Two-generator Kleinian orbifolds

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Abstract

We give a complete list of orbifolds uniformised by non-elementary two-generator subgroups of $\mathrm{PSL}(2,\mathbb{C})$ without invariant plane whose generators and their commutator have real traces.

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1 Introduction

Let Γ be a non-elementary Kleinian group, and let $\Omega(\Gamma)$ be the discontinuity set of Γ. Following [1], we say that the *Kleinian orbifold* $Q(\Gamma) = (\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$ is an orientable 3-orbifold with a complete hyperbolic structure on its interior $\mathbb{H}^3/\Gamma$ and a conformal structure on its boundary $\Omega(\Gamma)/\Gamma$.

First, we define the class of $\mathcal{RP}$ *groups* (two-generator groups with real parameters) by

$$\mathcal{RP} = \{\Gamma : \Gamma = \langle f, g\rangle \text{ for some } f, g \in \mathrm{PSL}(2,\mathbb{C}) \text{ with } \beta, \beta', \gamma \in \mathbb{R}\},$$

where $\beta = \beta(f) = \mathrm{tr}^2 f - 4$, $\beta' = \beta(g) = \mathrm{tr}^2 g - 4$, $\gamma = \gamma(f,g) = \mathrm{tr}[f,g] - 2$. A pair (f,g) such that $(\beta,\beta',\gamma) \in \mathbb{R}^3$ is called an $\mathcal{RP}$ *pair*.

Now denote by $\mathcal{D}$ the class of non-elementary Kleinian $\mathcal{RP}$ groups without invariant plane whose generators have real traces. The purpose of this paper is to describe the Kleinian orbifolds for the class $\mathcal{D}$.

Before we give an equivalent definition of $\mathcal{D}$ in terms of parameters, we recall that an element $f \in \mathrm{PSL}(2,\mathbb{C})$ with real $\beta = \beta(f)$ is *elliptic*, *parabolic*, *hyperbolic*, or π*-loxodromic* according to whether $\beta \in [-4, 0)$, $\beta = 0$, $\beta \in (0, +\infty)$,

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or $\beta \in (-\infty, -4)$. If $\beta \notin [-4, \infty)$, i.e. $\mathrm{tr} f$ is not real, then f is called *strictly loxodromic*. Among all strictly loxodromic elements, only π-loxodromics have real β.

The parameter γ is responsible for the mutual position of invariant planes of f and g. For the geometric meaning of γ we refer the reader to [11] and [13]. In particular, from [11, Theorem 4] we easily deduce the following

Proposition 1.1. *Let Γ be a two-generator subgroup of $\mathrm{PSL}(2, \mathbb{C})$ and let f and g be its generators with parameters (β, β', γ). Then $\beta > -4$, $\beta' > -4$, $\gamma < -\beta\beta'/4$, and $\gamma \neq 0$ if and only if all of the following hold:*

1. *Γ is an $\mathcal{RP}$ group;*
2. *Γ is non-elementary;*
3. *Γ has no invariant plane (in particular, Γ is not Fuchsian);*
4. *each of the generators f and g is either elliptic, parabolic, or hyperbolic.*

If we denote by $\mathcal{K}$ the class of all Kleinian groups, i.e. discrete subgroups of $\mathrm{PSL}(2, \mathbb{C})$, then Proposition 1.1 gives us the description of the class $\mathcal{D}$ in terms of parameters:

$$\mathcal{D} = \{\Gamma \in \mathcal{K} : \Gamma = \langle f, g\rangle \text{ for some } f, g \in \mathrm{PSL}(2, \mathbb{C}) \\ \text{with } \beta > -4, \beta' > -4,\ \gamma < -\beta\beta'/4, \text{ and } \gamma \neq 0\}.$$

The class $\mathcal{D}$ was studied in [8]–[14]. To lighten the reading, we collect the precise references to our previous results in Table 1.

Table 1: References for the class $\mathcal{D}$

β and f	β' and g	discreteness criteria	group presentations and orbifolds
$(-4, 0)$ elliptic	$(-4, 0)$ elliptic	Appendix: Theorem A.1 and Corollary A.1; [8, Theorem]	Appendix: Corollary A.2
0 parabolic	$(-4, +\infty)$ elliptic, parabolic, hyperbolic	[13, Theorem 2.3]	[13, Theorem 3.1]
$(0, +\infty)$ hyperbolic	$(0, +\infty)$ hyperbolic	[10, Theorem 2.1]	[10, Section 3]
$(-4, 0)$ elliptic	$(0, +\infty)$ hyperbolic	[9, Theorem 1], [11, Theorems A and B], [12, Theorems A and B]	[14, Proposition 3.1, Proposition 3.2, and Proposition 3.5]

An elliptic element f of order n is said to be *non-primitive* if it is a rotation through $2\pi q/n$, where q and n are coprime and $1 < q < n/2$. If f is a rotation through $2\pi/n$, then it is called *primitive*.

In this paper, we call a pair (f, g) of generators *primitive* if each of f and g is either hyperbolic, or parabolic, or a primitive elliptic element. It is clear that every $\Gamma \in \mathcal{D}$ has at least one primitive $\mathcal{RP}$ pair of generators.

The paper is organised as follows. Theorem 2.1 in Section 2 gives a presentation for each $\Gamma \in \mathcal{D}$. Theorem 2.2 lists the parameters (β, β', γ) for all primitive $\mathcal{RP}$ pairs of generators for each $\Gamma \in \mathcal{D}$. The list of parameters (without group presentations) first appeared in [12].

Theorem 3.1 in Section 3 lists all orbifolds $Q(\Gamma)$ with $\Gamma \in \mathcal{D}$. At the end of Section 3 we give a complete list of finite covolume groups in class $\mathcal{D}$ (cocompact and not).

In Appendix, we concentrate on primitive $\mathcal{RP}$ pairs of elliptics. We give necessary and sufficient conditions for discreteness of the non-elementary groups without invariant plane generated by such $\mathcal{RP}$ pairs (Theorem A.1), reformulate this theorem in terms of parameters (Corollary A.1), and describe corresponding orbifolds and group presentations (Corollary A.2).

Theorem A.1 was proved in [8], but there is a mathematical disorder in the English translation of the original result. So, we reproduce the proof here and complete the case of two elliptic generators by giving parameters for such groups and describing corresponding orbifolds. On the other hand, Appendix can be regarded as an example that helps to follow the general line of the study of $\mathcal{RP}$ groups.

We remark that the case of two elliptic generators whose commutator is also elliptic was studied in [15].

2 Combinatorics of the class $\mathcal{D}$: group presentations

In this section we will give a presentation for every group $\Gamma \in \mathcal{D}$ (Theorem 2.1) and a complete list of parameters (β, β', γ) for primitive $\mathcal{RP}$ pairs of generators for all $\Gamma \in \mathcal{D}$ (Theorem 2.2).

The parameters for $\mathcal{RP}$ pairs with non-primitive elliptics can be calculated using parameters for primitive elliptics as follows. Suppose that f is non-primitive elliptic of finite order n, i.e., $\beta(f) = -4\sin^2(q\pi/n)$, where $(q, n) = 1$, $1 < q < n/2$. Then there exists an integer r so that f^r is primitive of the same order. Obviously, $\langle f, g\rangle = \langle f^r, g\rangle$ and $\beta(f^r) = -4\sin^2(\pi/n)$. By [6], $\gamma(f^r, g) = (\beta(f^r)/\beta(f))\gamma(f, g)$.

