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This paper is devoted to the construction of regular min-density plane coverings with ellipses of one, two and three types. This problem is relevant, for example, to power-efficient surface sensing by autonomous above-grade sensors. A similar problem, for which discs are used to cover a planar region, has been well studied. On the one hand, the use of ellipses generalizes a mathematical problem; on the other hand, it is necessary to solve these types of problems in real applications of wireless sensor networks. This paper both extends some previous results and offers new regular covers that use a small number of ellipses to cover each regular polygon; these covers are characterized by having minimal known density in their classes and give the new upper bounds for densities in these classes as well.

Keywords: wireless sensor networks; plane covering; coverage density

AMS Subject Classifications: 52C15; 90C27; 51D20; 5115; 5125; 65K10; 74P20

1. Introduction

In wireless sensor networks (WSNs), the sensing area of each sensor is usually a disc with a sensor located at the centre of the disc.[1–5] It is said that the sensor *covers* the disc or that the disc is the sensing region of the sensor. In WSNs, each sensor is supplied with a limited amount of non-renewable energy, and its lifetime is inversely proportional to the covered area. Therefore, an optimal cover of a planar region with discs or ellipses (when every point of a region belongs to at least one disc or ellipse) is the cover that has minimal density, where the density of the plane region's cover is the ratio of the sum of the elements' areas in the cover to the region's area.[4–8,11,13]

Since every disc can be placed at any place of a plane, there is an infinite set of plane covers using discs.[4,7,8,13] The most studied covers are *regular* covers,[2–6,9,10] where a planar region is tiled by identical regular polygons (tiles), and all of the tiles are covered equally. In this case, the density of the planar cover is defined by the coverage density of one tile. Note that only three types of *regular* polygons tile a plane: triangle, square and hexagon. In [5], we introduced the classification of regular covers, according to which cover in class $COV_k(p,q)$ covers each regular k-angle polygon by p discs of q different radii. For example, Figure 1(a) shows a fragment (one triangular tile) of the regular plane cover in

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the class $COV_3(7,3)$. In this cover, we denote it as $C_3(7,3)$, the centres of equal pairwise intersecting discs are located in the nodes of a triangular tile, and a curvilinear triangle in the centre of the tile is covered by one disc with a lesser radius and by three equal discs. Note that when calculating the density of the planar cover for this case, one must consider that each disc that has its centre in the node of a tile participates in covering six tiles. If the centre of disc lies on the side of the tile, then the disc participates in covering two tiles. And the disc that has its centre inside the tile participates in covering only one tile. Then, the density of the cover $C_3(7,3)$ is the *minimum* (i.e. the radii of discs in $C_3(7,3)$ are optimal) ratio of the areas of three sectors (the parts of discs with centres in the nodes of a tile) plus area of one disc with the centre in the centre of a tile and plus the areas of three equal discs to the tile's area.

In regular covers, the sensors are located at certain points in the plane; but sometimes it is impossible to place the sensor exactly at a specific point (contaminated area, out-of-the-way place, enemy territory, etc.). However, even when sensors are distributed randomly in the sensing region, *regular* covers are used to evaluate the efficiency of the sensing (specifically, to estimate the lower bound of a WSN's lifetime).[11] Therefore, the investigation of regular covers remains topical.

In this paper, we consider the construction of min-density regular plane covers using *ellipses* of one, two and three types. This generalization of the problem (in comparison with the case in which the covers are made of discs) can be proved; for example, by the following consideration. If a sensor is equipped with a video camera that is located above the surface and that views the surface, then the covered region is an ellipse, and its form depends on such parameters as the height, the focus and the incidence of the object glass. If one knows the optimal (min-density) cover, then it is easy to estimate the parameter's values too.

The notation of the cover class $COV_k(p,q)$ remains the same as in [5], but in this paper p is the number of *ellipses* that cover one regular k-angle polygon (tile), and q is the quantity of different types of ellipses in the cover. Additionally, we consider that ellipses that have equal half-axles are equal (i.e. the congruous ellipses are equal). We will use notation $E_k(p,q)$ for min-density cover having special *structure* in the class $COV_k(p,q)$. Here, the structure is a coverage model which is determined by the geometric relationships of ellipses.

