A TWO-DIMENSIONAL SLICE THROUGH THE PARAMETER SPACE OF TWO-GENERATOR KLEINIAN GROUPS

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Abstract. We describe all real points of the parameter space of two-generator Kleinian groups with a parabolic generator, that is, we describe a certain two-dimensional slice through this space. In order to do this we gather together known discreteness criteria for two-generator groups and present them in the form of conditions on parameters. We complete the description by giving discreteness criteria for groups generated by a parabolic and a π-loxodromic elements whose commutator has real trace and present all orbifolds uniformized by such groups.

1. Introduction

A two-generator subgroup $\Gamma = \langle f, g\rangle$ of $\mathrm{PSL}(2, \mathbb{C})$ is determined up to conjugacy by its parameters $\beta = \beta(f) = \mathrm{tr}^2 f - 4$, $\beta' = \beta(g) = \mathrm{tr}^2 g - 4$, and $\gamma = \gamma(f, g) = \mathrm{tr}[f, g] - 2$ whenever $\gamma \neq 0$ [6]. So the conjugacy class of an ordered pair $\{f, g\}$ can be identified with a point in the parameter space $\mathbb{C}^3 = \{(\beta, \beta', \gamma)\}$ whenever $\gamma \neq 0$. The subspace $\mathcal{K}$ of $\mathbb{C}^3$ that corresponds to the discrete non-elementary groups $\Gamma = \langle f, g\rangle$ is called the *parameter space of two-generator Kleinian groups.* Note that a two-generator Kleinian group Γ can be represented by several points in $\mathcal{K}$, since the same group can have different generating pairs.

Among all two-generator subgroups of $\mathrm{PSL}(2, \mathbb{C})$, we distinguish the class of $\mathcal{RP}$ *groups* (two-generator groups with real parameters):

$$\mathcal{RP} = \{\Gamma : \Gamma = \langle f, g\rangle \text{ for some } f, g \in \mathrm{PSL}(2, \mathbb{C}) \text{ with } (\beta, \beta', \gamma) \in \mathbb{R}^3\}.$$

The aim of this paper is to completely determine all points in $\mathbb{C}^3$ that are parameters for the discrete non-elementary $\mathcal{RP}$ groups with one generator parabolic:

$$S_\infty = \{(\gamma, \beta) : (\beta, 0, \gamma) \text{ are parameters for some } \langle f, g\rangle \in \mathcal{DRP}\},$$

where $\mathcal{DRP}$ denotes the class of all discrete non-elementary $\mathcal{RP}$ groups. Geometrically, S_∞ is a two-dimensional slice through the six-dimensional parameter space $\mathcal{K}$.

The slice S_∞ intersects the well-known Riley slice $(0, 0, \gamma)$, $\gamma \in \mathbb{C}$, which consists of all Kleinian groups generated by two parabolics.

Consider the sequence of slices $\{S_n\}_{n=2}^{\infty}$, where

$$S_n = \{(\gamma, \beta) : (\beta, -4\sin^2(\pi/n), \gamma) \text{ are parameters for some } \langle f, g\rangle \in \mathcal{DRP}\}.$$

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The first slice S_2 of this sequence is of great interest in the theory of discrete groups. This slice consists of all parameters for discrete $\mathcal{RP}$ groups with an elliptic generator of order 2 and was investigated in [5]. It was shown that if $\langle f, g\rangle$ has parameters (β, β', γ), then there exists a group $\langle f, h\rangle$ with parameters $(\beta, -4, \gamma)$ such that if $\gamma \neq 0, \beta$, then $\langle f, h\rangle$ is discrete whenever $\langle f, g\rangle$ is. Hence, the slice S_2 gives necessary discreteness conditions for a group with parameters (β, β', γ), where β and γ are real. It follows that every S_n with $n > 2$, including S_∞, is a subset of S_2.

Since a parabolic element can be viewed as the limit of a sequence of primitive elliptic elements of order n as $n \to \infty$, the following two questions for $\{S_n\}$ and S_∞ naturally arise.

(1) Is it true that for every point $x \in S_\infty$ there exists a sequence $\{x_k\}_{k=2}^{\infty}$ with $x_k \in S_k$ that converges to x?
(2) Is it true that for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that the ε-neighbourhood of S_∞ contains S_n for all $n > N$?

Note that the structure of S_n for $n > 2$ is unknown.

We work out S_∞ by splitting the plane (γ, β) into several parts. It turns out that $\Gamma = \langle f, g\rangle$ has an invariant plane in one of the following cases: (1) $\gamma < 0$ and $\beta \leq -4$; (2) $\gamma > 0$ and $\beta \geq -4$. Such discrete groups were investigated, for example, in [13] and [8, 14, 15], respectively. If $\gamma < 0$ and $\beta > -4$, then Γ is truly spatial (non-elementary and without invariant plane) and this case is treated in [11]. We get these dicreteness criteria together and transform them into conditions on β and γ if it was not done before.

So the last case to consider is when $\gamma > 0$ and $\beta < -4$. In this case Γ is truly spatial with f π-loxodromic. We complete the study of the slice S_∞ by giving discreteness criteria for such groups.

The paper is organised as follows. In Section 2, discreteness criteria are given for truly spatial $\mathcal{RP}$ groups Γ generated by a π-loxodromic and a parabolic elements (Theorems 2.1 and 2.6). In Section 3, for each such discrete Γ we obtain a presentation and the Kleinian orbifold $Q(\Gamma)$ (Theorem 3.1). Section 4 is devoted to the analysis of the parameter space. We completely describe the slice S_∞ by giving explicit formulas for the parameters β and γ. We also program the obtained formulas in the package Maple 7.0 and plot a part of S_∞ on the (γ, β)-plane to give an idea of how it looks like.

2. Discreteness criteria

Recall that an element $f \in \mathrm{PSL}(2, \mathbb{C})$ with real $\beta(f)$ is *elliptic*, *parabolic*, *hyperbolic*, or *π-loxodromic* according to whether $\beta(f) \in [-4, 0)$, $\beta(f) = 0$, $\beta(f) \in (0, +\infty)$, or $\beta(f) \in (-\infty, -4)$. If $\beta(f) \notin [-4, +\infty)$, then f is called *strictly loxodromic*.

An elliptic element f of order n is said to be *non-primitive* if it is a rotation through $2\pi q/n$, where q and n are coprime $(1 < q < n/2)$. If f is a rotation through $2\pi/n$, then it is called *primitive*.

Theorem 2.1. *Let $f \in \mathrm{PSL}(2, \mathbb{C})$ be a π-loxodromic element, $g \in \mathrm{PSL}(2, \mathbb{C})$ be a parabolic element, and let $\Gamma = \langle f, g\rangle$ be a non-elementary $\mathcal{RP}$ group without invariant plane. Then*

(1) *there exist unique elements* $h_1, h_2 \in \mathrm{PSL}(2,\mathbb{C})$ *such that* $h_1^2 = fg^{-1}f^{-1}g^{-1}$ *and* $(h_1 g)^2 = 1$, $h_2^2 = f^{-1}g^{-1}f^2gf^{-1}$ *and* $(h_2 fg^{-1}f^{-1})^2 = 1$.

(2) *the group* Γ *is discrete if and only if one of the following conditions holds:*
   (i) h_1 *is either a hyperbolic, or parabolic, or primitive elliptic element of even order* $m \geq 4$, *and* h_2 *is either a hyperbolic, or parabolic, or primitive elliptic element of order* $p \geq 3$;
   (ii) h_1 *is a primitive elliptic element of odd order* $m \geq 3$, *and* $h_2 h_1$ *is either a hyperbolic, or parabolic, or primitive elliptic element of order* $k \geq 3$.

Basic geometric construction. We will construct a group Γ^* that contains $\Gamma = \langle f, g\rangle$ as a subgroup of finite index. The idea is to find Γ^* so that a fundamental polyhedron for a discrete Γ^* can be easily constructed. It will be clear from the construction that Γ is commensurable with a reflection group which either coincides with Γ^* or is an index 2 subgroup of Γ^*. The construction presented below will be used throughout Sections 2 and 3 and we shall use the notation introduced here.

Let f and g be as in the statement of Theorem 2.1. Since Γ is a non-elementary $\mathcal{RP}$ group without invariant plane, there exists an invariant plane of g, say η, which is orthogonal to the axis of f [9, Theorem 2].

Denote by M the fixed point of g and by ω the plane that passes through M and f (we denote elements and their axes by the same letters when it does not lead to any confusion). Note that f keeps ω invariant. Since f is orthogonal to η, ω is also orthogonal to η. Let e be the half-turn with the axis $\omega \cap \eta$. Then e passes through M and is orthogonal to f.

Let e_f and e_g be half-turns such that

$$f = e_f e \quad \text{and} \quad g = e_g e. \tag{2.1}$$

Then e_f is orthogonal to ω and e_g lies in η.

Let τ be the plane passing through e_g orthogonally to η and let $\sigma = e_f(\tau)$. The planes τ and ω are parallel and M is their common point on the boundary $\partial\mathbb{H}^3$. Since e_f is orthogonal to ω, the planes σ and ω are also parallel with the common point $e_f(M)$ on $\partial\mathbb{H}^3$. Since $e_f(M) \neq M$, the planes ω, σ, and τ do not have a common point in $\overline{\mathbb{H}^3} = \mathbb{H}^3 \cup \partial\mathbb{H}^3$. Therefore, there exists a unique plane δ orthogonal to all ω, σ, and τ. It is clear that $e_f \subset \delta$.

Consider two extensions of Γ: $\widetilde{\Gamma} = \langle f, g, e\rangle$ and $\Gamma^* = \langle f, g, e, R_\omega\rangle$. (We denote the reflection in a plane κ by R_κ.) One can show that $\widetilde{\Gamma} = \langle e_f, e_g, e\rangle$ and $\Gamma^* = \langle e_f, R_\eta, R_\omega, R_\tau\rangle$. From (2.1), it follows that $\widetilde{\Gamma}$ contains Γ as a subgroup of index at most 2. Moreover, $\widetilde{\Gamma}$ is the orientation preserving subgroup of Γ^* and, hence, Γ^* contains Γ as a subgroup of finite index. Therefore, Γ, $\widetilde{\Gamma}$, and Γ^* are either all discrete, or all non-discrete. We then concentrate on the group Γ^*.