We need the following group presentations:

1. $GT[n, m; q] = \langle f, g \mid f^n, g^m, [f, g]^q\rangle$
2. $PH[n, m, q] = \langle x, y, z \mid x^n, y^2, z^2, (xz)^2, [x, y]^m, (yxyz)^q\rangle$
3. $H[p; n, m; q] = \langle x, y, s \mid s^2, x^n, y^m, (xy^{-1})^p, (sxsy^{-1})^q, (sx^{-1}y)^2\rangle$
4. $P[n, m, q] = \langle w, x, y, z \mid w^n, x^2, y^2, z^2, (wx)^2, (wy)^2, (yz)^2, (zx)^q, (zw)^m\rangle$

5. $Tet[p_1,p_2,p_3;q_1,q_2,q_3] = \langle x,y,z \mid x^{p_1}, y^{p_2}, z^{p_3}, (xy^{-1})^{q_3}, (yz^{-1})^{q_1}, (zx^{-1})^{q_2}\rangle$. The group $Tet[2,2,n;2,q,m]$ is denoted by $Tet[n,m;q]$ for simplicity.

6. $GTet_1[n,m,q] = \langle x,y,z \mid x^n, y^2, (xy)^m, [y,z]^q, [x,z]\rangle$

7. $GTet_2[n,m,q] = \langle x,y,z \mid x^n, y^2, (xy)^m, (xz^{-1}y^{-1}zy)^q, [x,z]\rangle$

8. $\mathcal{S}_2[n,m,q] = \langle x,L \mid x^n, (xLxL^{-1})^m, (xL^2x^{-1}L^{-2})^q\rangle$

9. $\mathcal{S}_3[n,m,q] = \langle x,L \mid x^n, (xLxL^{-1})^m, (xLxLxL^{-2})^q\rangle$

10. $R[n,m;q] = \langle u,v \mid (uv)^n, (uv^{-1})^m, [u,v]^q\rangle$

In the presentations 1–10, the exponents $n, m, q, \ldots$ may be integers (greater than 1), ∞, or $\overline{\infty}$. We employ the symbols ∞ and $\overline{\infty}$ in the following way. If we have relations of the form $w^n = 1$, where $n = \overline{\infty}$, we remove them from the presentation (in fact, this means that the element w is hyperbolic in the Kleinian group). Further, if we keep the relations $w^\infty = 1$, we get a Kleinian group presentation where parabolics are indicated. To get an abstract group presentation, we need to remove such relations as well.

We assume that $\overline{\infty} > \infty > x$ and $x/\infty = x/\overline{\infty} = 0$ for every real x; $\infty/x = \infty$ and $\overline{\infty}/x = \overline{\infty}$ for every positive real x; in particular, $(\infty, n) = (\overline{\infty}, n) = n$ for every positive integer n.

Theorem 2.1. *$\Gamma \in \mathcal{D}$ if and only if Γ is isomorphic to one of the following groups:*

1. *$GT[n,m;q]$, where $3 \le n \le m$, and $\cos(\pi/(2q)) > \sin(\pi/n)\sin(\pi/m)$;*

2. *$PH[n,m,q]$, where $4 \le n < \infty$, n is even, $2/n + 1/m < 1$, and $q \ge 3$ is odd;*

3. *$H[p;n,m;q]$, where $[p;n,m;q]$ is one of the following:*
   $[2;2,3;5]$; $[2;2,5;3]$; $[2;3,3;p]$, where $p \ge 5$ is odd;
   $[2;3,n;2]$, where $n \ge 5$, $(n,6) = 1$;

4. *$P[n,m,q]$, where $4 \le n < \infty$, n is even, $1/n + 1/m < 1/2$, $m, q \ge 3$ are odd;*

5. *$Tet[2,3,3;2,3,m]$, where $m \ge 4$, $(m,3) = 1$; or*
   $Tet[n,m;q]$, where $3 \le n \le m$, $q \ge 3$ is odd and $\cos(\pi/q) > \sin(\pi/n)\sin(\pi/m)$;

6. *$GTet_1[n,m,q]$, where*
   either $4 \le n < \infty$, n is even, m is odd, $1/n + 1/m < 1/2$, $q \ge 2$,
   or $n \ge 7$ is odd, $m = 3$, $q = 2$;

7. *$GTet_2[n,m,q]$, where $n, m \ge 3$ are odd, $1/n + 1/m < 1/2$, and $q \ge 2$;*

8. $\mathcal{S}_2[n,m,q]$, *where* $4 \le n < \infty$, *n is even,* $2/n + 1/m < 1$, *and* $q \ge 2$*;*

9. $\mathcal{S}_3[n,m,q]$, *where* $n \ge 3$ *is odd,* $2/n + 1/m < 1$, *and* $q \ge 2$*;*

10. $R[n,2;2]$, *where* $n \ge 5$ *and* $(n,6) = 1$.

Proof easily follows from Theorem 2.2. □

Remark 2.1. Not all groups in Theorem 2.1 are distinct. For example, both $GT[n,\infty;\overline{\infty}]$ and $\mathcal{S}_2[n,\infty,\infty]$ are isomorphic to $\mathbb{Z}_n * \mathbb{Z}$.

If $f \in \mathrm{PSL}(2,\mathbb{C})$ is a loxodromic element with translation length d_f and rotation angle θ_f, then

$$\mathrm{tr}^2 f = 4\cosh^2 \frac{d_f + i\theta_f}{2}$$

and $\lambda_f = d_f + i\theta_f$ is called the *complex translation length* of f.

Note that if f is hyperbolic then $\theta_f = 0$ and $\mathrm{tr}^2 f = 4\cosh^2(d_f/2)$. If f is elliptic then $d_f = 0$ and $\mathrm{tr}^2 f = 4\cos^2(\theta_f/2)$. If f is parabolic then $\mathrm{tr}^2 f = 4$; by convention we set $d_f = \theta_f = 0$.

We define the set

$$\mathcal{U} = \{u : u = i\pi/p \text{ for some } p \in \mathbb{Z}, p \ge 2\} \cup [0,+\infty).$$

In other words, the set $\mathcal{U}$ consists of all complex translation half-lengths $u = \lambda_f/2$ for hyperbolic, parabolic, and primitive elliptic elements f. Furthermore, we define a function $t : \mathcal{U} \to \{2,3,4,\dots\} \cup \{\infty, \overline{\infty}\}$ as follows:

$$t(u) = \begin{cases} p & \text{if} \quad u = i\pi/p, \\ \infty & \text{if} \quad u = 0, \\ \overline{\infty} & \text{if} \quad u \in (0,+\infty). \end{cases}$$

Given $u \in \mathcal{U}$ and f with $\mathrm{tr}^2 f = 4\cosh^2 u$, $t(u)$ determines the type of f and, moreover, its order if f is elliptic. Note also that since we regard $\infty/n = \infty$ and $\overline{\infty}/n = \overline{\infty}$, an expression of the form $(t(u),n) = 1$ with $n > 1$ means, in particular, that $t(u)$ is finite (see Remark A.1 in Appendix as an example).

Now we are ready to introduce Table 2.

