/2$ (see Case (11) in Section 5 and Figure 9h). Recall that $\widetilde{\Gamma} = \langle f, g, e \rangle$ is the orientation preserving subgroup of $\Gamma^* = \langle G_T, e_g \rangle$ and so $\widetilde{\Gamma}$ is isomorphic to Γ_{353}.

It remains to prove that $\widetilde{\Gamma} = \Gamma$. All we need is to show that $\Gamma_{353} \cong \widetilde{\Gamma}$ is generated by f and g, i.e. that $e = W(f, g)$.

We have

$$h_1^2 = R'_\alpha R_\alpha = gfg^{-1}f \quad \text{and} \quad h_2^2 = R''_\alpha R'_\alpha = f^2g^{-1}f^{-1}gf^2gf^{-1}g^{-1}.$$

Moreover, in our case $h_1^4 = 1$ and $h_2^3 = 1$. Let $x = R_\zeta R_\alpha$ and $y = R_\zeta R_\omega$, where ζ is the bisector of the dihedral angle of $\widetilde{\mathcal{P}}$ made by α' and α'' . Then using

$(gfg^{-1}f)^2 = 1$ and $f^5 = 1$, we have

$$\begin{aligned} x &= R_\zeta R_\alpha = (R_\zeta R'_\alpha)(R'_\alpha R_\alpha) = h_2 h_1^2 = gfg^{-1}f^3g^{-1}fgf^3gfg^{-1}f \\ &= gfg^{-1}f^3g^{-1}fgf^2gf^{-1}g^{-1}, \\ y &= R_\zeta R_\omega = (R_\zeta R_\alpha)(R_\alpha R_\omega) = xf^{-1} = gfg^{-1}f^3g^{-1}fgf^2gf^{-1}g^{-1}f^{-1}. \end{aligned}$$

The element $z = yx$ is an element of order 3 and $z = R_\alpha R_\eta$, where η is the plane passing through e_1 orthogonally to ζ. So,

$$\begin{aligned} e_1 &= R_\zeta R_\eta = (R_\zeta R_\alpha)(R_\alpha R_\eta) = xz \\ &= gfg^{-1}f^3g^{-1}f^2gf^3gf^2g^{-1}f^3g^{-1}fgf^2gf^{-1}g^{-1}. \end{aligned}$$

Then since $e_1 = f^2 e$, we get

$$e = f^3gfg^{-1}f^3g^{-1}f^2gf^3gf^2g^{-1}f^3g^{-1}fgf^2gf^{-1}g^{-1}.$$

This completes the proof. $\square$

8 Parameters

Theorem A can be reformulated in terms of the parameters $\beta = \mathrm{tr}^2 f - 4$, $\beta' = \mathrm{tr}^2 g - 4$, and $\gamma = \mathrm{tr}[f,g] - 2$.

Theorem B. *Let $f, g \in \mathrm{PSL}(2,\mathbb{C})$, $\beta = -4\sin^2(\pi/n)$, $n \geq 3$, $(n,2) = 1$, $\beta' > 0$, and $0 < \gamma < -\beta\beta'/4$. Then $\Gamma = \langle f, g\rangle$ is discrete if and only if (β, β', γ) is one of the triples listed in rows 21–41 of Table 2 in Appendix.*

Proof. To prove the theorem it is sufficient to calculate values of the parameters (β, β', γ) for all discrete groups described in Theorem A. Since we know fundamental polyhedra for the discrete groups, it can be done by straightforward calculation.

Since f is a primitive elliptic element of order n, $\beta = \mathrm{tr}^2 f - 4 = -4\sin^2(\pi/n)$.

Compute $\gamma = \mathrm{tr}[f,g] - 2$. Using the identity (6) from Section 3 and the fact that $f = R_{\alpha^*} R_\omega$, where $\alpha^* = R_\omega(\alpha)$, we have

$$fgf^{-1}g^{-1} = f(f')^{-1} = (R_{\alpha^*}R_\omega)(R_\omega R'_\alpha) = R_{\alpha^*}R'_\alpha.$$

Note that α^* passes through f and makes an angle of π/n with ω. Moreover, α^* and α' are disjoint and δ is orthogonal to both α^* and α'. Therefore, $[f,g] = fgf^{-1}g^{-1}$ is a hyperbolic element with the axis lying in δ and the translation length $2d$, where d is the distance between α^* and α'.