Let $\mathcal{P}^*$ be the infinite volume polyhedron bounded by η, ω, τ, σ, and δ. $\mathcal{P}^*$ has five right dihedral angles (between faces lying in η and ω, η and τ, δ and ω, δ and τ, and δ and σ). The plane σ may either intersect with, or be parallel to, or be disjoint from each of τ and η.

If σ and τ intersect, then we denote the dihedral angle of $\mathcal{P}^*$ between them by $2\pi/m$, where $m > 2$, m is not necessary an integer. We keep the notation $2\pi/m$ taking $m = \infty$ and $m = \overline{\infty}$ for parallel or disjoint σ and τ, respectively. Similarly, we denote the "dihedral angle" between η and σ by π/p, where $p > 2$ is real, ∞,

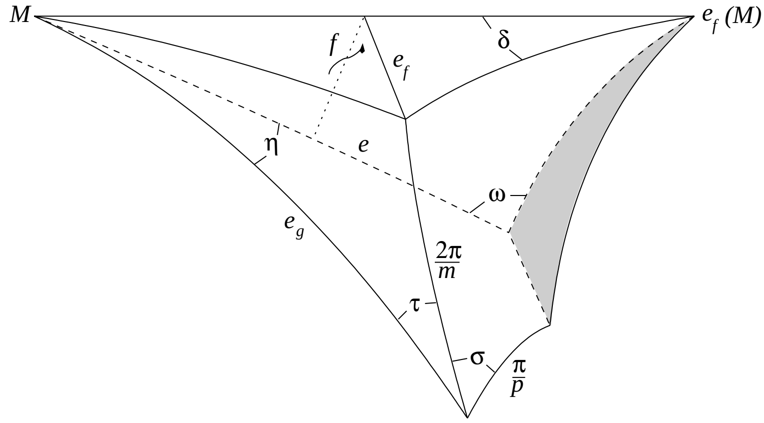


FIGURE 1. Polyhedron $\mathcal{P}^*$

or $\overline{\infty}$. (We regard $\overline{\infty} > \infty > x$, $x/\infty = x/\overline{\infty} = 0$, $\infty/x = \infty$, $\overline{\infty}/x = \overline{\infty}$ for any positive real x.) $\mathcal{P}^*$ exists in $\mathbb{H}^3$ for all $m > 2$ and $p > 2$ by [16].

In Figure 1, $\mathcal{P}^*$ is drawn under assumption that $m < \infty$, $p < \infty$, and $1/2 + 1/p + 2/m > 1$. The shaded triangle shows the hyperbolic plane orthogonal to η, σ, and ω. Note that this plane is not a face of $\mathcal{P}^*$ and is shown only to underline the combinatorial structure of $\mathcal{P}^*$. In figures, we do not label dihedral angles of $\pi/2$ in order to not overload the picture.

Suppose now that $m < \infty$, that is σ and τ intersect. Let ξ be the plane passing through e_f orthogonally to δ. Then ξ is orthogonal to ω. One can see that $\sigma = R_\xi(\tau)$ and ξ is the bisector of the dihedral angle of $\mathcal{P}^*$ made by τ and σ.

Let $\mathcal{Q}^*$ be the polyhedron bounded by η, τ, ω, δ, and ξ. $\mathcal{Q}^*$ has six dihedral angles of $\pi/2$; the dihedral angle between τ and ξ is equal to π/m with $2 < m < \infty$. Denote the "dihedral angle" between η and ξ by π/k, where $k > 2$ is real, $k = \infty$, or $k = \overline{\infty}$. $\mathcal{Q}^*$ exists in $\mathbb{H}^3$ for all $m > 2$ and $k > 2$ by [16]. Note that R_ξ is not necessary in Γ^*, but if it is and if Γ^* is discrete, then we will see that $\mathcal{Q}^*$ is a fundamental polyhedron for Γ^*. In Figure 2, $\mathcal{Q}^*$ is drawn under assumption that $1/2 + 1/k + 1/m > 1$.

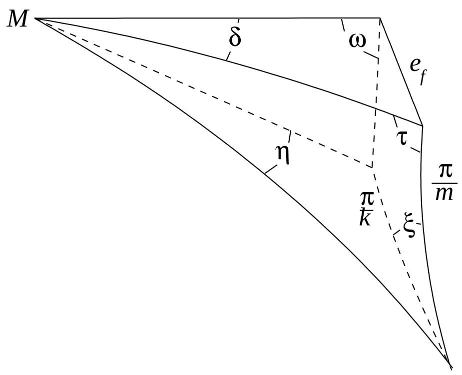


FIGURE 2. Polyhedron $\mathcal{Q}^*$

Lemma 2.2. *Let $f \in \mathrm{PSL}(2,\mathbb{C})$ be a π-loxodromic element, $g \in \mathrm{PSL}(2,\mathbb{C})$ be a parabolic element, and let $\Gamma = \langle f, g\rangle$ be a non-elementary $\mathcal{RP}$ group without invariant plane. Then there exist unique elements $h_1, h_2 \in \mathrm{PSL}(2,\mathbb{C})$ such that*

(1) $h_1^2 = fg^{-1}f^{-1}g^{-1}$ *and* $(h_1 g)^2 = 1$,
(2) $h_2^2 = f^{-1}g^{-1}f^2gf^{-1}$ *and* $(h_2fg^{-1}f^{-1})^2 = 1$.

Moreover, the elements h_1 and h_2 are not strictly loxodromic.

Proof. First, note that $R_\sigma = e_f R_\tau e_f$ and $g = R_\tau R_\omega$. Therefore,

$$R_\sigma R_\omega = e_f R_\tau e_f R_\omega = e_f R_\tau R_\omega e_f = e_f g e_f = fg^{-1}f^{-1}. \tag{2.2}$$

Let us show that if we take $h_1 = R_\xi R_\tau = R_\sigma R_\xi$, then the assertion (1) of the lemma hold. Indeed,

$$h_1^2 = R_\sigma R_\tau = (R_\sigma R_\omega)(R_\omega R_\tau) = fg^{-1}f^{-1}g^{-1}.$$

Moreover, $h_1 g = (R_\xi R_\tau)(R_\tau R_\omega) = R_\xi R_\omega$. Since ξ and ω are orthogonal, $(R_\xi R_\omega)^2 = 1$. Hence, $(h_1 g)^2 = 1$. Note also that since h_1 is a product of two reflections, h_1 is not strictly loxodromic.

Now let us show that h_1 is unique. The element $fg^{-1}f^{-1}g^{-1}$ is uniquely determined as an element of $\mathrm{PSL}(2,\mathbb{C})$.

If $fg^{-1}f^{-1}g^{-1}$ is parabolic, it has only one square root h_1. Suppose that $fg^{-1}f^{-1}g^{-1}$ is hyperbolic. Then it has exactly two square roots, one of which is h_1 defined above and the other, denoted $\overline{h}_1$, is a π-loxodromic element with the same axis and translation length as h_1. Clearly, $(\overline{h}_1 g)^2 \neq 1$.

If $fg^{-1}f^{-1}g^{-1}$ is elliptic, then it also has two square roots h_1 and $\overline{h}_1$, both are elliptic elements. The element $\overline{h}_1$ is elliptic with the same axis as h_1 and with rotation angle $(\pi - 2\pi/m)$, while h_1 is a rotation through $2\pi/m$ in the opposite direction. Again, $(\overline{h}_1 g)^2 \neq 1$.

Now we take

$$h_2 = R_\eta R_\sigma = (R_\eta R_\tau)(R_\tau R_\sigma) = e_g h_1^{-2} = efgf^{-1}.$$

Then

$$h_2^2 = f^{-1}g^{-1}f^2gf^{-1} \quad \text{and} \quad (fg^{-1}f^{-1}h_2)^2 = 1.$$

These two conditions determine h_2 uniquely. ☐

Note that the elements h_1, h_2 defined in Lemma 2.2 determine combinatorial and metric structures of $\mathcal{P}^*$. For example, if h_1 is elliptic, then its rotation angle is equal to the dihedral angle of $\mathcal{P}^*$ between σ and τ. If h_2 is elliptic, then its rotation angle is equal to the doubled dihedral angle of $\mathcal{P}^*$ between η and σ. Vice versa, if the metric structure of $\mathcal{P}^*$ is fixed, then the types of elements h_1 and h_2 can be determined.

The same can be said about $\mathcal{Q}^*$ and the elements h_1 and h_2h_1. The element h_2h_1 is responsible for the mutual position of the planes η and ξ (see the proof of Lemma 2.5).

Lemmas 2.3–2.5 below give some necessary conditions for discreteness of Γ via conditions on elements h_1 and h_2. One needs to keep in mind the connection between these elements and the polyhedra $\mathcal{P}^*$ and $\mathcal{Q}^*$.

Lemma 2.3. *If Γ is discrete, then h_1 is either a hyperbolic, or parabolic, or primitive elliptic element of order $m \geq 3$.*

Proof. The subgroup $H = \langle g, fgf^{-1}\rangle$ of Γ keeps δ invariant and is conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. Since Γ is discrete, H must be discrete. By [15] or [2], the group H is discrete if and only if either

(1) $fg^{-1}f^{-1}g^{-1} = h_1^2$ is a hyperbolic, or a parabolic, or a primitive elliptic element, or

(2) h_1 is a primitive elliptic element of odd order m, where $m \geq 3$.