Theorem 2.2. *Table 2 gives a complete list of parameters* (β,β',γ) *for primitive* $\mathcal{RP}$ *pairs* (f,g) *of generators for all* $\Gamma \in \mathcal{D}$*. Moreover, for each triple* (β,β',γ) *a presentation of the group* Γ *is given.*

Proof. In order to prove the theorem, we need only to summarise the results of our previous papers (see Table 1). □

Table 2: The parameters and presentations for the groups $\Gamma = \langle f, g \rangle \in \mathcal{D}$ so that (f, g) is a primitive $\mathcal{RP}$ pair

$u, v, w \in \mathcal{U}$ and n, m, p, q are positive integers

	$\beta = \beta(f)$	$\gamma = \gamma(f, g)$	$\beta' = \beta(g)$	$\Gamma = \langle f, g \rangle$
1	$4\sinh^2 u,\ t(u) \geq 3$	$-4\cosh^2 w,\ (t(w), 2) = 2,$ $\cos\frac{\pi}{t(w)} > \sin\frac{\pi}{t(u)}\sin\frac{\pi}{t(v)}$	$4\sinh^2 v,\ t(v) \geq 3$	$GT[t(u), t(v); t(w)/2]$
2	$4\sinh^2 u,\ t(u) \geq 3$	$-4\cosh^2 w,\ (t(w), 2) = 1,$ $\cos\frac{\pi}{t(w)} > \sin\frac{\pi}{t(u)}\sin\frac{\pi}{t(v)}$	$4\sinh^2 v,\ t(v) \geq 3$	$Tet[t(u), t(v); t(w)]$
3	$-4\sin^2\frac{\pi}{n},\ n \geq 5,\ (n, 2) = 1$	$-(\beta + 2)^2$	$4(\beta + 4)\cosh^2 u - 4,\ t(u) \geq 3$ $\{n, t(u)\} \neq \{5, 3\}$	$Tet[t(u), n; 3]$
4	-2	$2\cos(2\pi/m),\ m \geq 5,\ (m, 2) = 1$	$\gamma^2 + 4\gamma$	$Tet[4, m; 3]$
5	-3	$(\sqrt{5} - 1)/2$	$\sqrt{5} - 1$	$Tet[4, 5; 3]$
6	-3	$2\cos(2\pi/q),\ q \geq 7,\ (q, 4) = 1$	2γ	$Tet[3, 4; q]$
7	-3	$(\sqrt{5} - 3)/2$	$2(7 + 3\sqrt{5})\cosh^2 u - 4,\ t(u) \geq 3$	$Tet[3, t(u); 5]$
8	$(\sqrt{5} - 5)/2$	$(\sqrt{5} - 1)/2$	$(3\sqrt{5} - 1)/2$	$Tet[3, 3; 5]$
9	-3	$2\cos(\pi/m) - 1,\ m \geq 4,\ (m, 3) = 1$	$\gamma^2 + 4\gamma$	$Tet[2, 3, 3; 2, 3, m]$
10	$-4\sin^2\frac{\pi}{n},\ n \geq 5,\ (n, 6) = 1$	$\beta + 3$	$\frac{2}{\beta}\Big((\beta - 3)\cos\frac{\pi}{n} - 2\beta - 3\Big)$	$H[2; 3, n; 2]$
11	$(\sqrt{5} - 5)/2$	$(\sqrt{5} \pm 1)/2$	$3(\sqrt{5} + 1)/2$	$H[2; 5, 2; 3]$
12	-3	$(\sqrt{5} \pm 1)/2$	$\sqrt{5}$	$H[2; 3, 2; 5]$
13	$(\sqrt{5} - 5)/2$	$(\sqrt{5} - 1)/2$	$\sqrt{5}$	$H[2; 3, 2; 5]$
14	$(\sqrt{5} - 5)/2$	$\sqrt{5} + 2$	$(5\sqrt{5} + 9)/2$	$H[2; 3, 2; 5]$
15	-3	$2\cos(2\pi/q),\ q \geq 8,\ (q, 4) = 2$	2γ	$H[2; 3, 3; q]$

Table 2: (continued)

	$\beta = \beta(f)$	$\gamma = \gamma(f, g)$	$\beta' = \beta(g)$	$\Gamma = \langle f, g \rangle$
16	$-4\sin^2\frac{\pi}{n}$, $n \geq 5$, $(n, 6) = 1$	$2(\beta + 3)$	$-\dfrac{6}{\beta}\left(2\cos\dfrac{\pi}{n} + \beta + 2\right)$	$R[n, 2; 2]$
17	$-4\sin^2\frac{\pi}{n}$, $(n, 2) = 2$, $4 \leq n < \infty$	$4\cosh^2 u + \beta$, $(t(u), 2) = 2$, $1/n + 1/t(u) < 1/2$	$\dfrac{4}{\gamma}\cosh^2 v - \dfrac{4\gamma}{\beta}$, $t(v) \geq 3$, $(t(v), 2) = 1$	$PH[n, t(u)/2, t(v)]$
18	$-4\sin^2\frac{\pi}{n}$, $(n, 2) = 2$, $4 \leq n < \infty$	$4\cosh^2 u + \beta$, $(t(u), 2) = 2$, $1/n + 1/t(u) < 1/2$	$\dfrac{4}{\gamma}\cosh^2 v - \dfrac{4\gamma}{\beta}$, $t(v) \geq 4$, $(t(v), 2) = 2$	$\mathcal{S}_2[n, t(u)/2, t(v)/2]$
19	$-4\sin^2\frac{\pi}{n}$, $(n, 2) = 2$, $4 \leq n < \infty$	$4\cosh^2 u + \beta$, $(t(u), 2) = 1$, $1/n + 1/t(u) < 1/2$	$\dfrac{4(\gamma - \beta)}{\gamma}\cosh^2 v - \dfrac{4\gamma}{\beta}$, $t(v) \geq 3$, $(t(v), 2) = 1$	$P[n, t(u), t(v)]$
20	$-4\sin^2\frac{\pi}{n}$, $(n, 2) = 2$, $4 \leq n < \infty$	$4\cosh^2 u + \beta$, $(t(u), 2) = 1$, $1/n + 1/t(u) < 1/2$	$\dfrac{4(\gamma - \beta)}{\gamma}\cosh^2 v - \dfrac{4\gamma}{\beta}$, $t(v) \geq 4$, $(t(v), 2) = 2$	$GTet_1[n, t(u), t(v)/2]$
21	$-4\sin^2\frac{\pi}{n}$, $(n, 2) = 1$, $n \geq 3$	$4\cosh^2 u + \beta$, $(t(u), 2) = 2$, $1/n + 1/t(u) < 1/2$	$\dfrac{2}{\gamma}\left(\cosh v - \cos\dfrac{\pi}{n}\right) - \dfrac{2}{\gamma\beta}\left((\gamma - \beta)^2\cos\dfrac{\pi}{n} + \gamma(\gamma + \beta)\right)$	$\mathcal{S}_3[n, t(u)/2, t(v)]$
22	-3	$2\cos(2\pi/n) - 1$, $(n, 2) = 1$, $n \geq 7$	$\dfrac{2}{\gamma}(\gamma^2 + 2\gamma + 2)$	$GTet_1[n, 3, 2]$
23	$-4\sin^2\frac{\pi}{n}$, $(n, 2) = 1$, $n \geq 3$	$4\cosh^2 u + \beta$, $(t(u), 2) = 1$, $1/n + 1/t(u) < 1/2$	$\dfrac{2(\gamma - \beta)}{\gamma}\cosh v - \dfrac{2}{\gamma\beta}\left((\gamma - \beta)^2\cos\dfrac{\pi}{n} + \gamma(\gamma + \beta)\right)$	$GTet_2[n, t(u), t(v)]$
24	$-4\sin^2\frac{\pi}{n}$, $(n, 2) = 1$, $n \geq 7$	$(\beta + 4)(\beta + 1)$	$\dfrac{2(\beta + 2)^2}{\beta + 1}\left(\cosh v - \cos\dfrac{\pi}{n}\right) - \dfrac{2}{\beta}\left(\beta^2 + 6\beta + 4\right)$	$GTet_2[n, 3, t(v)]$

3 Geometry of the class $\mathcal{D}$: Kleinian orbifolds

In this section we describe all Kleinian orbifolds $Q(\Gamma)$ for which $\Gamma \in \mathcal{D}$.