It is known [18, Theorem 1 or Table 1] that for an elliptic and hyperbolic generators with intersecting axes, $0 < \gamma < -\beta\beta'/4$. Hence, we get

$$\mathrm{tr}[f,g] = +2\cosh d \qquad \text{and} \qquad \gamma = 2(\cosh d - 1).$$

Finally, we compute $\beta' = \mathrm{tr}^2 g - 4 = 4\sinh^2 T$, where T is the distance between e and e_g that can be measured in ω.

There are three essentially different fundamental polyhedra for the discrete groups from Theorem A: $\widetilde{\mathcal{P}}$, $\mathcal{P}_1$, and $\mathcal{P}_2$. All other fundamental polyhedra are just smaller polyhedra in decompositions of these three. We describe a procedure for calculating the parameters in case of $\widetilde{\mathcal{P}}$, for other polyhedra the procedure is similar.

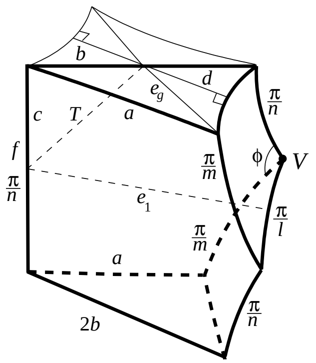


Figure 26:

To draw the non-compact polyhedron $\widetilde{\mathcal{P}}$ from Section 5 we add planes δ and $e_1(\delta)$, see Figure 26, where thin lines show lines in δ that are not edges of $\widetilde{\mathcal{P}}$, b is the distance between f and e_g, and c is the distance between e_1 and δ. If $m < \infty$ then from the plane δ we have

$$\cosh d = -\cos(2\pi/n)\cos(\pi/m) + \sin(2\pi/n)\sin(\pi/m)\cosh a,$$

where

$$\cosh a = \frac{\cos(\pi/n) + \cos(\pi/n)\cos(\pi/m)}{\sin(\pi/n)\sin(\pi/m)}.$$

Analogous calculations can be done for $m = \infty$ and $m = \overline{\infty}$. We obtain

$$\gamma = \begin{cases} 2(\cos(\pi/m) + \cos(2\pi/n)) & \text{if} \quad m < \infty, \\ 2(1 + \cos(2\pi/n)) & \text{if} \quad m = \infty, \\ 2(\cosh p + \cos(2\pi/n)) & \text{if} \quad m = \overline{\infty}, \end{cases}$$

where p is the distance between α and $\alpha' = e_g(\alpha)$ if they are disjoint.

Further,

$$\cosh T = \cosh b \cosh c, \quad \text{where} \quad \cosh b = \frac{\cos(\pi/(2m))}{\sin(\pi/n)}.$$

Find $\cosh c$. Suppose V is proper. The face of $\widetilde{\mathcal{P}}$ lying in ω is a pentagon with four right angles, so c is given by

$$\cosh 2c = \frac{\cos\phi + \cosh a \cosh 2b}{\sinh a \sinh 2b}, \quad \cos\phi = \frac{\cos(\pi/l) + \cos(\pi/n)\cos(\pi/m)}{\sin(\pi/n)\sin(\pi/m)}.$$

formulas are similar for ideal or imaginary vertex V and for $l \in \{\infty, \overline{\infty}\}$.

Combining the formulas above and simplifying them, we have

$$\beta' = \begin{cases} \dfrac{2\cos(\pi/l)}{\gamma} - \dfrac{\sqrt{\beta+4}(\beta+(\gamma-\beta)^2)}{\gamma\beta} - \dfrac{2\gamma}{\beta} - 2 & \text{if} \quad m < \infty, \\ \dfrac{2}{\gamma} - \dfrac{\sqrt{\beta+4}(\beta+(\gamma-\beta)^2)}{\gamma\beta} - \dfrac{2\gamma}{\beta} - 2 & \text{if} \quad m = \infty, \\ \dfrac{2\cosh t}{\gamma} - \dfrac{\sqrt{\beta+4}(\beta+(\gamma-\beta)^2)}{\gamma\beta} - \dfrac{2\gamma}{\beta} - 2 & \text{if} \quad m = \overline{\infty}, \end{cases}$$

where t is the distance between α' and α''. These results lead to rows (21)–(24) in Table 2.