If h_1^2 is parabolic of hyperbolic, then h_1 is parabolic or hyperbolic, respectively. If h_1^2 is a primitive elliptic element, then h_1 is a primitive elliptic of even order $m \geq 4$. □

Lemma 2.4. *If Γ is discrete, then h_2 is either a hyperbolic, or parabolic, or primitive elliptic element of order $p \geq 3$.*

Proof. Let κ be the plane orthogonal to η, σ, and ω. The subgroup $H = \langle e, fgf^{-1}\rangle$ of $\widetilde{\Gamma}$ keeps the plane κ invariant and is conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. By [15], H is discrete if and only if $h_2 = efgf^{-1}$ is either a hyperbolic, or parabolic, or primitive elliptic element of order $p \geq 3$. □

Lemma 2.5. *If Γ is discrete and h_1 is a primitive elliptic element of odd order, then h_2h_1 is either a hyperbolic, or parabolic, or primitive elliptic element of order $k \geq 3$.*

Proof. Recall that $\Gamma^* = \langle e_f, R_\eta, R_\tau, R_\omega\rangle$. Since h_1 has odd order and $h_1^2 \in \Gamma^*$, $h_1 \in \Gamma^*$. Since, moreover, $h_1 = R_\xi R_\tau$, $e_f = R_\delta R_\xi$, and $R_\tau \in \Gamma^*$, both R_ξ and R_δ are also in Γ^*. Further, since the plane ξ is orthogonal to ω, the group $\langle R_\eta R_\delta, e_f\rangle$ keeps ω invariant and is conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. It is clear that $\langle R_\eta R_\delta, e_f\rangle$ is discrete if and only if $R_\eta R_\xi = h_2h_1$ is a hyperbolic, parabolic, or primitive elliptic element of order $k \geq 3$ [15]. □

Proof of Theorem 2.1. Lemma 2.2 proves existence and uniqueness of elements h_1 and h_2. Now we prove part (2) of the theorem.

If Γ is discrete then h_1 is either a hyperbolic, or parabolic, or primitive elliptic element of order $m \geq 3$ by Lemma 2.3. We split the discrete groups Γ into two families. The first family consists of those groups for which h_1 is hyperbolic, parabolic, or primitive elliptic of even order. By Lemma 2.4, for these groups h_2 is a hyperbolic, parabolic, or primitive elliptic element.

The second family consists of the discrete groups with h_1 elliptic of odd order. Then by Lemma 2.5, h_2h_1 is a hyperbolic, or parabolic, or primitive elliptic element of order $k \geq 3$. (Note that in this case h_2 is necessarily hyperbolic or primitive elliptic.)

So if Γ is discrete, then either (2)(i) or (2)(ii) of Theorem 2.1 can occur. Clearly, if neither (2)(i) nor (2)(ii) holds, then Γ is not discrete by Lemmas 2.3–2.5.

Now prove that each of (2)(i) and (2)(ii) is a sufficient condition for Γ to be discrete. In each of the two cases we will give a fundamental polyhedron for Γ^* to show, by using the Poincaré polyhedron theorem [3], that Γ^* is discrete.

Suppose that (2)(i) holds. Then since m is even, the group G_1 generated by the side pairing transformations R_η, R_ω, R_σ, R_τ, and e_f and the polyhedron $\mathcal{P}^*$ satisfy the Poincaré polyhedron theorem, G_1 is discrete and $\mathcal{P}^*$ is its fundamental polyhedron. Obviously, $G_1 = \Gamma^*$.

Suppose that (2)(ii) holds. Then the group G_2 generated by the side pairing transformations R_η, R_ω, R_ξ, R_τ, and R_δ and the polyhedron $\mathcal{Q}^*$ satisfy the Poincaré theorem, G_2 is discrete, and $\mathcal{Q}^*$ is its fundamental polyhedron.

In the proof of Lemma 2.5 it was shown that, for m odd, $R_\xi \in \Gamma^*$ and $R_\delta \in \Gamma^*$. Moreover, $e_f = R_\xi R_\delta$. Hence, $G_2 = \Gamma^*$, so Γ^* is discrete.

Theorem 2.1 is proved. □

Our next goal is to compute parameters $(\beta(f), \beta(g), \gamma(f,g))$ for both series of discrete groups listed in Theorem 2.1.

If $f \in \mathrm{PSL}(2,\mathbb{C})$ is a loxodromic element with translation length d_f and rotation angle θ_f, then

$$\mathrm{tr}^2 f = 4\cosh^2 \frac{d_f + i\theta_f}{2}$$

and $\lambda_f = d_f + i\theta_f$ is called the *complex translation length* of f.

Note that if f is hyperbolic then $\theta_f = 0$ and $\mathrm{tr}^2 f = 4\cosh^2(d_f/2)$. If f is elliptic then $d_f = 0$ and $\mathrm{tr}^2 f = 4\cos^2(\theta_f/2)$. If f is parabolic then $\mathrm{tr}^2 f = 4$; by convention we set $d_f = \theta_f = 0$.

We define the set

$$\mathcal{U} = \{u : u = i\pi/p \text{ for some } p \in \mathbb{Z}, p \geq 2\} \cup [0, +\infty).$$

In other words, the set $\mathcal{U}$ consists of all complex translation half-lengths $u = \lambda_f/2$ for hyperbolic, parabolic, and primitive elliptic elements f. Furthermore, we define a function $t : \mathcal{U} \to \{2, 3, 4, \dots\} \cup \{\infty, \overline{\infty}\}$ as follows:

$$t(u) = \begin{cases} p & \text{if} \quad u = i\pi/p, \\ \infty & \text{if} \quad u = 0, \\ \overline{\infty} & \text{if} \quad u \in (0, +\infty). \end{cases}$$

Given $u \in \mathcal{U}$ and f with $\mathrm{tr}^2 f = 4\cosh^2 u$, $t(u)$ determines the type of f and, moreover, its order if f is elliptic. Note also that since we regard $\infty/n = \infty$ and $\overline{\infty}/n = \overline{\infty}$, an expression of the form $(t(u), n) = 1$ with $n > 1$ means, in particular, that $t(u)$ is finite.

Theorem 2.6. *Let $f, g \in \mathrm{PSL}(2,\mathbb{C})$ with $\beta(f) < -4$, $\beta(g) = 0$, and $\gamma(f,g) > 0$. Then $\Gamma = \langle f, g\rangle$ is discrete if and only if one of the following holds:*

(1) *$\gamma(f,g) = 4\cosh^2 u$ and $\beta(f) = -4\cosh^2 v/\gamma(f,g) - 4$, where $u, v \in \mathcal{U}$ with $t(u) \geq 4$, $(t(u), 2) = 2$, and $t(v) \geq 3$;*

(2) *$\gamma(f,g) = 4\cosh^2 u$ and $\beta(f) = -4\cosh^2 v - 4$, where $u, v \in \mathcal{U}$ with $t(u) \geq 3$, $(t(u), 2) = 1$, and $t(v) \geq 3$.*

Proof. Obviously, $\beta(f) < -4$ and $\beta(g) = 0$ if and only if f is π-loxodromic and g is parabolic. With this choice of $\beta(f)$ and $\beta(g)$, $\gamma(f,g) > 0$ if and only if the group $\Gamma = \langle f, g\rangle$ is a non-elementary $\mathcal{RP}$ group without invariant plane [9]. This means that the hypotheses of Theorem 2.6 are equivalent to the hypotheses of Theorem 2.1. Therefore, in order to prove Theorem 2.6 it is sufficient to calculate the parameters $\beta(f)$ and $\gamma(f,g)$ for both families of the discrete groups listed in Theorem 2.1.

Let σ' be the image of σ under R_ω, that is $R_{\sigma'} = R_\omega R_\sigma R_\omega$. Using the identity (2.2) and the fact that $g = R_\tau R_\omega$, we have

$$[f, g] = fgf^{-1}g^{-1} = (R_\omega R_\sigma)(R_\omega R_\tau) = (R_{\sigma'} R_\omega)(R_\omega R_\tau) = R_{\sigma'} R_\tau.$$

Note that σ' and τ are disjoint and δ is orthogonal to both of them. Therefore, $[f,g]$ is a hyperbolic element with the axis lying in δ and the translation length $2d$, where d is the distance between σ' and τ. Hence, since $\gamma(f,g)>0$,

$$\gamma(f,g)=\mathrm{tr}[f,g]-2=+2\cosh d-2.$$

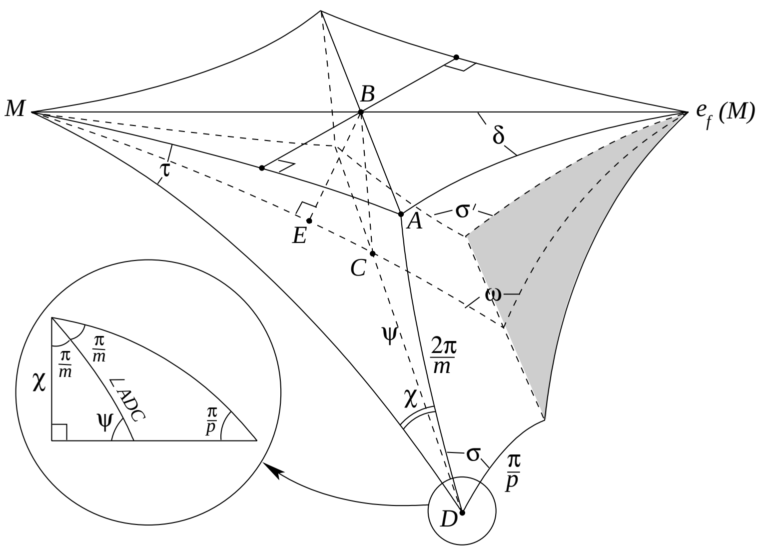


FIGURE 3

Now, using generalised triangles in the plane δ, it is not difficult to calculate that

$$\gamma(f,g)=\begin{cases}4\cos^2(\pi/m) & \text{if}\quad 3\le m<\infty,\\ 4 & \text{if}\quad m=\infty,\\ 4\cosh^2(d(\sigma,\tau)/2) & \text{if}\quad m=\overline{\infty},\end{cases}$$

where $d(\sigma,\tau)$ is the distance between σ and τ if they are disjoint. Hence,

$$\gamma(f,g)=4\cosh^2 u,$$

where $u\in\mathcal{U}$, $t(u)=m\ge 3$.

Let us calculate $\beta(f)$. The element f is π-loxodromic if and only if $\mathrm{tr}^2 f= 4\cosh^2(T+i\pi/2)=-4\sinh^2 T$, where $2T$ is the translation length of f. That is,

$$\beta(f)=-4\sinh^2 T-4.$$

Note that T is the distance between e and e_f. It is measured in ω and equals BE (see Figure 3).