Theorem 3.1. *All Kleinian orbifolds $Q(\Gamma)$ with $\Gamma \in \mathcal{D}$ are listed in Figure 1. The admissible values of n, m, p, q, p_i, and q_i are specified in Theorem 2.1.*

Remark 3.1. Notice that if the orbifolds in Figure 1 are labelled not like described in Theorem 2.1, this does not mean that the orbifolds are not hyperbolic. Sometimes, an orbifold remains hyperbolic, though its group does not belong to the class $\mathcal{D}$ anymore.

For example, $Q = Q(R[n, m; q])$ is hyperbolic for all $n, m, q \in \mathbb{Z}$ for which the determinant of the Gram matrix Δ is negative, where

$$\Delta = \begin{bmatrix} 1 & -\cos(\pi/q) & 0 & -\cos(\pi/2m) \\ -\cos(\pi/q) & 1 & -\cos(\pi/2m) & 0 \\ 0 & -\cos(\pi/2m) & 1 & -\cos(\pi/n) \\ -\cos(\pi/2m) & 0 & -\cos(\pi/n) & 1 \end{bmatrix}.$$

3.1 Fat edges and fat vertices

In figures, we schematically draw singular sets and boundary components of orbifolds using fat vertices and fat edges. In fact, each picture gives rise to an infinite series of orbifolds which might be compact or non-compact of finite or infinite volume.

We say that a finite 3-regular graph $\Sigma(Q)$ with fat vertices and fat edges *represents the singular set and/or boundary components of* $Q = Q(\Gamma)$ if

1. edges of $\Sigma(Q)$ are labelled by positive integers greater than 1 or symbols ∞ and $\overline{\infty}$;

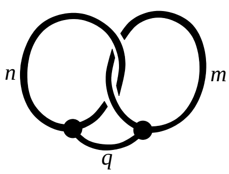
2. the endpoints of a fat edge are fat vertices;

3. if p, q, and r are labels on the edges incident to a non-fat vertex, then $1/p + 1/q + 1/r > 1$.

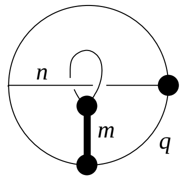
To reproduce the orbifold Q from a graph $\Sigma(Q)$ we first work out all fat vertices and then all fat edges as follows.

Let $v \in \Sigma(Q)$ be a fat vertex and p, q, and r be the indices at the edges incident to v.

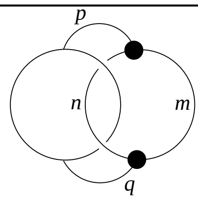
Suppose that all $p, q, r < \infty$. If $1/p + 1/q + 1/r > 1$ then the vertex v is a singular point of Q and the local group of v is one of the finite groups D_{2n}, S_4, A_4, A_5. If $1/p + 1/q + 1/r = 1$ then v represents a puncture. A cusp neighbourhood of v is a quotient of a horoball in $\mathbb{H}^3$ by a Euclidean triangle group $(2, 3, 6)$, $(2, 4, 4)$, or $(3, 3, 3)$. In case $1/p + 1/q + 1/r < 1$ the vertex v must be removed together with its open neighbourhood, which means that Q has a boundary component.



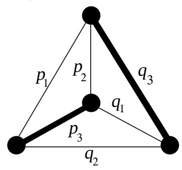
(a) $GT[n, m; q]$



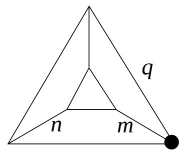
(b) $PH[n, m, q]$



(c) $H[p; n, m; q]$

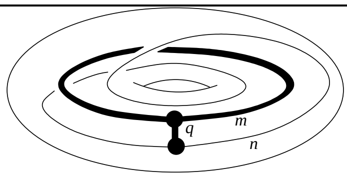


(d) $Tet[p_1, p_2, p_3; q_1, q_2, q_3]$

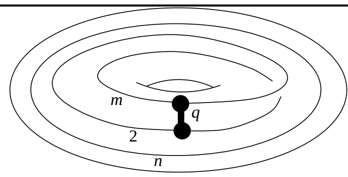


(e) $P[n, m, q]$

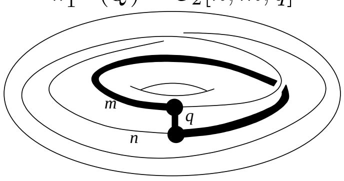
Part I: Orbifolds Q embedded in $\mathbb{S}^3$ and $\pi_1^{orb}(Q)$



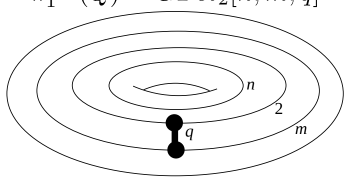
(f) Orbifolds embedded in $\mathcal{S}(2)$; $\pi_1^{orb}(Q) \cong \mathcal{S}_2[n, m, q]$



(g) Orbifolds embedded in $\mathcal{S}(2)$; $\pi_1^{orb}(Q) \cong GTet_2[n, m, q]$

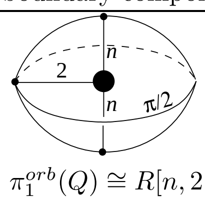


(h) Orbifolds embedded in $\mathcal{S}(3)$; $\pi_1^{orb}(Q) \cong \mathcal{S}_3[n, m, q]$



(i) Orbifolds embedded in $\mathbb{S}^2 \times \mathbb{S}^1$; $\pi_1^{orb}(Q) \cong GTet_1[n, m, q]$

Part II: Orbifolds Q embedded in Seifert fibre spaces; only the torus that contains the singular set or boundary components is shown



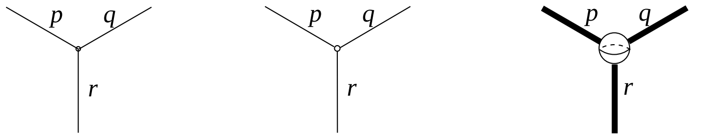
(j) $\pi_1^{orb}(Q) \cong R[n, 2; 2]$

Part III: Orbifolds Q embedded in $\mathbb{R}P^3$

Figure 1: All orbifolds $Q(\Gamma)$ with $\Gamma \in \mathcal{D}$

If one of the indices, say p, equals ∞ and $1/p+1/q+1/r=1$, then $q=r=2$ and v is a puncture.

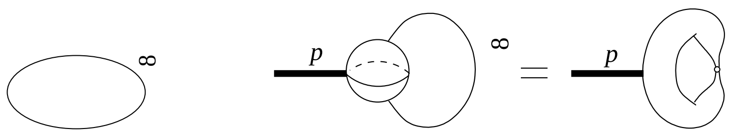
For all the other p, q, r, the vertex v is removed together with its open neighbourhood. Thus, we have three possible configurations for a fat vertex v, which are shown in Figure 2.