Definition 1.1 Consider a cover of one tile in the class $COV_k(p,q)$. Renumber the ellipses covering a tile and denote by e_i^t the i-th ellipse of type t, $t=1,\dots,q$, $i=1,\dots,n_t$, where n_t is a quantity of ellipses of type t covering one tile. Obviously, $\sum_{t=1}^{q} n_t = p$. The type t of an ellipse is defined by the sizes of its semi-axes $a(t)$ and $b(t)$. Denote by $A(e_i^t)$ tile area, where it can be located centre of the ellipse e_i^t. If one sets the number of each type of ellipses n_t and the placing domain of each ellipse $A(e_i^t)$, then define the *coverage model*. The choice of values $a(t),\ b(t),\ t=1,\dots,q$, the centre (from $A(e_i^t)$) and a tilt angle of each ellipse $e_i^t,\ i=1,\dots,n_t,\ t=1,\dots,q$, defines a concrete *cover*.

For example, the structure of the cover $C_3(7,3)$ is the coverage model when the centres of equal pairwise intersecting discs are located in the nodes of a triangular tile, and a curvilinear triangle in the centre of the tile is covered by one disc with a lesser radius and by three equal discs (Figure 1(a)). In this case, $n_1=3,\ n_2=3, n_3=1$, $A(e_i^1)$ – nodes of a tile, $A(e_1^3)$ – centre of a tile, and $A(e_i^2)$ – curvilinear triangles formed by discs of type 1 and 3.

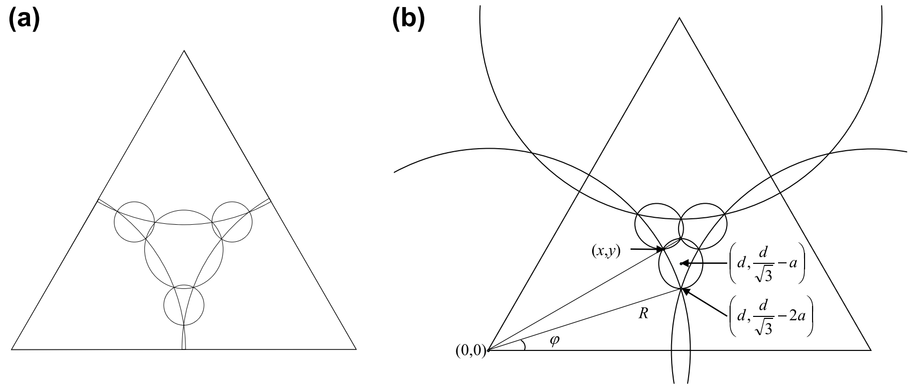

Figure 1. (a) Cover $C_3(7, 3)$ with density $D \approx 1.0677$; (b) Cover $E_3(6, 2)$ with density $D \approx 1.0786$.

Problem formulation. *In this paper, we consider the problem of the construction of regular plane covers with a minimal density in the classes $COV_k(p, q)$ when a tile is an equilateral triangle ($k = 3$) or a square ($k = 4$) and one tile is covered by p ellipses of q ($q = 1, 2, 3$) different types.*

It is evident that some covers using ellipses can be constructed from the covers that use discs by applying the *affine transformation* (AT). An AT is a transformation that preserves straight lines and ratios of distances between points lying on a straight line while keeping the coverage density the same. Examples of ATs include translation, expansion, reflection and rotation. An AT is equivalent to a linear transformation followed by a translation.

In [10], Fejes Tóth proposed a plane cover using discs of two radii for which the density tends to 1.0189, while the radii of small discs tend to zero. Although it is a strong theoretical result, such a cover is useless for WSNs because of the very large number of discs that cover one tile.