Suppose that we are in case (2)(i) of Theorem 2.1, that is $(t(u),2)=2$, and that σ and τ intersect. Recall that ξ is the bisector of the dihedral angle of $\mathcal{P}^*$ made by σ and τ. Let ψ be the angle that ξ makes with η. Note that $\psi=\angle BCE$. From the link of D, we have that

$$\cos\chi=\frac{\cos(\pi/p)}{\sin(2\pi/m)}=\frac{\cos\psi}{\sin(\pi/m)}$$

and, therefore,

$$\cos\psi = \frac{\cos(\pi/p)}{2\cos(\pi/m)}. \tag{2.3}$$

Further, from the link of D,

$$\cos\angle ADC = \frac{\cos\psi\cdot\cos(\pi/m)}{\sin\psi\cdot\sin(\pi/m)}. \tag{2.4}$$

From the $\triangle ABM$, $\cosh^2 AB = 1/\sin(\pi/m)$ and, from the quadrilateral $ABCD$,

$$\sinh BC = \frac{\cos\angle ADC}{\sinh AB} \tag{2.5}$$

Finally, from $\triangle BCE$,

$$\sinh T = \sinh BE = \sin\psi\cdot\sinh BC. \tag{2.6}$$

Combining (2.3)–(2.6), we have that

$$\sinh^2 T = \frac{\cos^2(\pi/p)}{4\cos^2(\pi/m)} = \frac{\cos^2(\pi/p)}{\gamma(f,g)}.$$

Similar calculations can be done for parallel or disjoint σ and τ. Hence, $\beta(f) = -\sinh^2 T - 4 = -\cosh^2 v/\gamma(f,g) - 4$, where $v \in \mathcal{U}$, $t(v) \geq 3$.

Now note that in case (2)(ii) of Theorem 2.1, the angle $\psi = \angle BCE$ must be of the form π/k, $k \geq 3$ is an integer, ∞, or $\overline{\infty}$. Then we need to recompute the formulas (2.4)–(2.6) with $\psi = \pi/k$:

$$\cos\angle ADC = \frac{\cos(\pi/k)\cdot\cos(\pi/m)}{\sin(\pi/k)\cdot\sin(\pi/m)}, \quad \sinh BC = \frac{\cos\phi}{\sinh a} = \frac{\cos(\pi/k)}{\sin(\pi/k)}.$$

Then

$$\sinh T = \sin\psi\cdot\sinh BC = \cos(\pi/k).$$

Hence, $\beta(f) = -4\cosh^2 v - 4$, where $v \in \mathcal{U}$, $t(v) \geq 3$. $\square$

3. Orbifolds

Denote by $\Omega(\Gamma)$ the discontinuity set of a Kleinian group Γ. The *Kleinian orbifold* $Q(\Gamma) = (\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$ is said to be an orientable 3-orbifold with a complete hyperbolic structure on its interior $\mathbb{H}^3/\Gamma$ and a conformal structure on its boundary $\Omega(\Gamma)/\Gamma$.

We need the following (Kleinian) group presentations:

- $PH[\infty,m;q] = \langle x,y,s \mid x^\infty = s^2 = (xs)^2 = (ys)^2 = (xyxy^{-1})^m = (y^{-1}xys)^q = 1\rangle$,
- $P[\infty,m;q] = \langle w,x,y,z \mid w^\infty = x^2 = y^2 = z^2 = (wx)^2 = (wy)^2 = (yz)^2 = (zx)^q = (zw)^m = 1\rangle$,
- $\mathcal{S}_2[\infty,m;q] = \langle x,L \mid x^\infty = (xLxL^{-1})^m = (xL^2x^{-1}L^{-2})^q = 1\rangle$,
- $GTet_1[\infty,m;q] = \langle x,y,z \mid x^\infty = y^2 = z^\infty = (xy)^m = (yzy^{-1}z^{-1})^q = [x,z] = 1\rangle$.

Here m and q are integers greater than 1, or ∞ or $\overline{\infty}$ with the following convention. If we have a relation of the form $w^n = 1$ with $n = \overline{\infty}$, then we simply remove the relation $w^n = 1$ from the presentation (in fact, this means that the element w is hyperbolic). Further, if $n = \infty$ and we keep the relation $w^n = 1 \sim w^\infty = 1$, we get a Kleinian group presentation where parabolics are indicated. To get an abstract group presentation, we need to remove all relations of the form $w^\infty = 1$.

Theorem 3.1. *Let $\Gamma = \langle f, g\rangle$ be a non-elementary discrete $\mathcal{RP}$ group without invariant plane. Let $\beta(f) \in (-\infty, -4)$ and let $\beta(g) = 0$. Then $\gamma(f, g) = 4\cosh^2 u$, where $u \in \mathcal{U}$, $t(u) \geq 3$, and one of the following holds:*

(1) *If $(t(u), 2) = 2$ and $\beta(f) = -4\cosh^2 v/\gamma(f, g) - 4$, where $v \in \mathcal{U}$, $t(v) \geq 3$, $(t(v), 2) = 1$, then Γ is isomorphic to $PH[\infty, t(u)/2; t(v)]$.*
(2) *If $(t(u), 2) = 2$ and $\beta(f) = -4\cosh^2 v/\gamma(f, g) - 4$, where $v \in \mathcal{U}$, $t(v) \geq 4$, $(t(v), 2) = 2$, then Γ is isomorphic to $\mathcal{S}_2[\infty, t(u)/2; t(v)/2]$.*
(3) *If $(t(u), 2) = 1$ and $\beta(f) = -4\cosh^2 v - 4$, where $v \in \mathcal{U}$, $t(v) \geq 3$, $(t(v), 2) = 1$, then Γ is isomorphic to $P[\infty, t(u); t(v)]$.*
(4) *If $(t(u), 2) = 1$ and $\beta(f) = -4\cosh^2 v - 4$, where $v \in \mathcal{U}$, $t(v) \geq 4$, $(t(v), 2) = 2$, then Γ is isomorphic to $GTet_1[\infty, t(u); t(v)/2]$.*

Proof. Suppose $(t(u), 2) = 2$, that is the dihedral angle of $\mathcal{P}^*$ between σ and τ is $2\pi/m$ with m even, ∞, or $\overline{\infty}$. Consider a polyhedron $\widetilde{\mathcal{P}}$ bounded by σ, τ, $\sigma' = R_\omega(\sigma)$, $\tau' = R_\omega(\tau)$, η, and δ. Applying the Poincaré theorem to $\widetilde{\mathcal{P}}$ and the side pairing transformations g, $g' = R_\sigma R_\omega$, e, and e_f, one can see that $\langle g, g', e_f, e\rangle$ is isomorphic to $\widetilde{\Gamma}$ and has the presentation

$$\langle f, g, e \,|\, g^\infty = e^2 = (ef)^2 = (eg)^2 = (gfgf^{-1})^{m/2} = (f^{-1}gfe)^p = 1\rangle.$$

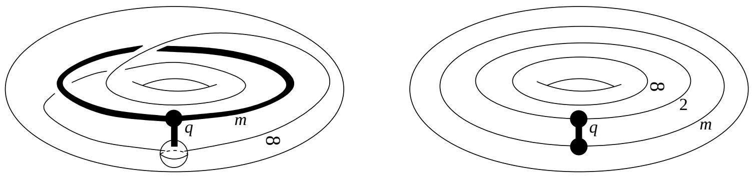
If p is odd, then $e \in \langle f, g\rangle$ and $\widetilde{\Gamma} = \Gamma \cong PH[\infty, m/2; p]$.

If p is even, ∞, or $\overline{\infty}$, then $\widetilde{\Gamma}$ contains Γ as a subgroup of index 2 and has presentation $\mathcal{S}_2[\infty, m/2; p/2]$. In order to see this, one can apply the Poincaré theorem to a polyhedron $\mathcal{P}$ bounded by τ, σ, τ', σ', η, and $e_f(\eta)$, and side-pairing transformations f, g, and $g' = fg^{-1}f^{-1}$.

The proof for $(t(u), 2) = 1$ is analogous. In this case we need to use the polyhedron $\mathcal{Q}^*$ as the starting point. □

(a) $\pi_1^{orb}(Q) \cong PH[\infty, m; q]$ $m \geq 2$, $q \geq 3$ (b) $\pi_1(Q)^{orb} \cong P[\infty, m; q]$ $m \geq 3$, $q \geq 3$

FIGURE 4. Orbifolds embedded in $\mathbb{S}^3$



(a) $\pi_1^{orb}(Q) \cong \mathcal{S}_2[\infty, m; q]$ $m \geq 2$, $q \geq 2$ (b) $\pi_1^{orb}(Q) \cong GTet_1[\infty, m; q]$ $m \geq 3$, $q \geq 2$

FIGURE 5. Orbifolds embedded in Seifert fibred spaces

The orbifolds $Q(\Gamma)$ for the groups described in Theorem 3.1 can be obtained from corresponding fundamental polyhedra. In Figures 4 and 5, we schematically draw singular sets, cusps, and boundary components of $Q(\Gamma)$ by using fat vertices and fat edges. Roughly speaking, a fat vertex is either an interior point, or is removed, or removed together with its regular neighbourhood depending on the indices. A fat edge can be labelled by ∞ or $\overline{\infty}$. If the index at a fat edge is ∞, then the egde corresponds to a cusp, and if the index is $\overline{\infty}$, the edge is removed together with its regular neighbourhood. For details, see [12].

In Figure 4, orbifolds are embedded in $\mathbb{S}^3$ so that ∞ is a non-singular interior point of $Q(\Gamma)$. Note that the volume of $Q(PH[\infty,m;q])$ is always infinite and $Q(P[\infty,m;q])$ is always non-compact.

Let $T(n)$ be a Seifert fibred solid torus obtained from a trivial fibred solid torus $D^2\times\mathbb{S}^1$ by cutting it along $D^2\times\{x\}$ for some $x\in\mathbb{S}^1$, rotating one of the discs through $2\pi/n$ and glueing back together.

Denote by $\mathcal{S}(n)$ a space obtained by glueing two copies of $T(n)$ along their boundaries fibre to fibre. Clearly, $\mathcal{S}(n)$ is homeomorphic to $\mathbb{S}^2\times\mathbb{S}^1$ and is n-fold covered by trivially fibred $\mathbb{S}^2\times\mathbb{S}^1$. There are two critical fibres whose length is n times shorter than the length of a regular fibre.