$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>1$, $p,q,r<\infty$ | $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$, $p,q,r\leq\infty$ | $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$ or $p=q=2$, $r=\overline{\infty}$

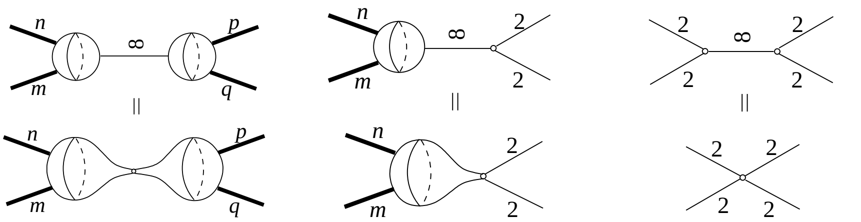
Figure 2: Fat vertices

Now we proceed with the fat edges. If an edge e is labelled by an integer $p<\infty$, then e is a part of the singular set of the orbifold Q and consists of cone points of index p.



non-rigid rank 2 cusp; F is a torus
(a)

rank 1 cusp; F is an annulus
(b)



rank 1 cusp; F is an annulus
(c)

rank 1 cusp; F is a disc with two cone points of order 2
(d)

non-rigid rank 2 cusp; F is a pillow
(e)

Figure 3: Cusps

Edges marked by ∞ represent cusps of Q. A cusp neighbourhood is the quotient of a horoball by an elementary parabolic group. Topologically it is $F\times[0,\infty)$, where F is a Euclidean orbifold called the cross-section of the cusp (see, e.g., [2] for geometric structures on orbifolds). All cusps are drawn in Figure 3. Note that for the orbifolds $Q(\Gamma)$ with $\Gamma \in \mathcal{D}$, the cases shown in Figures 3(a) and (e) never occur.

If e is labelled by $\overline{\infty}$, then it must be deleted together with its open regular neighbourhood.

3.2 Geometry of the orbifolds

Orbifolds embedded in $\mathbb{S}^3$ or $\mathbb{R}P^3$

The orbifolds Q shown in Figures 1(a)–1(e) are embedded in $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$ so that ∞ is a non-singular interior point of Q.

In Figure 1(j), $\mathbb{R}P^3$ is shown as a lens with antipodal points on the boundary identified. The angle at the edge of the lens is $\pi/2$ and, therefore, the edge is mapped onto a singular loop with index 2.

Orbifolds embedded in $\mathcal{S}(n)$ or $\mathbb{S}^2 \times \mathbb{S}^1$

Let $T(n)$ be a Seifert fibred solid torus obtained from a trivial fibred solid torus $D^2 \times \mathbb{S}^1$ by cutting it along $D^2 \times \{x\}$ for some $x \in \mathbb{S}^1$, rotating one of the discs through $2\pi/n$ and gluing back together.

Denote by $\mathcal{S}(n)$ a space obtained by gluing two copies of $T(n)$ along their boundaries fibre to fibre. Clearly, $\mathcal{S}(n)$ is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ and is n-fold covered by trivially fibred $\mathbb{S}^2 \times \mathbb{S}^1$. There are two critical fibres whose length is n times shorter than the length of a regular fibre.

The orbifolds shown in Figures 1(f)–1(i) are embedded in Seifert fibre spaces $\mathcal{S}(n) = T(n) \cup T(n)$ or $\mathbb{S}^2 \times \mathbb{S}^1$ trivially fibred. We draw only the solid torus that contains singular points (or boundary components). The other fibred torus is meant to be attached and is not shown.

For example, if $\Gamma = \mathcal{S}_2[n, m, q]$, then $Q(\Gamma)$ is embedded into $\mathcal{S}(2)$ and $\Sigma(Q)$ is placed in $\mathcal{S}(2)$ so that the edge of order m (if $m < \infty$) lies on a critical fibre of $\mathcal{S}(2)$ and the edge of order n lies on a regular one. Note that the generator L of the group $\mathcal{S}_2[n, m, q]$ is π-loxodromic [14].

3.3 Orbifolds of finite volume

In this section, we list all finite volume orbifolds $Q(\Gamma)$ for $\Gamma \in \mathcal{D}$.

Compact orbifolds

(b) $\pi_1^{orb}(Q) \cong PH[n, m, q]$: $n = 4$, $3 \leq m \leq 5$, $q = 3$

(c) $\pi_1^{orb}(Q) \cong H[p; n, m; q]$: $[p; n, m; q]$ is $[2; 2, 3; 5]$, or $[2; 2, 5; 3]$, or $[2; 3, 5; 2]$

(d) $\pi_1^{orb}(Q) \cong Tet[2, 3, 3; 2, 3, q_3]$: $q_3 = 4, 5$
$\pi_1^{orb}(Q) \cong Tet[n, m; q]$: $n = 5$, $m = 4, 5$, $q = 3$ or $n = m = 3$, $q = 5$

(e) $\pi_1^{orb}(Q) \cong P[n,m,q]$:

$8 \leq n < \infty$, n is even, $m = 3$, $q = 3, 5$;

$4 \leq n < \infty$, n is even, $m = 5$, $q = 3$

(g) $\pi_1^{orb}(Q) \cong GTet_2[n,m,q]$:

$n \geq 7$ is odd, $m = 3$, $3 \leq q \leq 5$;

$n \geq 5$ is odd, $m = 5$, $q = 3$;

$n, m \geq 3$ are odd, $1/n + 1/m < 1/2$, $q = 2$

(h) $\pi_1^{orb}(Q) \cong \mathcal{S}_3[n,m,q]$:

$n \geq 5$ is odd, $m = q = 2$;

$n = 5$, $m = 2$, $q = 3$;

$n = 5$, $m = 3$, $q = 2$;

$n = 3$, $m = 4, 5$, $q = 2$

(i) $\pi_1^{orb}(Q) \cong GTet_1[n,m,q]$: $7 \leq n < \infty$, $m = 3$, $q = 2$

There are no compact orbifolds in the class $\mathcal{D}$ with $\pi_1^{orb}(Q) \cong GT[n,m;q]$, $\mathcal{S}_2[n,m,q]$, or $R[n,2;2]$.

Non-compact orbifolds of finite volume

(a) $\pi_1^{orb}(Q) \cong GT[n,m;q]$: $[n,m;q]$ is $[3,3;3]$, or $[4,4;2]$, or $[4,3;2]$

(b) $\pi_1^{orb}(Q) \cong PH[n,m,q]$: $n = 4$, $m = 6$, $q = 3$ or $n = 6$, $2 \leq m \leq 6$, $q = 3$

(d) $\pi_1^{orb}(Q) \cong Tet[n,m;q]$: $n = 6$, $3 \leq m \leq 6$, $q = 3$

(f) $\pi_1^{orb}(Q) \cong \mathcal{S}_2[n,m,q]$: $n = 4$, $m = 3, 4$, $q = 2$

(g) $\pi_1^{orb}(Q) \cong GTet_2[n,m,q]$: $n \geq 7$ is odd, $m = 3$, $q = 6$

(h) $\pi_1^{orb}(Q) \cong \mathcal{S}_3[n,m,q]$: $n = 3$, $m = 6$, $q = 2$

(i) $\pi_1^{orb}(Q) \cong GTet_1[n,m,q]$: $8 \leq n < \infty$, $(n,2) = 2$, $m = 3$, $q = 3$

There are no non-compact orbifolds of finite volume in the class $\mathcal{D}$ with $\pi_1^{orb}(Q) \cong H[p;n,m;q]$, $P[n,m,q]$, or $R[n,2;2]$.

A Both generators are elliptic

Suppose an $\mathcal{RP}$ group Γ is generated by two elliptic elements f and g. By [11, Theorem 2], the axes of f and g either lie in one plane or are mutually orthogonal skew lines. Note that such a group is elementary or has an invariant plane if and only if the axes of f and g lie in one plane or one of the generators has order 2. So we do not consider such groups.

Theorem A.1 gives necessary and sufficient discreteness conditions for non-elementary $\mathcal{RP}$ groups without invariant plane generated by two primitive elliptics. The proof of Theorem A.1 is constructive and we will use it later to find parameters for $\Gamma \in \mathcal{D}$ generated by two primitive elliptics and describe corresponding orbifolds.