We proposed several new coverage models with a *small* number of ellipses, which are used to cover one tile. Also, we constructed several new regular plane covers using ellipses from covers that use discs by applying an AT. Some of these covers are optimal in their classes, another ones have a minimal known densities and used to find upper bounds for coverage density in the classes.

If the number of types of elements in a cover is fixed, then the usage of a square or hexagon tile in the regular cover with *discs* usually leads to a density increase in comparison with the covers that use triangular tiles.[5,6] However, if we use *ellipses*, then we will see that it is possible to obtain the same density for the covers that use either triangle or square, or hexagonal tiles.

The rest of the paper is organized as follows. We introduce new coverage models in Section 2, and theoretical analysis of density of each coverage model is conducted. Section 3 introduces an approach to generate a set of different regular plane covers in different classes by applying AT. Finally, we conclude our work in the last section.

2. New coverage models

In this section, we present three new coverage models by ellipses of two and three types.

2.1. *Coverage by ellipses of two types*

Let us consider the next coverage model in the class $COV_3(6, 2)$. In every cover in this coverage model, the centres of equal pairwise intersecting discs are located in the nodes of a triangular tile, and a curvilinear triangle in the centre of the tile is covered by three equal ellipses with one common point (Figure 1(b)). The density of each cover depends on the angle φ, and every cover indeed belongs to the one-parameter family of covers. We suppose that the cover $E_3(6, 2)$ has minimal density in the family (the value $\varphi \in [0, \pi/6]$ is optimal).

THEOREM 2.1 *The minimal density of the covers in the class $COV_3(6, 2)$ is upper bounded by* 1.0786.

Proof In order to proof the theorem, we will show that the density of the cover $E_3(6, 2) \in COV_3(6, 2)$ is at most 1.0786. Let R be the radius of the disc that has its centre in the node of a regular triangular tile, $2d$ is the length of the side of the triangular tile, a is the length of semi-major axis, and b is the length of the semi-minor axis of the ellipses that cover a curvilinear triangle in the centre of a tile. The latter three ellipses have one point in common, and the points of pairwise intersections of these ellipses lie on the sides of the curvilinear triangle. Then,

$$x = \frac{R\sqrt{3}}{2}; \quad y = \frac{R}{2}; \quad d = R\cos\varphi; \quad a = \frac{R}{2}\left(\frac{\cos\varphi}{\sqrt{3}} - \sin\varphi\right).$$

From equation of the ellipse, which reads

$$\frac{(x-d)^2}{b^2} + \frac{\left(y - d/\sqrt{3} + a\right)^2}{a^2} = 1$$

it can be inferred that

$$b = \frac{R(\cos\varphi/\sqrt{3} - \sin\varphi)(\cos\varphi - \sqrt{3}/2)}{\sqrt{(2\cos\varphi/\sqrt{3} - 1)(1 - 2\sin\varphi)}}, \qquad \varphi < \frac{\pi}{6}.$$

Then, the density of the cover can be calculated by the formula

$$D(R, a, b, d) = \frac{\pi}{2\sqrt{3}} \cdot \frac{R^2 + 6ab}{d^2}.$$

Or, after substitution,

$$D(\varphi) = \frac{\pi}{2\sqrt{3}\cos^2\varphi}\left[1 + 3^{1/4}\left(1 - \cos\left(\frac{\pi}{3} - 2\varphi\right)\right)\sqrt{\tan\left(\frac{\pi}{12} + \frac{\varphi}{2}\right)}\right].$$

The numerically found minimum is $D(\varphi) \approx 1.0786$ when $\varphi \approx 0.3055 \approx \pi/10.2835$. Since the cover $E_3(6, 2)$ is in $COV_3(6, 2)$, then the theorem is proved. □

2.2. *Coverage by ellipses of three types*

Let us consider a new coverage model (structure) in the class $COV_3(7, 3)$. In any cover with such a structure, each regular triangular tile is partially covered by equal discs with radius R and with their centres in the nodes of a tile; furthermore, a curvilinear triangle in the centre of a tile is covered by one disc of radius r that has its centre in the centre of a tile and by three equal ellipses that have a semi-major axis of length a and a semi-minor axis of length b. Denote by $E_3(7,3)$ the cover with given structure having minimal density.