In Figure 5(a), orbifolds are embedded in Seifert fibre spaces $\mathcal{S}(2)=T(2)\cup T(2)$. We draw only the solid torus that contains singular points (or boundary components). The other fibred torus is meant to be attached and is not shown. If $m<\infty$, the orbifold $Q(\mathcal{S}_2[\infty,m;q])$ is embedded in $\mathcal{S}(2)$ in such a manner that the axis of order m lies on a critical fibre of $\mathcal{S}(2)$. The removed regular fibre gives rise to a cusp.

In Figure 5(b), orbifolds are embedded in trivially fibred space $\mathbb{S}^2\times\mathbb{S}^1$. The rank 2 cusp corresponds to the subgroup of $GTet_1[\infty,m;q]$ generated by x and z.

4. Structure of the slice S_∞

Recall that

$$S_\infty=\{(\gamma,\beta):(\beta,0,\gamma)\text{ are parameters for some }\langle f,g\rangle\in\mathcal{DRP}\},$$

where $\mathcal{DRP}$ denotes the class of all non-elementary discrete $\mathcal{RP}$ groups.

To investigate the slice S_∞, we split the plane (γ,β) as follows.

1. If $\beta=-4$ then by [9, Theorem 2], the group $\langle f,g\rangle$ has an invariant plane. We use [5] to find all discrete groups on the line $\beta=-4$.
2. If $\beta>-4$ and $\gamma>0$ then the group $\langle f,g\rangle$ is conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. More precisely, if $-4<\beta<0$ then f is elliptic and the axis of f is orthogonal to an invariant plane of g and if $\beta=0$ then the fixed points of f and g lie in their common invariant plane. Discreteness criteria in terms of traces of f, g, and fg were given in [14]. For $\beta>0$, an algorithm to decide whether f and g generate a discrete group was given in [8].
3. If $\beta>-4$ and $\gamma<0$ then f is elliptic, parabolic, or hyperbolic and the group $\langle f,g\rangle$ is known to be truly spatial. Discrete such groups are described in [11], where β and γ are found explicitly.
4. If $\beta<-4$ and $\gamma<0$ then f is π-loxodromic whose axes lies in an invariant plane of g. Then this plane is invariant under action of $\langle f,g\rangle$ and f acts as

a glide-reflection on it. A geometrical description of such discrete groups was given in [13].

5. The case of $\beta < -4$ and $\gamma > 0$ was treated in Section 2 of the present paper.

We will obtain explicit formulas for β and γ in the cases 2 and 4 above and completely describe the structure of the slice S_∞. We will pay special attention to the subsets of S_∞ corresponding to free groups.

First, we need the following elementary facts.

Lemma 4.1. *If $f, g \in \mathrm{PSL}(2,\mathbb{C})$ and g is parabolic, then*

$$\gamma(f,g) = (\mathrm{tr}(fg) - \mathrm{sign}(\mathrm{tr}g) \cdot \mathrm{tr}f)^2.$$

Proof. By the Fricke identity, we have

$$\begin{aligned}\gamma(f,g) &= \mathrm{tr}[f,g] - 2\\ &= \mathrm{tr}^2 f + \mathrm{tr}^2 g + \mathrm{tr}^2(fg) - \mathrm{tr}f \cdot \mathrm{tr}g \cdot \mathrm{tr}(fg) - 4\\ &= (\mathrm{tr}(fg) - \mathrm{sign}(\mathrm{tr}g) \cdot \mathrm{tr}f)^2,\end{aligned}$$

since $\mathrm{tr}^2 g = 4$. □

Lemma 4.2. *If $f, g \in \mathrm{PSL}(2,\mathbb{C})$ and $\mathrm{tr}g = 2$, then*

$$\mathrm{tr}(fg^k) = k(\mathrm{tr}(fg) - \mathrm{tr}f) + \mathrm{tr}f.$$

Proof. By substituting $\mathrm{tr}g = 2$ into the recurrent formula

$$\mathrm{tr}(fg^k) = \mathrm{tr}(fg^{k-1})\mathrm{tr}g - \mathrm{tr}(fg^{k-2}),$$

we immediately get the result. □

Remark 4.3. *Suppose that f is non-primitive elliptic of finite order n, i.e., $\beta(f) = -4\sin^2(q\pi/n)$, where $(q,n) = 1$, $1 < q < n/2$. Then there exists an integer r so that f^r is primitive of the same order. Obviously, $\langle f, g\rangle = \langle f^r, g\rangle$ and $\beta(f^r) = -4\sin^2(\pi/n)$. By* [7], *$\gamma(f^r, g) = (\beta(f^r)/\beta(f))\gamma(f,g)$.*

It is natural to introduce the constant

$$C(q,n) = \frac{\sin^2(q\pi/n)}{\sin^2(\pi/n)} = \frac{\beta(f)}{\beta(f^r)} \geq 1$$

that plays an important role in parameters calculation concerning groups with elliptic elements. It is also convenient to consider a parabolic element f as a limit rotation of order $n = \infty$ and write $0 = \beta(f) = -4\sin^2(\pi/n)$ with $C(q,n) = C(1,n) = 1$.

4.1. $-4 \leq \beta \leq 0$**.** This means that f is either elliptic or parabolic. Obviously, if f is elliptic of infinite order, then $\langle f, g\rangle$ is not discrete. So we assume that $\beta = -4\sin^2(q\pi/n)$, where $(q,n) = 1$ and $1 \leq q < n/2$, including $\beta = 0$.

Theorem 4.4. *Let $\Gamma = \langle f, g\rangle \subset \mathrm{PSL}(2,\mathbb{C})$ have parameters $(\beta, 0, \gamma)$ with $\gamma \in \mathbb{R}\backslash\{0\}$. Let $\beta = -4\sin^2(q\pi/n)$, where $(q,n) = 1$ and $1 \leq q < n/2$, including $\beta = 0$. Then Γ is discrete if and only if one of the following holds:*

(1) *$\gamma = -4C(q,n)\cosh^2 u$, where $u \in \mathcal{U}$ and $t(u) \geq 3$;*
(2) *$\gamma = 4C(q,n)(\cos(\pi/n) + \cosh u)^2$, where $u \in \mathcal{U}$;*
(3) *$\beta = 0$ and $\gamma = 4(1 + \cos(2\pi/k))^2$, where $k \geq 3$ is odd.*

Proof. Let us prove the theorem for $q = 1$; in order to get the result for $q > 1$, we only need to apply Remark 4.3.

If $n = 2$ then $\beta = -4$ and, by [5, Theorem 4.15], Γ is discrete if and only if $\gamma = \pm 4\cosh^2 u$, where $u \in \mathcal{U}$ with $t(u) \geq 3$.

If $2 < n \leq \infty$ and $\gamma < 0$, then, by [11, Corollary 2.5], Γ is discrete if and only if $\gamma = -4\cosh^2 u$, where $u \in \mathcal{U}$ and $t(u) \geq 3$.

Assume that $2 < n < \infty$ and $\gamma > 0$. In this case Γ is conjugate to a subgroup of $\mathrm{PSL}(2, \mathbb{R})$ and we can apply Knapp's results [14] to compute γ. Conjugate Γ so that ∞ is the fixed point of g. By replacing, if necessary, f with f^{-1} and g with g^{-1}, we may assume that

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} -1 & \tau \\ 0 & -1 \end{pmatrix},$$

where $ad - bc = 1$, $a + d = -2\cos(\pi/n)$ with $n \in \mathbb{Z}$, $b > 0$, and $\tau > 0$.

One can show that $\mathrm{tr}(fg) < 2$. By [14, Proposition 4.1], Γ is discrete if and only if $\mathrm{tr}(fg) \leq -2$ or $\mathrm{tr}(fg) = -2\cos(\pi/k)$, where $k \geq 2$ is an integer, that is $\mathrm{tr}(fg) = -2\cosh u$, where $u \in \mathcal{U}$. Hence, by Lemma 4.1, $\gamma = (\mathrm{tr}(fg) + \mathrm{tr} f)^2 = (2\cosh u + 2\cos(\pi/n))^2$.

So it remains to consider the case when $n = \infty$ (i.e., $\beta = 0$) and $\gamma > 0$. Again, we normalize Γ so that g is as above and $f = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$. By [14, Proposition 4.2], such a group is discrete if and only if $\tau \geq 4$ or $\tau = 2 + 2\cos(2\pi/k)$ for an integer $k \geq 3$. Since in this case $\gamma = \tau^2$, we have that $\gamma \geq 16$ or $\gamma = (2 + 2\cos(2\pi/k))^2$, which can be written as $\gamma = 4(1 + \cosh u)^2$, where $u \in \mathcal{U}$, or $\gamma = 4(1 + \cos(2\pi/k))^2$ for odd $k \geq 3$. □

Remark 4.5. *If $-4 \leq \beta \leq 0$ then Γ is discrete and free if and only if $\beta = 0$ and $\gamma \in (-\infty, -4] \cup [16, +\infty)$.*

The parameters from the infinite strip $-4 \leq \beta \leq 0$ are displayed in Figure 6. If $\beta = -4\sin^2(q\pi/n)$ is fixed, then there exist values $\gamma_1(\beta) < 0$ and $\gamma_2(\beta) > 0$ so that Γ is discrete in the union of two rays $(-\infty, \gamma_1(\beta)] \cup [\gamma_2(\beta), +\infty)$. There are only countably many discrete groups in $(\gamma_1(\beta), \gamma_2(\beta))$ with accumulation points $\gamma_1(\beta)$ and $\gamma_2(\beta)$.

Moreover, if we denote $\beta_n^q = -4\sin^2(q\pi/n)$, then

$$\gamma_1(\beta_n^q) < \gamma_1(\beta_n^1) < \gamma_2(\beta_n^1) < \gamma_2(\beta_n^q) \quad \text{for all } 1 < q < n/2.$$

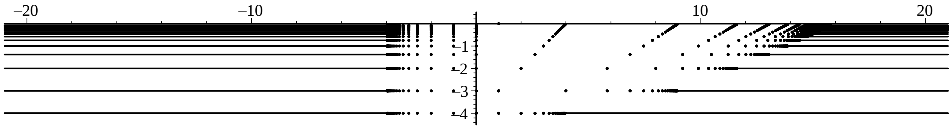
4.2. $\beta > 0$**.** In this case f is hyperbolic.