Theorem A.1 ([8]). *Let f and g be primitive elliptic elements of orders $n \geq 3$ and $m \geq 3$, respectively, and let their axes be mutually orthogonal skew lines. Then:*

(1) *there exists a unique element $h \in \mathrm{PSL}(2, \mathbb{C})$ such that $h^2 = fgf^{-1}g^{-1}$ and $(hg)^2 = 1$, and*

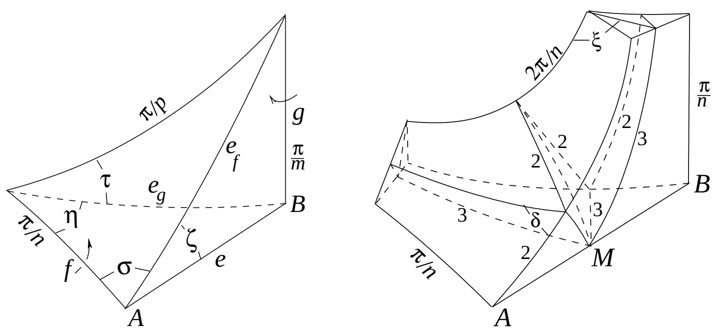
(2) *$\Gamma = \langle f, g \rangle$ is discrete if and only if one of the following holds:*

  (2.*i*) *h is hyperbolic, parabolic, or a primitive elliptic element of order p, where $\cos(\pi/p) > \sin(\pi/n)\sin(\pi/m)$;*

  (2.*ii*) *$m = n \geq 7$, $(n, 2) = 1$, and h is a square of a primitive elliptic element of order n.*

Proof. **1.** *Construction of* Γ^*. We start with construction of a reflection group Γ^* containing Γ as a subgroup of finite index. Such a group is discrete if and only if so is Γ.

Let f and g be primitive elliptic elements of orders $n \geq 3$ and $m \geq 3$, respectively, and let the axes of f and g be mutually orthogonal skew lines. We will denote elements and their axes by the same letters when it does not lead to any confusion.



(a) (b)

Figure 4:

Let e be a half-turn whose axis is orthogonal to both f and g. Let e_f and e_g be half-turns such that $f = e_f e$ and $g = e_g e$. Then e_f and e intersect at an angle of π/n and e_g and e intersect at an angle of π/m.

Denote by η the plane through e_g, e, and f and by ζ the plane through e_f, e, and g. Denote by σ the plane through e_f and f and by τ the plane through e_g and g (see Figure 4(a)).

For the group $\Gamma = \langle f, g \rangle$ we define two finite index extensions of it as follows: $\widetilde{\Gamma} = \langle f, g, e \rangle$ and $\Gamma^* = \langle f, g, e, R_\eta \rangle$ (we denote the reflection in a plane κ by R_κ).

It is easy to see that $\widetilde{\Gamma} = \Gamma \cup \Gamma e$. If $e \in \Gamma$ then $\widetilde{\Gamma} = \Gamma$ and if $e \notin \Gamma$ then Γ is a subgroup of index 2 in $\widetilde{\Gamma}$. (As we will see in Corollary A.2, both possibilities are realised.) Since, moreover, $\widetilde{\Gamma}$ is the orientation preserving subgroup of index 2 in Γ^*, the groups Γ, $\widetilde{\Gamma}$, and Γ^* are either all discrete or all non-discrete. It is clear that $\Gamma^* = \langle R_\eta, R_\zeta, R_\sigma, R_\tau \rangle$.

2. *Existence and uniqueness of h.* First, we show that if $h = R_\sigma R_\tau$ then $h^2 = [f, g]$ and $(hg)^2 = 1$. Indeed,

$$\begin{aligned} fgf^{-1}g^{-1} &= (e_f e)(e_g e)(e e_f)(e e_g) = (e_f e e_g)^2 = (f e_g)^2 \\ &= (R_\sigma R_\eta R_\eta R_\tau)^2 = (R_\sigma R_\tau)^2 = h^2. \end{aligned}$$

Moreover, $hg = (R_\sigma R_\tau)(R_\tau R_\zeta) = R_\sigma R_\zeta = e_f$. So, $(hg)^2 = e_f^2 = 1$. Hence, the element h required in assertion (1) of Theorem A.1 exists.

Now show that h is unique. If $[f, g]$ is parabolic, then the square root of $[f, g]$ is unique in $\mathrm{PSL}(2, \mathbb{C})$ and we are done. Let $[f, g]$ be elliptic or hyperbolic (it cannot be strictly loxodromic, because $[f, g] = (R_\sigma R_\tau)^2 = R_\sigma(R_\tau R_\sigma R_\tau)$ is the product of two reflections). Denote by $\overline{h}$ the second square root of $[f, g]$. If $[f, g]$ is hyperbolic, then $\overline{h}$ is π-loxodromic with the same axis and translation length as h. If $[f, g]$ is elliptic, then $\overline{h}$ is elliptic with the same axis as h, but the rotation angle of $\overline{h}$ is $(\pi - 2\pi/p)$. It is clear that in both cases $\overline{h}^2 = [f, g]$ and $(\overline{h}g)^2 \neq 1$. Uniqueness is proved.

3. *Polyhedron $\mathcal{T}$.* We first consider the polyhedron $\mathcal{T}$ bounded by the planes η, ζ, σ, and τ (we know that reflections in these planes generate Γ^*) and determine the dihedral angles of $\mathcal{T}$. The line f is the line of intersection of the planes η and σ and the dihedral angle made by η and σ equals π/n. Similarly, g is the line of intersection of ζ and τ and the dihedral angle made by them equals π/m. The plane η is orthogonal to both ζ and τ, and the plane σ is orthogonal to ζ. As for σ and τ, they may either intersect, or be parallel or disjoint. Denote the dihedral angle of $\mathcal{T}$ made by σ and τ by π/p (p is not necessarily an integer). We keep the notation π/p with $p = \infty$ and $p = \overline{\infty}$ for the cases of parallel and disjoint σ and τ, respectively. From [17], $\mathcal{T}$ exists in $\mathbb{H}^3$ if and only if

$$\cos(\pi/p) > \sin(\pi/n)\sin(\pi/m) \qquad \text{(A.1)}$$

Note that (A.1) implies $\pi/p < \pi/2$. Therefore, there is an obvious connection between $h = R_\sigma R_\tau$ and the dihedral angle of $\mathcal{T}$ made by σ and τ. In particular, h is parabolic (hyperbolic) if and only if σ and τ are parallel (disjoint, respectively).

4. *Discreteness of Γ implies (2.i) or (2.ii).*

Assume that Γ is discrete, then Γ^* is also discrete and, therefore, $h = R_\sigma R_\tau$ either satisfies (2.i) or is a non-primitive elliptic element of finite order, i.e., $p = q/k$, where q and k are coprime integers, $k > 1$. Let us show that then (2.ii) holds.

Consider a surface S that is orthogonal to the planes η, σ, and τ. If η, σ, and τ intersect at a point $x \in \mathbb{H}^3$, then S is a sphere with center x; if η, σ, and τ meet at a point $x \in \partial\mathbb{H}^3$, then S is a horosphere; if η, σ, and τ do not have a common point in $\overline{\mathbb{H}^3}$, then S is a hyperbolic plane. Consider the intersection Δ of S with the polyhedron $\mathcal{T}$, that is a link made by η, σ, and τ (cf. [11]).