Similarly, considering the polyhedron $\mathcal{P}_1$ from Section 4 we calculate the parameters for the groups from item (2)(v) of Theorem A to get rows (25)–(26). Rows (27)–(28) come from Theorem A(2)(vii) (see Section 4 and Figure 7b for $\mathcal{P}_2$).

If a fundamental polyhedron appears as a result of decomposition of $\widetilde{\mathcal{P}}$, $\mathcal{P}_1$, or $\mathcal{P}_2$, then it suffices to substitute corresponding values of dihedral angles or distances into general formulas. So, rows (29), (32), and (35)–(40) correspond to Theorem A(2)(ii); rows (33) and (34) correspond to Theorem A(2)(iii); rows (30) and (41) correspond to Theorem A(2)(iv); the row (31) corresponds to Theorem A(2)(vi). □

Appendix

For simplicity, in the statement of Theorem B and in Table 2 all elliptic generators are assumed to be primitive. If one (or both) generator(s) of an $\mathcal{RP}$ group is non-primitive elliptic, Table 2 and Theorem B still can be used to verify discreteness of the group, but first we must replace the triple (β, β', γ), where $\beta = -4\sin^2(q\pi/n)$, $(q, n) = 1$, $1 < q < n/2$, with a new triple $(\widetilde{\beta}, \beta', \widetilde{\gamma})$, where $\widetilde{\beta} = -4\sin^2(\pi/n)$ and $\widetilde{\gamma} = (\widetilde{\beta}/\beta)\gamma$. The new triple corresponds to the same group by Gehring and Martin [9] (cf. [18, Remark 2, p. 262]).

Remark A.1. A part of Table 2 first appeared in [18], but unfortunately, there was a misprint. Here we correct it.

Table 2: All truly spatial discrete $\mathcal{RP}$ groups whose generators have real traces. Here all numbers n, m, p are positive integers

	$\beta = \beta(f)$	$\gamma = \gamma(f,g)$	$\beta' = \beta(g)$
Both generators are elliptic, mutually orthogonal skew axes			
1	$-4\sin^2\frac{\pi}{n},\ n \geq 3$	$-4\cos^2\frac{\pi}{p}$, $\cos\frac{\pi}{p} > \sin\frac{\pi}{n}\sin\frac{\pi}{m}$	$-4\sin^2\frac{\pi}{m},\ m \geq 3$
2	$-4\sin^2\frac{\pi}{n},\ n \geq 3$	$(-\infty, -4]$	$-4\sin^2\frac{\pi}{m},\ m \geq 3$

Table 2: (continued)