Theorem 4.6 ([11, Corollary 2.5])**.** *Let $\Gamma = \langle f, g \rangle \subset \mathrm{PSL}(2, \mathbb{C})$ have parameters $(\beta, 0, \gamma)$ with $\beta > 0$ and $\gamma < 0$. Then Γ is discrete if and only if $\gamma = -4\cosh^2 u$, where $u \in \mathcal{U}$, $t(u) \geq 3$.*

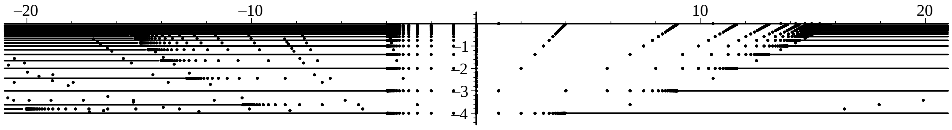
Remark 4.7. *From* [11]*, Γ with parameters $(\beta, 0, \gamma)$, where $\beta \geq 0$ and $\gamma < 0$ is free if and only if (γ, β) lies in the region*

$$A = \{(\gamma, \beta) : \gamma \leq -4, \beta \geq 0\}.$$

Theorem 4.8. *Let $\Gamma = \langle f, g \rangle \subset \mathrm{PSL}(2, \mathbb{C})$ have parameters $(\beta, 0, \gamma)$ with $\beta > 0$ and $\gamma > 0$. Let $k = \left\lceil \dfrac{\sqrt{\beta + 4} - 2}{\sqrt{\gamma}} \right\rceil$. The group Γ is discrete if and only if one of the following holds:*



$\beta = -4\sin^2(\pi/n)$ with $n \in \mathbb{Z}$



$\beta = -4\sin^2(q\pi/n)$ with $(q,n)=1$ and $1 \le q < n/2$

FIGURE 6. Structure of the strip $-4 \le \beta \le 0$

(1) *$\beta = (k\sqrt{\gamma}+2)^2-4$ and $\gamma = 16\cosh^4 u$, where $u \in \mathcal{U}$ and $t(u) \ge 3$;*
(2) *$\beta = (k\sqrt{\gamma} \pm 2\cos(q\pi/n))^2 - 4$ and $\gamma = 4C(q,n)(\cos(\pi/n)+\cosh u)^2$, where $(q,n)=1$, $1 \le q < n/2$, and $u \in \mathcal{U}$;*
(3) *$\beta = (k\sqrt{\gamma} - 2\cosh u)^2 - 4$ and $\gamma > 4(1+\cosh u)^2$, where $u \ge 0$.*

Proof. Since $\gamma > 0$, the axis of f lies in an invariant plane of g, so $\Gamma = \langle f, g\rangle$ is conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. In [8], an algorithm for determining whether such a group is discrete was given. We will apply this algorithm and calculate parameters for each discrete group.

Normalize Γ so that ∞ is the fixed point of g and ± 1 are the fixed points of f. Then we can write

$$f = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}, \text{ where } a^2 - b^2 = 1,\ a > 1,\ b, \tau \in \mathbb{R}.$$

By replacing f with f^{-1} and g with g^{-1}, we may assume that $b < 0$ and $\tau > 0$.

Let k be a positive integer such that $\mathrm{tr}(fg^k) \le 2$ and $\mathrm{tr}(fg^\ell) > 2$ for all ℓ with $0 \le \ell < k$.

By Lemmas 4.1 and 4.2, we have that $k^2\gamma = k^2(\mathrm{tr}(fg) - \mathrm{tr}f)^2 = (\mathrm{tr}(fg^k) - \mathrm{tr}f)^2$. Since $\mathrm{tr}(fg^k) \le 2$ and $\mathrm{tr}f > 2$,

$$\mathrm{tr}f = k\sqrt{\gamma} + \mathrm{tr}(fg^k). \tag{4.7}$$

We distinguish three cases:

1. $\mathrm{tr}(fg^k) = 2$, that is fg^k is parabolic. From (4.7),

$$\beta = (k\sqrt{\gamma}+2)^2 - 4.$$

By Theorem 4.4, $\langle fg^k, g\rangle$ and, hence, $\langle f, g\rangle$ is discrete if and only if

$\gamma = \gamma(fg^k, g) = 4(1+\cosh v)^2$, where $v \in \mathcal{U}$, or
$\gamma = 4(1+\cos(2\pi/k))^2$, where $k \ge 3$ is odd.

These expressions can be rearranged and combined as $\gamma = 16\cosh^4 u$, where $u \in \mathcal{U}$ and $t(u) \ge 3$.

2. $-2 < \mathrm{tr}(fg^k) < 2$, that is fg^k is elliptic and $\mathrm{tr}(fg^k) = \pm 2\cos(q\pi/n)$, where $(q,n) = 1$ and $1 \le q < n/2$. Hence, from (4.7),

$$\beta = (k\sqrt{\gamma} \pm 2\cos(q\pi/n))^2 - 4.$$

By Theorem 4.4, $\langle fg^k, g\rangle$ and, hence, $\langle f, g\rangle$ is discrete if and only if

$$\gamma = 4C(q,n)(\cos(\pi/n)+\cosh u)^2, \quad \text{where } u \in \mathcal{U}.$$

3. $\mathrm{tr}(fg^k) \le -2$, that is fg^k is hyperbolic or parabolic so we can write $\mathrm{tr}(fg^k) = -2\cosh u$, where $u \ge 0$. Then

$$\beta = (k\sqrt{\gamma} - 2\cosh u)^2 - 4.$$

Consider the group $\langle g^{k-1}f, g\rangle$. The element $g^{k-1}f$ is hyperbolic with $\mathrm{tr}(g^{k-1}f) > 2$. Therefore, one can normalize $\langle g^{k-1}f, g\rangle$ so that the attracting and repelling fixed points of $g^{k-1}f$ are x_a and x_r, respectively, and $x_a < x_r$. Since $\mathrm{tr}(g^k f) \le -2$, such a group is dicrete and free by [8, Case II]. So by Lemma 4.1, we have that

$$\begin{aligned}\gamma = \gamma(fg^{k-1}, g) &= (\mathrm{tr}(fg^k) - \mathrm{tr}(fg^{k-1}))^2\\ &= (2\cosh u + 2\cosh v)^2,\end{aligned}$$

where v is any positive real number.

It remains to compute k. Since $\mathrm{tr}(fg^k) = 2a + b\tau k \le 2$, we have that $k \ge (-2a+2)/(b\tau)$. Computing $\gamma = b^2\tau^2$, we get $b\tau = -\sqrt{\gamma}$. So $k = \left\lceil \dfrac{\sqrt{\beta+4}-2}{\sqrt{\gamma}} \right\rceil$. □

It follows from [8] that Γ is free if and only if (γ, β) lies in one of the regions

$$C_k = \{(\gamma,\beta) : \gamma \ge 16, ((k-1)\sqrt{\gamma}+2)^2 \le \beta + 4 \le (k\sqrt{\gamma}-2)^2\},\ k = 1,2,3\dots$$

4.3. $\beta < -4$. First, consider $\gamma < 0$. In this case the axis of the π-loxodromic generator f lies in an invariant plane of g [9], so $\langle f, g\rangle$ keeps this plane invariant.

Theorem 4.9. *Let* $\Gamma = \langle f, g\rangle \subset \mathrm{PSL}(2,\mathbb{C})$ *have parameters* $(\beta, 0, \gamma)$ *with* $\beta < -4$ *and* $\gamma < 0$. *Let* $k = \left\lceil \dfrac{\sqrt{-\beta-4}}{\sqrt{-\gamma}} \right\rceil$. *Then the group* $\langle f, g\rangle$ *is discrete if and only if one of the following holds:*

(1) $-4(\beta+4) = \big((2k-1)\sqrt{-\gamma} \pm \sqrt{-\gamma - 8(1+\cosh u)}\big)^2$, *where* $u \in \mathcal{U}$;
(2) $4(\beta+4) = (2k-1)^2\gamma$ *and* $\gamma = -16\cos^2(\pi/p)$, *where* $p \ge 3$ *is odd;*
(3) $\beta = k^2\gamma - 4$ *and* $\gamma = -4\cosh^2 u$, *where* $u \in \mathcal{U}$ *and* $t(u) \ge 3$.

Proof. Let $\delta = \{(z,t) : \mathrm{Im}\ z = 0\}$ be the invariant plane of Γ. Since the axis of f lies in δ, we can normalize Γ so that the fixed point of g is ∞, the fixed points of f are ± 1, and

$$f = \begin{pmatrix} ai & bi \\ bi & ai \end{pmatrix},\quad g = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix},\quad \text{where } b^2 - a^2 = 1,\ a > 1,\ b, \tau \in \mathbb{R}.$$

Further, replacing f with f^{-1} and g with g^{-1}, we can assume that $b < 0$ and $\tau > 0$. Since b is negative, $+1$ is the repelling fixed point of f and -1 is attracting.

Let e be the half-turn whose axis passes through the fixed point of g orthogonally to the axis of f. That is e fixes 0 and ∞. Let e_f and e_1 be half-turns such that $f = ee_f$ and $g = e_1 e$. Since f is π-loxodromic, the axis of e_f intersects the axis of f (and the plane δ) orthogonally; denote the intersection point by A. Further, since g is parabolic and keeps δ invariant, the axis of e_1 fixes ∞ and lies in the plane δ. It is easy to calculate that

$$e = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\quad e_f = \begin{pmatrix} a & b \\ -b & -a \end{pmatrix},\quad e_1 = \begin{pmatrix} i & \tau \\ 0 & -i \end{pmatrix}.$$

Consider half-turns $e_{k-1} = g^{k-1}e$ and $e_k = g^k e$ such that A lies in the region bounded by the axes of e_{k-1} and e_k in the plane δ, see Figure 7. It is easy to calculate that $A = -a/b - j/b$. Since e_k fixes ∞ and $\tau k/2$, we have that

$$A \in \left\{(z,t) : \frac{\tau(k-1)}{2} < \mathrm{Re} z \le \frac{\tau k}{2},\ \mathrm{Im}\, z = 0,\ t > 0\right\}.$$

Hence, we can immediately determine k.

$$\frac{\tau(k-1)}{2} < -\frac{a}{b} \le \frac{\tau k}{2}. \tag{4.8}$$

Therefore, since $2a = -i\mathrm{tr} f = \sqrt{-\beta-4}$ and $b\tau = -\sqrt{-\gamma}$,

$$k = \left\lceil -\frac{2a}{b\tau} \right\rceil = \left\lceil \frac{\sqrt{-\beta-4}}{\sqrt{-\gamma}} \right\rceil.$$

It is easy to see that Γ is discrete if and only if $\widetilde{\Gamma} = \langle e_f, e_{k-1}, e_k\rangle$ is. Following [13], we give geometric conditions for $\widetilde{\Gamma}$ to be discrete.