THEOREM 2.2 *The minimal density of the covers in the class $COV_3(7,3)$ is upper bounded by* 1.056.

Proof It is sufficient to proof that the density of the cover $E_3(7,3) \in COV_3(7,3)$ is at most 1.056. To find the density of the cover $E_3(7,3)$, we set (for ease of operation) $R = 1$ and use the notations in Figure 2(a). Point $(R\cos(\alpha+\beta), R\sin(\alpha+\beta))$ belongs both to the disc of radius r and to the ellipse. Then,

$$(\cos\alpha - \cos(\alpha+\beta))^2 + (\cos\alpha/\sqrt{3} - \sin(\alpha+\beta))^2 = r^2;$$
$$\frac{(\cos\alpha - \cos(\alpha+\beta))^2}{b^2} + \frac{(\sin\alpha + a - \sin(\alpha+\beta))^2}{a^2} = 1.$$

Express the minor semi-axis as

$$b = \frac{a(\cos\alpha - \cos(\alpha+\beta))}{\sqrt{a^2 - (\sin\alpha + a - \sin(\alpha+\beta))^2}}.$$

Minimizing the area of the feasible ellipses (ellipses passing through certain points and ensuring a tile coverage)

$$\frac{a^2}{\sqrt{a^2 - (\sin\alpha + a - \sin(\alpha+\beta))^2}} \to \min_a,$$

we obtain

$$a = 2(\sin(\alpha+\beta) - \sin\alpha)/3, \quad b = 2(\cos\alpha - \cos(\alpha+\beta))/\sqrt{3}.$$

Then, the density is

$$D(\alpha,\beta) = \frac{\pi\left(\sqrt{3} + 2\sqrt{3}\left((\cos\alpha - \cos(\alpha+\beta))^2 + \left(\cos\alpha/\sqrt{3} - \sin(\alpha+\beta)\right)^2\right)\right)}{6\cos^2\alpha}$$

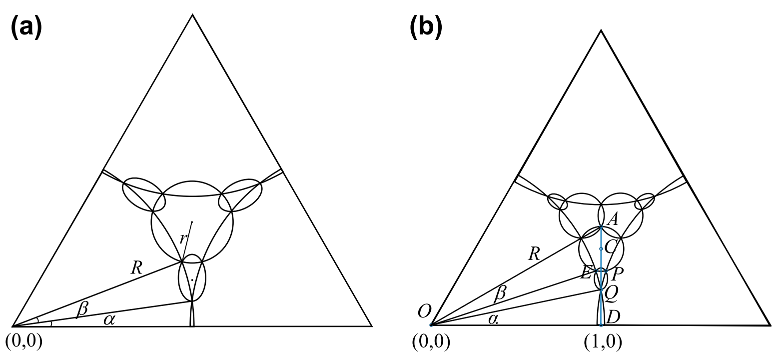

Figure 2. (a) Cover $E_3(7,3)$ with density $D \approx 1.056$; (b) Cover $E_3(9,3)$ with density $D \approx 1.0442$.

$$+\frac{4\,(\sin(\alpha+\beta)-\sin\alpha)\,(\cos\alpha-\cos(\alpha+\beta))}{3\cos^2\alpha}.$$

One can find (numerically) that $\min D(\alpha,\beta)\approx 1.056$ when $\alpha\approx 0.247\approx\pi/12.72$ and $\beta\approx 0.163\approx\pi/19.27$. □

If in the cover $E_3(7,3)$ one substitutes the disc of radius r by three equal discs with common point in the centre of a tile, then one gets a cover in the class $COV_3(9,3)$ (Figure 2(b)).

THEOREM 2.3 *The minimal density of the covers in the class $COV_3(9,3)$ is upper bounded by* 1.0442.