	$\beta = \beta(f)$	$\gamma = \gamma(f,g)$	$\beta' = \beta(g)$
3	$-4\sin^2\frac{\pi}{n},\ n \geq 7$, $(n,2) = 1$	$-(\beta+2)^2$	β
f is elliptic and g is parabolic, the axis of f lies in an invariant plane of g			
4	$-4\sin^2\frac{\pi}{n},\ n \geq 3$	$-4\cos^2\frac{\pi}{p},\ p \geq 3$	0
5	$-4\sin^2\frac{\pi}{n},\ n \geq 3$	$(-\infty, -4]$	0
Each of f and g is either parabolic or hyperbolic, each of the generators has an invariant plane which is orthogonal to all invariant planes of the other generator			
6	$[0, +\infty)$	$-4\cos^2\frac{\pi}{p},\ p \geq 3$	$[0, +\infty)$
7	$[0, +\infty)$	$(-\infty, -4]$	$[0, +\infty)$
f is elliptic, g is hyperbolic, disjoint axes			
8	$-4\sin^2\frac{\pi}{n},\ n \geq 3$	$-4\cos^2\frac{\pi}{p},\ p \geq 3$	$(0, +\infty)$
9	$-4\sin^2\frac{\pi}{n},\ n \geq 3$	$(-\infty, -4]$	$(0, +\infty)$
10	$-4\sin^2\frac{\pi}{n},\ n \geq 5$, $(n,2) = 1$	$-(\beta+2)^2$	$4(\beta+4)\cos^2\frac{\pi}{p} - 4$, $p \geq 4$
11	$-4\sin^2\frac{\pi}{n},\ n \geq 5$, $(n,2) = 1$	$-(\beta+2)^2$	$[4(\beta+3), +\infty)$
12	-3	$(\sqrt{5}-3)/2$	$2\cos^2\frac{\pi}{p}(7+3\sqrt{5}) - 4$, $p \geq 3$
13	-3	$(\sqrt{5}-3)/2$	$[2(5+3\sqrt{5}), +\infty)$
f is elliptic of even order n, g is hyperbolic, the axes intersect non-orthogonally; $1/n + 1/m < 1/2$			
14	$-4\sin^2\frac{\pi}{n},\ n \geq 4$	$2(\cos\frac{2\pi}{m} + \cos\frac{2\pi}{n})$, $(m,2) = 2$	$\dfrac{4\cos^2\frac{\pi}{p}}{\gamma} - \dfrac{4\gamma}{\beta},\ p \geq 3$
15	$-4\sin^2\frac{\pi}{n},\ n \geq 4$	$2(\cos\frac{2\pi}{m} + \cos\frac{2\pi}{n})$ $(m,2) = 2$	$\left[\dfrac{4}{\gamma} - \dfrac{4\gamma}{\beta}, +\infty\right)$
16	$-4\sin^2\frac{\pi}{n},\ n \geq 4$	$[\beta+4, +\infty)$	$\dfrac{4\cos^2\frac{\pi}{p}}{\gamma} - \dfrac{4\gamma}{\beta},\ p \geq 3$
17	$-4\sin^2\frac{\pi}{n},\ n \geq 4$	$[\beta+4, +\infty)$	$\left[\dfrac{4}{\gamma} - \dfrac{4\gamma}{\beta}, +\infty\right)$
18	$-4\sin^2\frac{\pi}{n},,\ n \geq 4$	$2(\cos\frac{2\pi}{m} + \cos\frac{2\pi}{n})$, $(m,2) = 1$	$\frac{4(\gamma-\beta)}{\gamma}\cos^2\frac{\pi}{p} - \frac{4\gamma}{\beta}$, $p \geq 3$

Table 2: (continued)