Suppose that $A \notin axis(e_k)$; see Figure 7(a). By [13], $\widetilde{\Gamma}$ is discrete if either

(a) the angle ϕ between e_{k-1} and $e_f(e_k)$ is of the form π/p, where $p \ge 2$ is an integer, ∞, or $\overline{\infty}$; or

(b) $\phi = 2\pi/p$, where $p \ge 3$ is odd and the bisector of ϕ passes through A.

Suppose that $A \in axis(e_k)$; see Figure 7(b). By [13], $\widetilde{\Gamma}$ is discrete if

(c) the angle ψ made by $axis(e_{k-1})$ and $axis(\tilde{e}_f)$ is of the form π/p, $p \ge 3$ is an integer, ∞, or $\overline{\infty}$, where $\tilde{e}_f = e_k e_f$ is the half-turn whose axis passes through A orthogonally to $axis(e_k)$ in the plane δ.

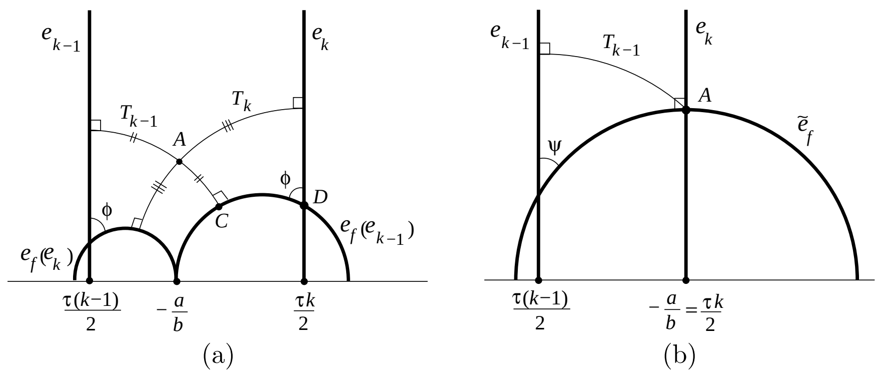


FIGURE 7. The invariant plane δ

There are no other discrete groups. So, we need to calculate the parameters β and γ in each of the cases (a), (b), and (c).

Assume that we are in case (a) or (b). Then each $g^\ell f = e_\ell e_f$, $\ell \in \mathbb{Z}$, is a π-loxodromic element with translation length $2T_\ell$ and $\mathrm{tr}(g^\ell f) = \pm 2i \sinh T_\ell$, where T_ℓ is the distance between e_ℓ and A. Moreover, from the matrix representation, $\mathrm{tr}(g^\ell f) = 2ai + b\tau\ell i$. The inequalities (4.8) enable us to determine the signs of $\mathrm{tr}(fg^{k-1})$ and $\mathrm{tr}(fg^k)$:

$$\mathrm{tr}(fg^k) = -2i\sinh T_k \quad \text{and} \quad \mathrm{tr}(fg^{k-1}) = +2i\sinh T_{k-1}.$$

Suppose that $p < \infty$. Simple calculations in the plane δ show that

$$\sinh CD = \frac{1 + \cos\phi \cosh(2T_{k-1})}{\sin\phi \sinh(2T_{k-1})}$$

and, on the other hand,

$$\sinh CD = \frac{\sinh T_k + \cos\phi \sinh T_{k-1}}{\sin\phi \cosh T_{k-1}}.$$

So, we obtain

$$2(1 + \cos\phi) = 4 \sinh T_{k-1} \sinh T_k = \mathrm{tr}(fg^{k-1})\mathrm{tr}(fg^k).$$

Applying Lemmas 4.1 and 4.2 and the facts that $\mathrm{tr}f = i\sqrt{-\beta - 4}$ and $\mathrm{tr}(fg) - \mathrm{tr}f = b\tau i = -i\sqrt{-\gamma}$, we get

$$\begin{aligned} 2(1+\cos\phi) &= [(k-1)(\mathrm{tr}(fg) - \mathrm{tr}f) + \mathrm{tr}f] \cdot [k(\mathrm{tr}(fg) - \mathrm{tr}f) + \mathrm{tr}f] \\ &= k(k-1)(\mathrm{tr}(fg) - \mathrm{tr}f)^2 + (2k-1) \cdot \mathrm{tr}f \cdot (\mathrm{tr}(fg) - \mathrm{tr}f) + \mathrm{tr}^2 f \\ &= k(k-1)\gamma + (2k-1)\sqrt{-\beta-4}\sqrt{-\gamma} + \beta + 4. \end{aligned}$$

Hence, $-4(\beta+4) = ((2k-1)\sqrt{-\gamma} \pm \sqrt{-8(1+\cos\phi) - \gamma})^2$, where $\phi = \pi/p$, $p \geq 2$ is an integer. Analogous calculation can be done for $p = \infty$ and $p = \overline{\infty}$, and we obtain item (1) of the theorem.

In case (b), in addition, $T_{k-1} = T_k$. Then $\mathrm{tr}(fg^k) = -\mathrm{tr}(fg^{k-1})$ and by Lemmas 4.1 and 4.2 we have

$$2\sqrt{-\beta-4} = (2k-1)\sqrt{-\gamma}.$$

Therefore, $2(1+\cos\phi) = -\mathrm{tr}^2(fg^k) = (-k\sqrt{-\gamma} + \sqrt{-\beta-4})^2 = -\gamma/4$. Hence, since $\phi = 2\pi/p$, $\gamma = -16\cos^2(\pi/p)$.

Now assume that we are in case (c) and $p < \infty$. Since in this case $e_k e_f = \tilde{e}_f$ is an ellitic element of order 2, $\mathrm{tr}(g^k f) = 0$. Therefore, since $\mathrm{tr}(g^k f) = -ki\sqrt{-\gamma} + i\sqrt{-\beta-4}$, we have that $\beta = k^2\gamma - 4$.

Further, since $\mathrm{tr}(fg^{k-1}) = 2i\sinh T_{k-1}$ and, from the plane δ, $\sinh T_{k-1} = \cos\psi$, we have that

$$\begin{aligned} 4\cos^2\psi = 4\sinh^2 T_{k-1} &= -((k-1)(\mathrm{tr}(fg) - \mathrm{tr}f) + \mathrm{tr}f)^2 \\ &= (-(k-1)\sqrt{-\gamma} + \sqrt{-\beta-4})^2 \\ &= (-(k-1)\sqrt{-\gamma} + k\sqrt{-\gamma})^2 \\ &= -\gamma. \end{aligned}$$

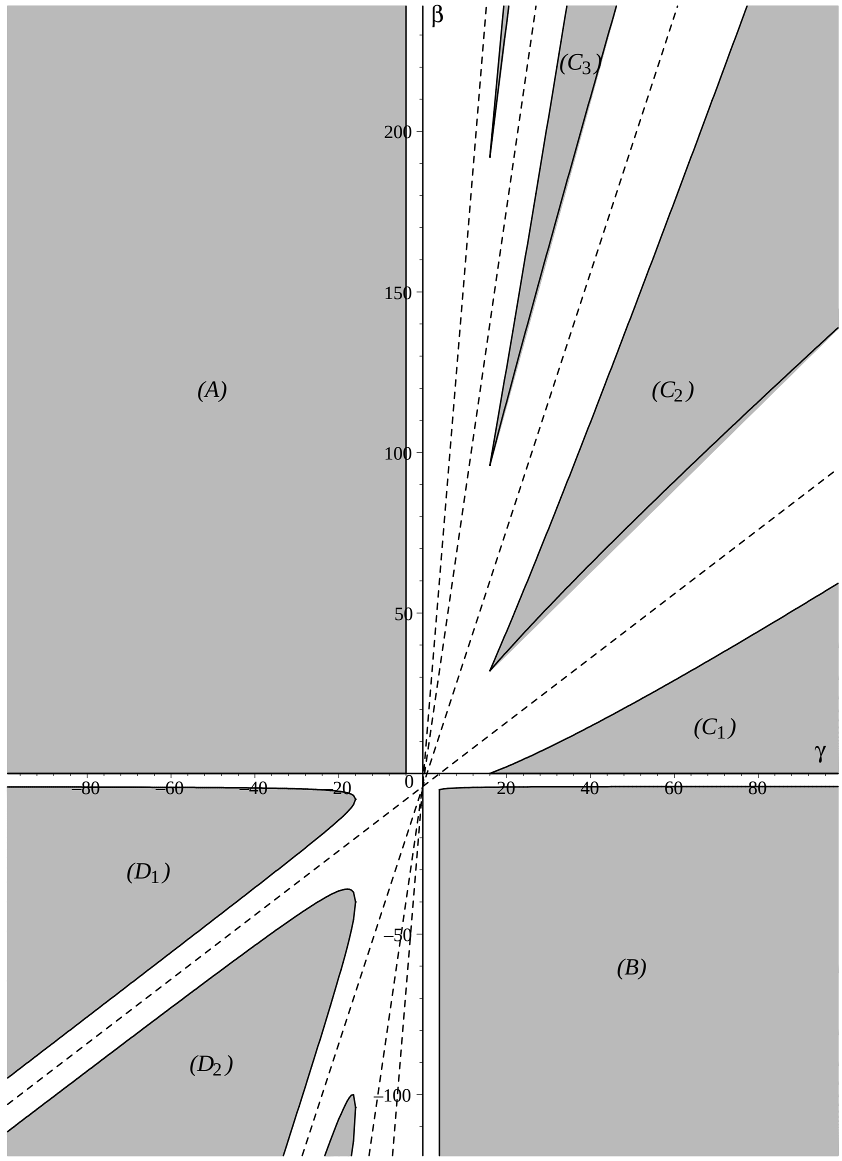
Thus, $\gamma = -4\cos^2(\pi/p)$, where $p \geq 3$ is an integer. Analogous calculations can be done for $p = \infty$ and $p = \overline{\infty}$ and we obtain item (3) of the theorem. □