The link Δ is then a spherical, Euclidean, or hyperbolic triangle (depending on the type of S) with angles of $\pi/2$, π/n, and $\pi/p = k\pi/q$, where k and q are coprime and

(a) Suppose that Δ is spherical. From the list of all spherical triangles with two primitive and one non-primitive angles [4], there are two possibilities for Δ: $(\pi/2, \pi/5, 2\pi/5)$ and $(\pi/2, \pi/3, 2\pi/5)$. Since in both cases $k\pi/q$=$2\pi/5$, the link of the vertex made by ζ, σ, and τ is also spherical, that is $n = 3$ or 5, and $m = 3$ or 5. However, all triples (n, m, p), where $n, m \in \{3, 5\}$ and $p = 5/2$, do not satisfy (A.1). Contradiction. So, Δ cannot be spherical.

(b) Δ also cannot be Euclidean, because there are no Euclidean triangles with angles $\pi/2$, π/n, $k\pi/q$ with $(k, q) = 1$ that lead to a discrete group.

(c) Suppose that Δ is a hyperbolic triangle with angles $\pi/2$, π/n, $k\pi/q$. From the list of all hyperbolic triangles with one non-primitive and two primitive angles [16], we have that $k = 2$ and $q = n \geq 7$ is odd.

Since $q \geq 7$, the link of the vertex made by ζ, σ, and τ is also a hyperbolic triangle and again from [16] we conclude that $m = q = n$.

From (a), (b), and (c) above, it follows that the only possibility for dihedral angles of $\mathcal{T}$ is that with $n = m \geq 7$, $(n, 2) = 1$, and the dihedral angle between σ and τ equals $2\pi/n$, that is $h = R_\sigma R_\tau$ is a square of a primitive elliptic element of order n. Hence, (2.*ii*) holds.

5. (2.*i*) *implies discreteness of* Γ. Assume that (2.*i*) holds. Then the inequality (A.1) implies that the polyhedron $\mathcal{T}$ exists in $\mathbb{H}^3$. Moreover, since h is a hyperbolic, parabolic, or primitive elliptic element of order p, the planes σ and τ are either disjoint, or parallel, or the dihedral angle between them is π/p, $p \in \mathbb{Z}$. Therefore, $\mathcal{T}$ and reflections R_η, R_ζ, R_σ, and R_τ satisfy the hypotheses of the Poincaré theorem [3], Γ^* is discrete, and $\mathcal{T}$ is its fundamental polyhedron.

6. (2.*ii*) *implies discreteness of* Γ. Since h is a square of a primitive elliptic element of order n and $m = n$, the dihedral angles of $\mathcal{T}$ are as described in part 3 of the proof with $\pi/m = \pi/n$ and $\pi/p = 2\pi/n$.

Note that for each n, $\mathcal{T} = \mathcal{T}(n)$ is uniquelly determined by its dihedral angles. (To see this, we apply Andreev's thorem to the compact polyhedron that has two more faces: the hyperbolic plane S orthogonal to η, σ, and τ, and the plane $\widetilde{S}$ orthogonal to ζ, σ, τ.) In particular, the length of AB in Figure 4(b) is determined by the dihedral angles of $\mathcal{T}$ uniquely, and so it depends only on n.

Since the dihedral angle between σ and τ is $2\pi/n$, there is one more reflection R_ξ in Γ^*, $R_\xi \in \langle R_\sigma, R_\tau \rangle$, where ξ is the bisector of the dihedral angle of $\mathcal{T}$ made by σ and τ. The plane ξ divides $\mathcal{T}$ into two symmetric polyhedra, each of which is further subdivided (by planes of reflections from Γ^*) into three tetrahedra $T(n) = T[2,2,3;2,3,n]$ with primitive dihedral angles. To see this, look at the tesselation of $\mathbb{H}^3$ by the infinite volume tetrahedra $T(n)$ as above and find the polyhedron $\mathcal{T}(n)$ which is formed by six tetrahedra $T(n)$.

On the other hand, reflections in the faces of $\mathcal{T}(n)$ generate all reflections in the faces of $T(n)$. Since $\mathcal{T}(n)$ is determined by its dihedral angles uniquely, we conclude that Γ^* is the group of reflections in the faces of $T(n)$ and, therefore, both Γ^* and Γ are discrete. □

The following corollary is just a reformulation of Theorem A.1 in terms of parameters.

Corollary A.1. *Let $f, g \in \mathrm{PSL}(2,\mathbb{C})$, $\beta(f) = -4\sin^2(\pi/n)$, $\beta(g) = -4\sin^2(\pi/m)$, where $n, m \in \mathbb{Z}$ and $n, m \geq 3$, and let $\gamma(f,g) < -\beta(f)\beta(g)/4$. Then $\Gamma = \langle f, g\rangle$ is discrete if and only if one of the following holds:*

1. $\gamma(f,g) \in (-\infty; -4]$;
2. $\gamma(f,g) = -4\cos^2(\pi/p)$, $p \in \mathbb{Z}$, $\cos(\pi/p) > \sin(\pi/n)\sin(\pi/m)$;
3. $m = n \geq 7$, $(n,2) = 1$, $\gamma(f,g) = -(\beta+2)^2$.

Proof. Since $\beta(f) = -4\sin^2(\pi/n)$, $\beta(g) = -4\sin^2(\pi/m)$, $n, m \in \mathbb{Z}$, f and g are primitive elliptic elements. Since $\gamma(f,g) < -\beta(f)\beta(g)/4$, the axes of f and g are mutually orthogonal skew lines [11, Theorem 4]. So the hypotheses of Corollary A.1 are equivalent to the hypotheses of Theorem A.1. Therefore, in order to prove Corollary A.1 it suffices to compute parameters for each discrete group described in Theorem A.1.

We need to rewrite conditions (2.i) and (2.ii) via $\gamma(f,g)$. Since $\gamma(f,g) = \mathrm{tr}[f,g] - 2$ and h is a square root of $[f,g]$, it is not difficult to find $\gamma(f,g)$.

The element h is hyperbolic if and only if the planes σ and τ are disjoint. (We denote planes (and lines) as in the proof of Theorem A.1.) Let d be the hyperbolic distance between them. Since $[f,g] = h^2 = (R_\sigma R_\tau)^2$,

$$\gamma(f,g) = \mathrm{tr}[f,g] - 2 = -2\cosh(2d) - 2 < -4$$

(we must take $\mathrm{tr}[f,g]$ to be negative, because $\gamma(f,g)$ is strictly negative for all values $\beta(f)$, $\beta(g)$).

The element h is parabolic if and only if $[f,g]$ is parabolic and if and only if $\mathrm{tr}[f,g] = -2$, that is, $\gamma(f,g) = \mathrm{tr}[f,g] - 2 = -4$.

Thus, h is hyperbolic or parabolic if and only if $\gamma(f,g) \in (-\infty, -4]$, and part 1 of Corollary A.1 is proved.

Now suppose that h is an elliptic element with rotation angle ϕ, where $\phi/2 = \pi/p < \pi/2$ is the dihedral angle of $\mathcal{T}$ between σ and τ. Then $[f,g] = h^2$ is also elliptic with rotation angle 2ϕ. Since $\mathrm{tr}[f,g]$ is well-defined (does not depend on the choice of representatives of f and g in $\mathrm{SL}(2,\mathbb{C})$) we can determine which formula, $\mathrm{tr}[f,g] = +2\cos\phi$ or $\mathrm{tr}[f,g] = -2\cos\phi$, is correct. The easiest way to do this is by using continuity of $\mathrm{tr}[f,g]$ as a function of ϕ and the limit condition $\mathrm{tr}[f,g] \to -2$ as $\phi \to 0$. So we must take $\mathrm{tr}[f,g] = -2\cos\phi$ where $\phi < \pi$ is the doubled dihedral angle of $\mathcal{T}$.