Proof In order to prove the theorem, one can find the optimal forms of discs and ellipses in the coverage model shown in Figure 2(b). Let us denote the corresponding cover (having minimal density in the coverage model) as $E_3(9,3)\in COV_3(9,3)$ and find its density. Suppose that the size of a triangular tile equals 2. Denote by R the radius of the discs with the centres in the nodes of a tile, and by r the radius of the three discs in the centre of a tile. The both radii depend on the angle α and

$$R(\alpha)=\frac{1}{\cos\alpha},\quad r(\alpha)=\frac{2}{\sqrt{3}}-\frac{1}{\cos\alpha}.$$

Moreover,

$$|CD|=\frac{1}{\sqrt{3}}-r=\frac{1}{\cos\alpha}-\frac{1}{\sqrt{3}},\quad \beta=\frac{\pi}{3}-2\arctan|CD|=\frac{\pi}{3}-2\arctan\left(\frac{1}{\cos\alpha}-\frac{1}{\sqrt{3}}\right).$$

There is a unique circumscribing ellipse (it is called Steiner ellipse) about triangle which (ellipse) has the minimal area $S=4\pi S_\Delta/\sqrt{27}$, where S_Δ is the area of the triangle.[12] Then, the ellipse circumscribing about the triangle with the vertices E, P and Q (Figure 2(b)) has the minimal area

$$S(\alpha)=\frac{4\pi}{\sqrt{27}\cos\alpha}\left(1-\frac{1}{\cos\alpha}\cos\left(2\arctan\left(\frac{1}{\cos\alpha}-\frac{1}{\sqrt{3}}\right)-\frac{\pi}{6}\right)\right)$$
$$\times\left(\sin\left(2\arctan\left(\frac{1}{\cos\alpha}-\frac{1}{\sqrt{3}}\right)-\frac{\pi}{6}\right)-\sin\alpha\right)$$

for every feasible alpha.

The coverage density is a ratio of twelfth area of the disc with radius R plus half area of ellipse and plus half area of the disc with radius r to the area of triangle OAD. In this case, the density depends on one variable – angle α and equals

$$D(\alpha)=2\sqrt{3}\left(\frac{\pi R^2}{12}+\frac{\pi r^2}{2}+\frac{S}{2}\right).$$

Minimum density is $D(\alpha)\approx 1.0442$ when $\alpha\approx 0.14$. □

3. Covers constructed by applying an AT

3.1. *Coverage by equal ellipses*

In [9], it was proved that, in the optimal (min-density) plane coverage by equal discs, the centres of three adjacent discs with one common point are the nodes of a regular triangle.

The density of such covers, denoted as $C_3(3, 1) \in COV_3(3, 1)$, does not depend on the radius and equals $2\pi/\sqrt{27} \approx 1.2091$. If a tile is a square, then the minimal density of the cover with a square tile $C_4(4, 1) \in COV_4(4, 1)$ using equal discs is $\pi/2 \approx 1.5708$.[5]

Applying an AT (a suitable vertical compression) to the cover $C_3(3, 1)$, we obtain an isosceles right-angled triangle as a tile. Two such triangles with a common hypotenuse form a square ($EBCF$ in Figure 3(a)). Then, the minimal density of a regular cover with ellipses $E_4(4, 1) \in COV_4(4, 1)$ using a square tile equals the density of a regular cover $C_3(3, 1) \in COV_3(3, 1)$ using discs, for which a triangle tile is used; this density equals $2\pi/\sqrt{27} \approx 1.2091$.

Moreover, one can find another cover with ellipses $E_4(2, 1) \in COV_4(2, 1)$ with the same density, where a square $ABDC$ is a tile (Figure 3(a)).