	$\beta=\beta(f)$	$\gamma=\gamma(f,g)$	$\beta'=\beta(g)$
19	$-4\sin^2\frac{\pi}{n},\ n\geq 4$	$2(\cos\frac{2\pi}{m}+\cos\frac{2\pi}{n})$, $(m,2)=1$	$\left[\frac{4(\gamma-\beta)}{\gamma}-\frac{4\gamma}{\beta},+\infty\right)$
20	-2	$2\cos\frac{2\pi}{m}$, $m\geq 5,\ (m,2)=1$	$\gamma^2+4\gamma$
	f is elliptic of odd order n, g is hyperbolic, the axes intersect non-orthogonally; $1/n+1/m<1/2$, $U=-2\frac{(\gamma-\beta)^2\cos\frac{\pi}{n}+\gamma(\gamma+\beta)}{\gamma\beta},\ \ V=-2\frac{(\beta+2)^2\cos\frac{\pi}{n}}{\beta+1}-2\frac{(\beta^2+6\beta+4)}{\beta}$		
21	$-4\sin^2\frac{\pi}{n},\ n\geq 3$	$2(\cos\frac{2\pi}{m}+\cos\frac{2\pi}{n})$, $(m,2)=2$	$\frac{2}{\gamma}(\cos\frac{\pi}{p}-\cos\frac{\pi}{n})+U$, $p\geq 2$
22	$-4\sin^2\frac{\pi}{n},\ n\geq 3$	$2(\cos\frac{2\pi}{m}+\cos\frac{2\pi}{n})$, $(m,2)=2$	$\left[\frac{2(1-\cos\frac{\pi}{n})}{\gamma}+U,+\infty\right)$
23	$-4\sin^2\frac{\pi}{n},\ n\geq 3$	$[\beta+4,+\infty)$	$\frac{2}{\gamma}(\cos\frac{\pi}{p}-\cos\frac{\pi}{n})+U$, $p\geq 2$
24	$-4\sin^2\frac{\pi}{n},\ n\geq 3$	$[\beta+4,+\infty)$	$\left[\frac{2(1-\cos\frac{\pi}{n})}{\gamma}+U,+\infty\right)$
25	$-4\sin^2\frac{\pi}{n},\ n\geq 3$	$2(\cos\frac{2\pi}{m}+\cos\frac{2\pi}{n})$, $(m,2)=1$	$\frac{2(\gamma-\beta)}{\gamma}\cos\frac{\pi}{p}+U,\ p\geq 2$
26	$-4\sin^2\frac{\pi}{n},\ n\geq 3$	$2(\cos\frac{2\pi}{m}+\cos\frac{2\pi}{n})$, $(m,2)=1$	$\left[\frac{2(\gamma-\beta)}{\gamma}+U,+\infty\right)$
27	$-4\sin^2\frac{\pi}{n},\ n\geq 7$	$(\beta+4)(\beta+1)$	$\frac{2(\beta+2)^2\cos\frac{\pi}{p}}{\beta+1}+V,\ p\geq 2$
28	$-4\sin^2\frac{\pi}{n},\ n\geq 7$	$(\beta+4)(\beta+1)$	$\left[\frac{2(\beta+2)^2}{\beta+1}+V,+\infty\right)$
29	$-4\sin^2\frac{\pi}{n},\ n\geq 5$, $(n,3)=1$	$\beta+3$	$2\frac{(\beta-3)\cos\frac{\pi}{n}-2\beta-3}{\beta}$
30	$-4\sin^2\frac{\pi}{n},\ n\geq 5$, $(n,3)=1$	$2(\beta+3)$	$-6\frac{2\cos\frac{\pi}{n}+\beta+2}{\beta}$
31	-3	$2\cos(2\pi/m)-1$, $m\geq 7, (m,2)=1$	$2\frac{\gamma^2+2\gamma+2}{\gamma}$
32	-3	$2\cos(\pi/m)-1$, $m\geq 4, (m,3)=1$	$\gamma^2+4\gamma$
33	-3	$2\cos(2\pi/m)$, $m\geq 7, (m,4)\leq 2$	2γ
34	-3	$(\sqrt{5}+1)/2$	$\sqrt{5}$
35	-3	$(\sqrt{5}-1)/2$	$\sqrt{5}$

Table 2: (continued)

	$\beta=\beta(f)$	$\gamma=\gamma(f,g)$	$\beta'=\beta(g)$
36	-3	$(\sqrt{5}-1)/2$	$\sqrt{5}-1$
37	$(\sqrt{5}-5)/2$	$(\sqrt{5}-1)/2$	$\sqrt{5}$
38	$(\sqrt{5}-5)/2$	$(\sqrt{5}-1)/2$	$(3\sqrt{5}-1)/2$
39	$(\sqrt{5}-5)/2$	$(\sqrt{5}-1)/2$	$3(\sqrt{5}+1)/2$
40	$(\sqrt{5}-5)/2$	$(\sqrt{5}+1)/2$	$3(\sqrt{5}+1)/2$
41	$(\sqrt{5}-5)/2$	$\sqrt{5}+2$	$(5\sqrt{5}+9)/2$

References

[1] A. F. Beardon, *The geometry of discrete groups*, Springer-Verlag, New York–Heidelberg–Berlin, 1983.

[2] T. Chinburg and E. Friedman, *The smallest arithmetic hyperbolic three-orbifold*, Invent. Math. **86** (1986), no. 3, 507–527.