On the other hand, if $\mathrm{tr}[f,g]$ is given, we can use the formula $\mathrm{tr}[f,g] = -2\cos\phi$, $\phi < \pi$, to obtain the rotation angle ϕ of the element h from Theorem A.1.

Thus, h is a primitive elliptic element of order p if and only if $\phi = 2\pi/p$ which is equivalent to

$$\gamma(f,g) = \mathrm{tr}[f,g] - 2 = -2\cos(2\pi/p) - 2 = -4\cos^2(\pi/p),\ p \in \mathbb{Z},$$

which is part 2 of Corollary A.1.

At last, h is a square of a primitive elliptic element of order n if and only if $\phi = 4\pi/n$, which corresponds to $\gamma(f,g) = -4\cos^2(2\pi/n) = -(\beta+2)^2$.

Corollary A.1 is proved. □

Remark A.1. The parts 1 and 2 of Corollary A.1 can be rewritten in terms of the function $t(w)$ as $\gamma(f,g) = -4\cosh^2 w$, where $w \in \mathcal{U}$ and $\cos(\pi/t(w)) > \sin(\pi/n)\sin(\pi/m)$. Indeed,

$$\begin{aligned}
\gamma = -4\cos^2(\pi/p),\ p \in \mathbb{Z} \quad &\Leftrightarrow \quad \gamma = -4\cosh^2 w,\ w \in \mathcal{U},\ t(w) < \infty,\\
\gamma = -4 \quad &\Leftrightarrow \quad \gamma = -4\cosh^2 w,\ w \in \mathcal{U},\ t(w) = \infty,\\
\gamma < -4, \quad &\Leftrightarrow \quad \gamma = -4\cosh^2 w,\ w \in \mathcal{U},\ t(w) = \overline{\infty}.
\end{aligned}$$

Keep in mind that in this notation, $(t(w),2) = 1$ implies that $t(w)$ is finite and odd, while $(t(w),2) = 2$ means that $t(w)$ can be even, ∞, or $\overline{\infty}$.

The following corollary gives group presentations for all groups from class $\mathcal{D}$ generated by two elliptics. The corresponding orbifolds are drawn in Figures 1(a) and 1(d).

Corollary A.2. *Let* $\Gamma = \langle f,g\rangle \in \mathcal{D}$, $\beta(f) = -4\sin^2(\pi/n)$, $\beta(g) = -4\sin^2(\pi/m)$, *where* $n,m \in \mathbb{Z}$, $n,m \geq 3$.

1. *If* $\gamma(f,g) = -4\cosh^2 w$, *where* $w \in \mathcal{U}$, $(t(w),2) = 2$, *and* $\cos(\pi/t(w)) > \sin(\pi/n)\sin(\pi/m)$, *then* $\Gamma \cong GT[n,m;t(w)/2]$.
2. *If* $\gamma(f,g) = -4\cosh^2 w$, *where* $w \in \mathcal{U}$, $(t(w),2) = 1$, *and* $\cos(\pi/t(w)) > \sin(\pi/n)\sin(\pi/m)$, *then* $\Gamma \cong Tet[n,m;t(w)]$.
3. *If* $m = n \geq 7$, $(n,2) = 1$, *and* $\gamma(f,g) = -(\beta+2)^2$, *then* $\Gamma \cong Tet[3,n;3]$.

Proof. All parameters for discrete groups in the statement of Corollary A.2 are described in Corollary A.1. We will obtain a presentation for each case by using the Poincaré polyhedron theorem.

We start with construction of a fundamental polyhedron and a presentation for the group $\widetilde{\Gamma}$ defined in the proof of Theorem A.1. Since $\widetilde{\Gamma}$ is an orientation preserving index 2 subgroup in Γ^* and $\mathcal{T}$ is a fundamental polyhedron for Γ^*, a fundamental polyhedron $\widetilde{\mathcal{T}}$ for $\widetilde{\Gamma}$ consists of two copies of $\mathcal{T}$. We take $\widetilde{\mathcal{T}}$ to be the polyhedron bounded by η, σ, τ, and $R_\zeta(\tau)$.

By applying the Poincaré theorem to $\widetilde{\mathcal{T}}$ and face pairing transformations e, e_f, and g, we get

$$\widetilde{\Gamma} = \langle e, e_f, g \,|\, e^2, e_f^2, g^m, (e_f e)^n, (ge)^2, (ge_f)^p \rangle,$$

where p is an integer, ∞, or $\overline{\infty}$. Since $f = e_f e$,

$$\widetilde{\Gamma} = \langle f, g, e \,|\, f^n, g^m, e^2, (fe)^2, (ge)^2, (gfe)^p \rangle.$$

Note that if p is odd, then using the relations $(fe)^2 = (ge)^2 = 1$, we see that from $(gfe)^p = 1$ it follows that $e = f^{-1}g^{-1}(fgf^{-1}g^{-1})^{(p-1)/2}$. Hence, in this case $\widetilde{\Gamma} = \Gamma$ and $\Gamma \cong Tet[n, m; p]$. Identifying faces of $\widetilde{\mathcal{T}}$, we get an orbifold $\mathbb{H}^3/\Gamma$ (see Figure 1(d)).

If p is even, ∞, or $\overline{\infty}$, then Γ is a subgroup of index 2 in $\widetilde{\Gamma}$. To see this we apply the Poincaré theorem to a polyhedron bounded by σ, τ, $R_\eta(\sigma)$, and $R_\zeta(\tau)$. Then

$$\Gamma = \langle f, g \,|\, f^n, g^m, (fgf^{-1}g^{-1})^{p/2} \rangle.$$

The orbifold $\mathbb{H}^3/\Gamma$ is shown in Figure 1(a). The parts 1 and 2 are proved.

Now suppose that we are in the case 3 of Corollary A.1 and obtain a presentation for Γ.

Let δ be the plane shown in Figure 4(b) and let $u = R_\sigma R_\delta$. It is clear that

$$G = \langle f, e_f, u \,|\, f^n, e_f^2, u^2, (fe_f)^2, (ue_f)^3, (fu)^3 \rangle \cong Tet[3, n; 3].$$

We claim that G is isomorphic to $\Gamma = \langle f, g \rangle$. From the link made by ζ, σ, and τ, we have that $g = uf^2ue_f \in G$. Hence, $\Gamma = \langle f, g \rangle$ is a subgroup of G.

Let us show that G is a subgroup of Γ, i.e., that e_f and u belong to Γ. From the link made by η, σ, and τ, we get $uf^2u = h^{-1}$. Since $h^n = 1$ and n is odd,

$$h = (h^2)^{-\frac{n-1}{2}} = [f, g]^{-\frac{n-1}{2}}.$$

Moreover, since $f^n = 1$,

$$ufu = (uf^{-2}u)^{\frac{n-1}{2}} = h^{\frac{n-1}{2}} = [f, g]^{-\frac{(n-1)^2}{4}}. \qquad \text{(A.2)}$$

Further, combining (A.2) with the relation $(fu)^3 = 1$, we have

$$\begin{aligned} u &= f(ufu)f = f[f, g]^{-\frac{(n-1)^2}{4}} f \in \Gamma \quad \text{and} \\ e_f &= uf^{-2}ug = [f, g]^{\frac{(n-1)^2}{2}} g \in \Gamma. \end{aligned}$$

Thus, G is a subgroup of Γ and so $\Gamma \cong G$. $\square$

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