We conclude the subsection by

Proposition 3.1 *Let C is a plane cover by equal discs. Then its density $D(C) \geq 2\pi/\sqrt{27}$, and the equality is achieved for the covers $C_3(3, 1) \in COV_3(3, 1)$, $E_4(2, 1) \in COV_4(2, 1)$, and $E_4(4, 1) \in COV_4(4, 1)$.*

3.2. *Coverage by ellipses of two types*

In [5], we proposed a regular plane cover $C_3(4, 2) \in COV_3(4, 2)$ using discs of two radii with density $11\pi/(18\sqrt{3}) \approx 1.1084$, and we showed the optimality of this cover for the classes $COV_3(4, q)$, $q = 2, 3, 4$. The cover $C_4(5, 2) \in COV_4(5, 2)$ was considered as well, and its density is $3\pi/8 \approx 1.1787$.

Similarly, one can apply suitable ATs to the cover $C_3(4, 2)$ using discs [5] and obtain a regular cover $E_4(6, 2) \in COV_4(6, 2)$ using ellipses with the same density $11\pi/(18\sqrt{3}) \approx 1.1084$ (Figure 3(b)). A tile in this cover is a square $BCDE$ in Figure 3(b). However, in Figure 3(b), one can find another square tile $ABFC$, which is the determinant for the cover $E_4(4, 2) \in COV_4(4, 2)$.

Let us apply an AT (vertical compression) to the cover $E_3(6, 2)$ (Figure 1(b)) to obtain an isosceles right-angled triangle and a square tile, as we have done above. Then, a square tile is covered by ten ellipses of four types. In this way, we leave a certain class of covers (in this paper, we consider covers that use at most three types of ellipses). Nevertheless, in the classes $COV_4(6, 2)$ and $COV_4(10, 2)$ there are the regular plane covers that use

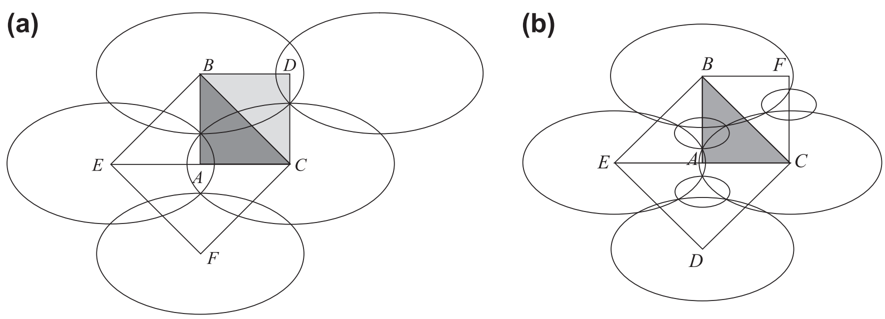

Figure 3. (a) Covers $E_4(4, 1)$ and $E_4(2, 1)$ with density $2\pi/\sqrt{27} \approx 1.2091$; (b) Covers $E_4(6, 2)$ and $E_4(4, 2)$ with density $D \approx 1.1084$.

ellipses of two types with a density $D \approx 1.0883$. In order to prove this proposition, let us consider a regular cover using discs, assuming that $a = b$ in the cover $E_3(6, 2)$, that have a minimal density in the coverage model that is approximately equal to 1.0883. After vertical compression, it appears that a right-angled isosceles triangle is covered by six ellipses of two types, and a square, which is generated by two such triangles with a common hypotenuse, is the determinant for the cover in the class $COV_4(10, 2)$. Moreover, one can find another square tile, which is the determinant for the cover in the class $COV_4(6, 2)$, with the same density.

Remark 1 A square tile in the last cover in the class $COV_4(6, 2)$ comprises two equal right-angled isosceles triangles with a common hypotenuse, and each triangle is covered by four ellipses of two types. If one apply the AT to obtain a regular triangular tile, then a half of the tiles will be covered in one way and another half in a different way. But six such tiles form a regular hexagon, and all hexagons are covered equally, and one gets a regular cover in the class $COV_6(14, 2)$ using hexagonal tiles with density $D \approx 1.0883$ (Figure 4(a)).