[3] D. A. Derevnin and A. D. Mednykh, *Discrete extensions of the Lanner groups (Russian)*, Dokl. Akad. Nauk **361** (1998), no. 4, 439–442; translation in: Dokl. Math. **58** (1998), no. 1, 78–80.

[4] D. B. A. Epstein and C. Petronio, *An exposition of Poincaré's polyhedron theorem*, L'Enseignement Mathématique **40** (1994), 113–170.

[5] A. A. Felikson, *Coxeter decompositions of spherical tetrahedra*, preprint, 99–053, Bielefeld, 1999.

[6] W. Fenchel, *Elementary geometry in hyperbolic space*, de Gruyter Studies in Mathematics, **11**. Walter de Gruyter & Co., Berlin, 1989.

[7] B. Fine and G. Rosenberger, *Classification of generating pairs of all two generator Fuchsian groups*, in: Groups'93 (Galway/St.Andrews, v.1, Galway, 1993), 205–232; London Math. Soc. Lecture Note Ser., 211, Cambridge Univ. Press, Cambridge, 1995.

[8] F. W. Gehring, J. P. Gilman, and G. J. Martin, *Kleinian groups with real parameters*, Commun. Contemp. Math. **3**, no. 2 (2001), 163–186.

[9] F. W. Gehring and G. J. Martin, *Chebyshev polynomials and discrete groups*, Proc. of the Conf. on Complex Analysis (Tianjin, 1992), 114–125, Conf. Proc. Lecture Notes Anal., I, Internat. Press, Cambridge, MA, 1994.

[10] F. W. Gehring and G. J. Martin, *Commutators, collars and the geometry of Möbius groups*, J. Anal. Math. **63** (1994), 175–219.

[11] F. W. Gehring and G. J. Martin, *On the minimal volume hyperbolic* 3-*orbifold*, Math. Res. Lett. **1** (1994), no. 1, 107–114.

[12] F. W. Gehring, F. W. Marshall, and G. J. Martin, *The spectrum of elliptic axial distances in Kleinian groups*, Indiana Univ. Math. J. **47** (1998), no. 1, 1–10.

[13] J. Gilman, *Two-generator discrete subgroups of* $\mathrm{PSL}(2,\mathbb{R})$, Mem. Amer. Math. Soc. **117** (1995), no. 561.

[14] J. Gilman and B. Maskit, *An algorithm for* 2*-generator Fuchsian groups*, Mich. Math. J. **38** (1991), no. 1, 13–32.

[15] E. Klimenko, *Discrete groups in three-dimensional Lobachevsky space generated by two rotations*, Siberian Math. J. **30** (1989), no. 1, 95–100.

[16] E. Klimenko, *Some remarks on subgroups of* $\mathrm{PSL}(2,\mathbb{C})$, Q & A in General Topology **8** (1990), no. 2, 371–381.

[17] E. Klimenko, *Some examples of discrete groups and hyperbolic orbifolds of infinite volume*, J. Lie Theory **11** (2001), 491–503.

[18] E. Klimenko and N. Kopteva, *Discreteness criteria for* $\mathcal{RP}$ *groups*, Israel J. Math. **128** (2002), 247–265.

[19] E. Klimenko and N. Kopteva, *Discrete* $\mathcal{RP}$ *groups with a parabolic generator*, to appear in Siberian Math. J. in 2005.

[20] E. Klimenko and M. Sakuma, *Two-generator discrete subgroups of* $\mathrm{Isom}(\mathbb{H}^2)$ *containing orientation-reversing elements*, Geometriae Dedicata **72** (1998), 247–282.

[21] A. W. Knapp, *Doubly generated Fuchsian groups*, Mich. Math. J. **15** (1968), no. 3, 289–304.

[22] B. Maskit, *Kleinian groups*, Grundlehren der Math. Wiss. 287, Springer-Verlag, 1988.

[23] J. P. Matelski, *The classification of discrete 2-generator subgroups of* $\mathrm{PSL}(2,\mathbb{R})$, Israel J. Math. **42** (1982), no. 4, 309–317.

[24] G. Rosenberger, *All generating pairs of all two-generator Fuchsian groups*, Arch. Math. **46** (1986), no. 3, 198–204.

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