Remark 4.10. *If $\beta < -4$ and $\gamma < 0$, then $\langle f, g\rangle$ is free if and only if (γ, β) lies in one of the regions D_k, $k = 1, 2, 3, \ldots$, given by*

$$D_k = \{(\gamma,\beta) : \gamma \leq -16, \\ \frac{((2k-1)\sqrt{-\gamma} - \sqrt{-\gamma-16})^2}{-4} \geq \beta + 4 \geq \frac{((2k-1)\sqrt{-\gamma} + \sqrt{-\gamma-16})^2}{-4}\}.$$

When $\gamma > 0$, the parameters were described in Theorem 2.6. Here we just note that for $\gamma > 0$ and $\beta < 0$, the group $\langle f, g\rangle$ is free if and only if (γ, β) lies in the region

$$B = \{(\gamma, \beta) : \gamma \geq 4,\ \beta + 4 \leq -4/\gamma\}.$$



$A = \{(\gamma,\beta) : \gamma \leq -4,\ \beta \geq 0\}$

$B = \{(\gamma,\beta) : \gamma \geq 4,\ \beta + 4 \leq -4/\gamma\}$

$C_k = \{(\gamma,\beta) : \gamma \geq 16,\ ((k-1)\sqrt{\gamma}+2)^2 \leq \beta + 4 \leq (k\sqrt{\gamma}-2)^2\}$

$$D_k = \{(\gamma,\beta) : \gamma \leq -16,\ \frac{((2k-1)\sqrt{-\gamma}+\sqrt{-\gamma-16})^2}{-4} \leq \beta+4 \leq \frac{((2k-1)\sqrt{-\gamma}-\sqrt{-\gamma-16})^2}{-4}\}$$

Dashed lines $\beta = k^2\gamma - 4$, $k = 1, 2, 3, \ldots$

Figure 8. The discrete free groups

Finally, we are able to draw those subsets of S_∞ that correspond to discrete free groups. These subsets are shown in Figure 8. The dashed lines $\beta = k^2\gamma - 4$ are plotted to show a certain symmetry of S_∞.

The other discrete groups contain elliptic elements. Their parameters are represented by lines, parabolas, hyperbolas, and points accumulating, as orders of elliptic elements tend to ∞, to the regions of free groups.

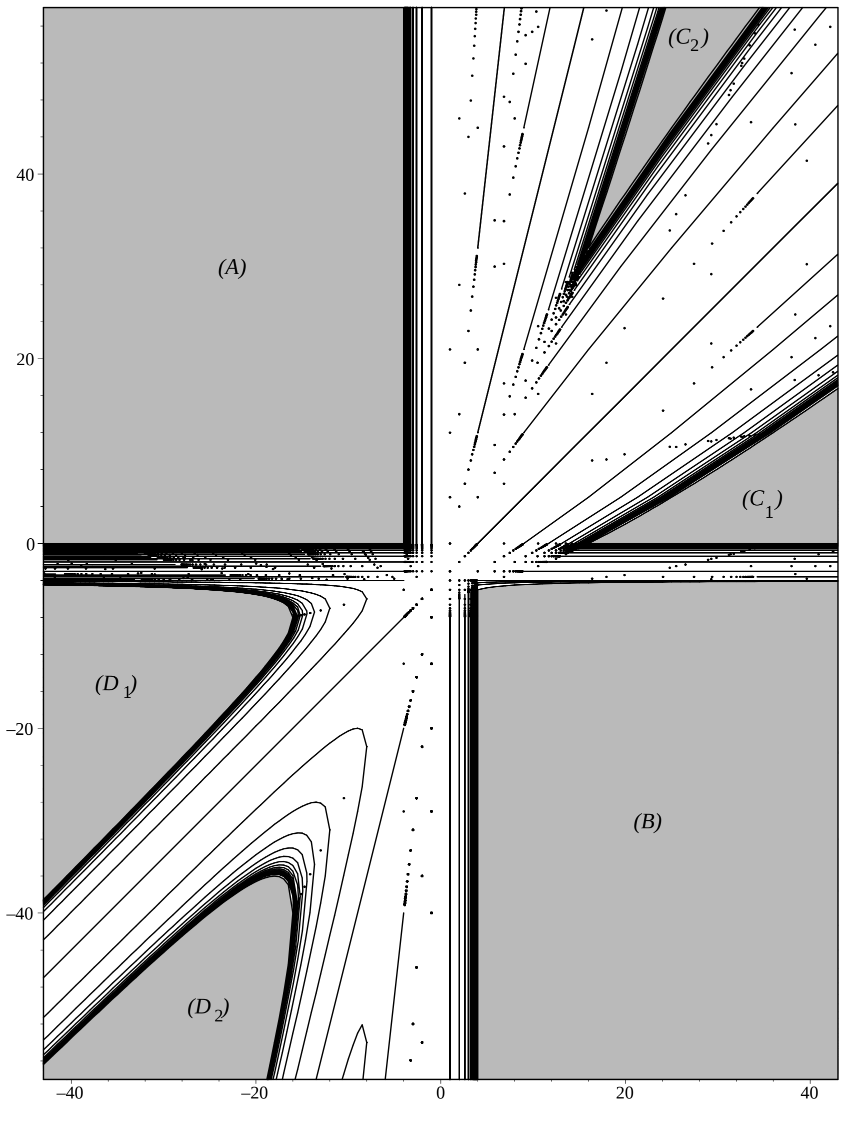


FIGURE 9. The structure of the slice S_∞

In Figure 9, the whole picture for the slice S_∞ is shown to give an idea of the structure of S_∞. The formulas for β and γ obtained in Theorems 2.6, 4.4, 4.6, 4.8, and 4.9, were programmed with the package Maple 7.0 for some (sufficiently large) values of independent variables like $n, q \in \mathbb{Z}$ and $u, v \in \mathcal{U}$ and plotted on the plane (γ, β).

The most interesting families of parameters appear when γ and β are of the same sign. For a fixed k, the hyperbolas

$$-4(\beta+4)=\Big((2k-1)\sqrt{-\gamma}\pm\sqrt{-\gamma-8(1+\cos(\pi/p))}\Big)^2,$$

where $p\geq 2$ is an integer, form a one-parameter family of curves converging to the boundary of D_k as $p\to\infty$. Each hyperbola has the asymptotes $\beta=(k-1)^2\gamma-4k(1+\cos(\pi/p))+4$ and $\beta=k^2\gamma+4k(1+\cos(\pi/p))-4$, which are obviously parallel to $\beta=(k-1)^2\gamma-4$ and $\beta=k^2\gamma-4$, respectively.

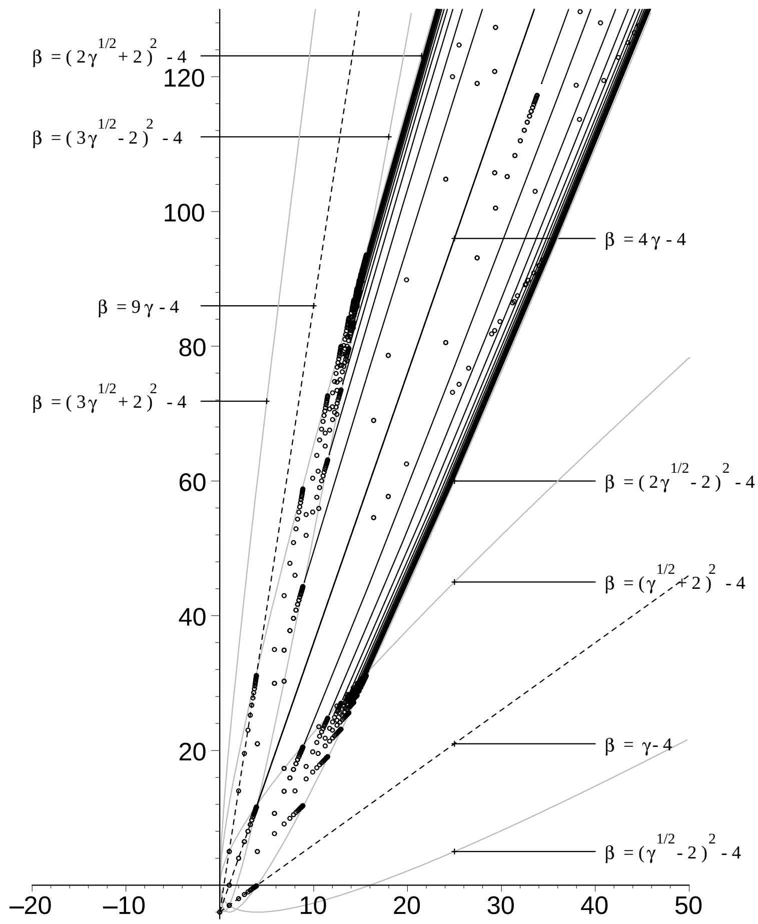


FIGURE 10. The structure of Σ_2

For $\gamma>0$ and $\beta>0$, consider a one-parameter family of parabolas $\beta_k=(k\sqrt{\gamma}\pm 2)^2-4$. Let Σ_k be the domain bounded by β_k:

$$\Sigma_k=\{(\gamma,\beta):(k\sqrt{\gamma}-2)^2\leq\beta+4\leq(k\sqrt{\gamma}+2)^2\}.$$

Within each Σ_k, the parameters for discrete groups are given by

$$\begin{cases}\beta=(k\sqrt{\gamma}\pm 2\cos(q\pi/n))^2-4,\\ \gamma=4C(q,n)(\cos(\pi/n)+\cosh u)^2,\end{cases}$$

where $(q,n)=1$, $1\le q<n/2$, and $u\in\mathcal{U}$. Note that for $n=2$, we have $\beta=k^2\gamma-4$ and $\gamma=4\cosh^2 u$. As $n\to\infty$, the curves $\beta=(k\sqrt{\gamma}\pm 2\cos(q\pi/n))^2-4$ accumulate to the boundary of Σ_k, i.e., to the boundaries of C_{k-1} and C_k (see Figure 10 for an example of Σ_k for $k=2$).

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