Similarly, one can construct a cover in the class $COV_6(5, 1)$ using triangles like ABC in Figure 3(a), or a cover in the class $COV_6(9, 2)$ using triangles like ABC in Figure 3(b).

3.3. *Coverage by ellipses of three types*

Let us consider a cover $C_3(7, 3) \in COV_3(7, 3)$ that uses discs (Figure 1(a)). The minimal density of such a cover is not more than 1.0677. The application of an AT allows us to obtain two regular covers in different classes, although they have the same density, namely, the covers $E_4(8, 3) \in COV_4(8, 3)$ and $E_4(12, 3) \in COV_4(12, 3)$.

PROPOSITION 3.2 *The minimal density of the regular covers in the classes $COV_4(8, 3)$ and $COV_4(12, 3)$ is upper bounded by* 1.0677.

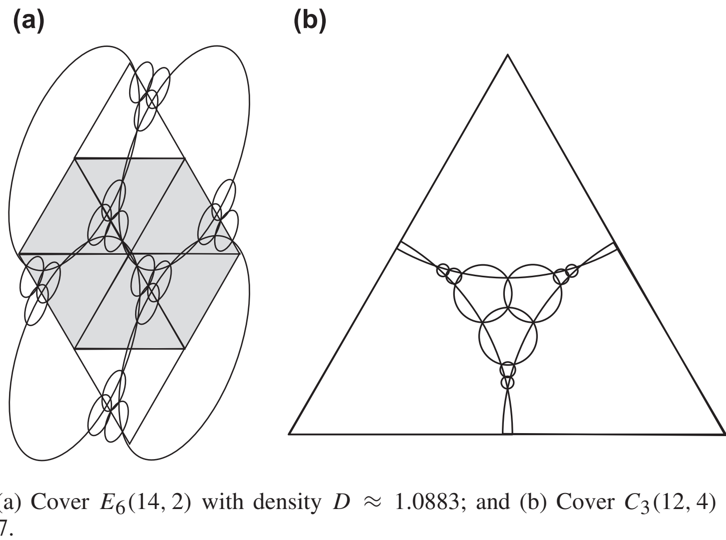

Figure 4. (a) Cover $E_6(14, 2)$ with density $D \approx 1.0883$; and (b) Cover $C_3(12, 4)$ with density $D \approx 1.0547$.

4. Conclusions

The problem of min-density plane coverage by ellipses has not yet been studied sufficiently well, what could be explained by the complexity of density minimisation problems. As usual, these problems are nonlinear and multi-extremal.

Thus, the *minimal* density of regular plane coverage with equal ellipses is $2\pi/\sqrt{27}$, and we proposed several such covers in different classes.

If two types of ellipses are used in a regular cover, then the density depends both on the class and the structure (coverage model) of the cover. Thus, for example, in class $COV_3(6, 2)$, we proposed a cover $E_3(6, 2)$ that has a density that is approximately equal to 1.0786. The cover $E_3(6, 2)$ is optimal in $COV_3(6, 2)$ among the covers with the same structure. However, if we consider the whole class $COV_3(6, 2)$, then the density of cover $E_3(6, 2)$ is the least known of the densities in the class $COV_3(6, 2)$.

Minimum density of the proposed regular covers is approximately equal to 1.0442. This is the least-known density in the class $COV_3(9, 3)$ and is minimal density of the covers that have the same structure. But we cannot prove its minimality in the whole class. Nevertheless, we believe that this paper contributes to the field by disseminating these novel results. In any case, we found the upper bounds for the minimum densities of covers in the different classes.

With an AT, we constructed several different nontrivial covers and estimated their densities. It is very difficult to find some of the covers in a different way.

The covers that have more than three types of ellipses are not considered in this paper because in this case, the density reduction is insignificant (for example, the minimal density of the cover $C_3(12, 4)$ by discs with four different radii in Figure 4(b) not less than 1.0547) but the optimisation problems become rather complex. Note that, in the majority of the considered cases, it was impossible to solve the corresponding optimisation problems analytically, and the minimal density was estimated numerically.

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